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On a Solvable Systems of Third Order Rational Difference Equations

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Abstract: The existence of solutions to several third-order rational difference equation systems having nonzero real values as initial conditions, as well as their periodicity, are discussed in this study. Theoretically, seven systems are introduced and discussed in detail. Finally, selected systems characterized by the initial conditions are presented numerically.

Keywords: Periodic solution; Stability, System of difference equations; Difference equation solution; Difference equations; Recursive sequences.

1 Introduction

For a variety of reasons, rational difference equations recently caught the attention of many scholars. In one sense, They provide illustrations of nonlinear problems that, while occasionally treatable, exhibit new dynamics as compared to the linear situation. However, because they are commonly used in various biological models, rational equations also make for interesting study topics. Large-scale planar rational systems are quickly investigated in the study of Camouzis et al. [1] for systems of equations with nonnegative parameters in this kind of study:

$$\left.\begin{aligned} Q_{n+1} &= \frac{\epsilon_1 + \epsilon_2 Q_n + \epsilon_3 R_n}{A_1 + B_1 Q_n + C_1 R_n} \\ R_{n+1} &= \frac{\epsilon_4 + \epsilon_5 Q_n + \epsilon_6 R_n}{A_2 + B_2 Q_n + C_2 R_n} \end{aligned}\right\}, \quad n = 0, 1, 2, ...,$$

they offer some results and open questions.

According to the referenced paper, it is possible to reduce some of these structures to recently explored second-order rational equations called Ricatti equations. Additionally, Camouzis et al. came up with an index of 325 non-comparable systems to which emphasis should be directed because for some parameter selections, one acquires a system that is the same as the circumstance plus some additional factors. These systems are listed as pairs with the notation k, l, where k and l denote the number of the associated equation. One of many publications on difference equation structures is one of many that deals with the periodic positive solutions of rational difference equations:

$$Q_{n+1} = \frac{1}{R_n}, \quad R_{n+1} = \frac{R_n}{Q_{n-1}R_{n-1}},$$

was acquired by Cinar in [2].

The following system of differential equations has been solved by Elsayed [3]:

$$Q_{n+1} = \frac{1}{R_{n-k}}, \quad R_{n+1} = \frac{R_{n-k}}{Q_n R_n}.$$

Elsayed and Gafel [4] dealt with periodic and systems of difference equations and their solutions:

$$Q_{n+1} = \frac{1 \pm (R_n + Q_{n-1})}{R_{n-2}}, \quad R_{n+1} = \frac{1 \pm (Q_n + R_{n-1})}{Q_{n-2}}.$$

The way the following system's constructive solutions behave:

$$Q_{n+1} = \frac{Q_{n-1}}{1 + Q_{n-1}R_n}, \quad R_{n+1} = \frac{R_{n-1}}{1 + R_{n-1}Q_n}.$$

Kurbanli et al. [5] have examined the subject.

Kurbanli [6] explored how the difference equation system's solution behaved:

$$Q_{n+1} = \frac{Q_{n-1}}{Q_{n-1}R_n - 1}, \quad R_{n+1} = \frac{R_{n-1}}{R_{n-1}Q_n - 1},$$
$$S_{n+1} = \frac{1}{S_n R_n}.$$

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Özban [7] has looked into A set of rational difference formulas' positive solution:

$$Q_{n+1} = \frac{\alpha}{R_{n-3}}, \; R_{n+1} = \frac{\beta R_{n-3}}{Q_{n-q}R_{n-q}}.$$

The periodicity of the following systems was also explored by Touafek et al. [8] who provided the form of the solutions:

$$Q_{n+1} = \frac{R_n}{Q_{n-1}(\pm 1 \pm R_n)}, \; R_{n+1} = \frac{Q_n}{R_{n-1}(\pm 1 \pm Q_n)}.$$

In [9, 10] Yalçınkaya looked into the necessary conditions for the systems listed below of difference equations to be globally asymptotically stable:

$$S_{n+1} = \frac{t_n S_{n-1} + \alpha}{t_n + S_{n-1}}, \; t_{n+1} = \frac{S_n t_{n-1} + \alpha}{S_n + t_{n-1}},$$

and

$$Q_{n+1} = \frac{Q_n + R_{n-1}}{Q_n R_{n-1} - 1}, \; R_{n+1} = \frac{R_n + Q_{n-1}}{R_n Q_{n-1} - 1}.$$

In [11, 12], Global asymptotic stability, persistence, and boundedness of positive solutions to systems of difference equations were all topics of study by Zhang et al.

$$Q_n = A + \frac{1}{R_{n-p}}, \quad R_n = A + \frac{R_{n-1}}{Q_{n-r}R_{n-s}},$$

and

$$Q_{n+} = A + \frac{R_{n-m}}{Q_n}, \quad R_{n+1} = A + \frac{Q_{n-m}}{R_n}.$$

Reasonable difference equations in nonlinear structures and difference equations have both been researched; more details can be found in [13]- [38].

Definition 1.*(Periodicity) If $Q_{n+p} = Q_n$ for all $n \geq -k.$, a sequence $\{Q_n\}_{n=-k}^{\infty}$ is said to be periodic with period p.*

The periodicity and shape of several nonlinear difference equation systems of order three percent are examined in this research:

$$Q_{n+1} = \frac{Q_n R_{n-2}}{R_{n-1}(\pm 1 \pm Q_n R_{n-2})}, \qquad R_{n+1} = \frac{R_n Q_{n-2}}{Q_{n-1}(\pm 1 \pm R_n Q_{n-2})},$$

with initially requirements that $Q_{-2}, Q_{-1}, Q_0, R_{-2}, R_{-1}$, and R_0 are real, nonzero values.

2 The First System

This section provides the format of the system of difference equations solutions:

$$Q_{n+1} = \frac{Q_n R_{n-2}}{R_{n-1}(-1 - Q_n R_{n-2})}, \qquad R_{n+1} = \frac{R_n Q_{n-2}}{Q_{n-1}(-1 + R_n Q_{n-2})}, \tag{1}$$

in which $n = 0, 1, 2, \cdots$ and the initial circumstances $Q_{-2},\ Q_{-1},\ Q_0,\ R_{-2},\ R_{-1}$ *and* R_0 are arbitrary nonzero real numbers with $Q_0 R_{-2} \neq -1,\ Q_{-2}R_0 \neq 1$.

Theorem 1.*If $\{Q_n, R_n\}$ are differences in the solutions formula system (1). Then for $n = 0, 1, 2, \cdots,$:*

$$Q_{4n-2} = \frac{\alpha^n w^n}{\gamma^{n-1}\delta^n}, \; Q_{4n-1} = \frac{\beta\alpha^n w^n}{\gamma^n \delta^n}\frac{(-1+\gamma\delta)^n}{(-1-\alpha w)^n},$$
$$Q_{4n} = \frac{\alpha^{n+1} w^n}{\gamma^n \delta^n}, \; Q_{4n+1} = \frac{\alpha^{n+1} w^{n+1}(-1+\gamma\delta)^n}{e\gamma^n \delta^n (-1-\alpha w)^{n+1}},$$

and

$$R_{4n-2} = \frac{\gamma^n \delta^n}{\alpha^n w^{n-1}}, \; R_{4n-1} = \frac{e\gamma^n \delta^n}{\alpha^n w^n}\frac{(-1-\alpha w)^n}{(-1+\gamma\delta)^n},$$
$$R_{4n} = \frac{\gamma^n \delta^{n+1}}{\alpha^n w^n}, \; R_{4n+1} = \frac{\gamma^{n+1}\delta^{n+1}(-1-\alpha w)^n}{\beta\alpha^n w^n (-1+\gamma\delta)^{n+1}}.$$

with $Q_{-2} = c,\ Q_{-1} = b,\ Q_0 = a,\ R_{-2} = w,\ R_{-1} = e$ and $R_0 = d$.

*Proof.*For $n = 0$, the conclusion is valid. Let's assume that $n > 1$ and $n - 1$ are both consistent with our assumption. Meaning:

$$Q_{4n-6} = \frac{\alpha^{n-1}w^{n-1}}{\gamma^{n-2}d^{n-1}},$$
$$Q_{4n-5} = \frac{(\beta\alpha)^{n-1}w^{n-1}(-1+\gamma\delta)^{n-1}}{\gamma^{n-1}d^{n-1}(-1-\alpha w)^{n-1}},$$
$$Q_{4n-4} = \frac{\alpha^n w^{n-1}}{\gamma^{n-1}\delta^{n-1}},$$
$$Q_{4n-3} = \frac{\alpha^n w^n (-1+\gamma\delta)^{n-1}}{e\gamma^{n-1}\delta^{n-1}(-1-\alpha w)^n}, \tag{2}$$

and

$$R_{4n-6} = \frac{\gamma^{n-1}\delta^{n-1}}{\alpha^{n-1}w^{n-2}},$$
$$R_{4n-5} = \frac{e\gamma^{n-1}\delta^{n-1}(-1-\alpha w)^{n-1}}{\alpha^{n-1}w^{n-1}(-1+\gamma\delta)^{n-1}},$$
$$R_{4n-4} = \frac{\gamma^{n-1}\delta^n}{\alpha^{n-1}w^{n-1}},$$
$$R_{4n-3} = \frac{\gamma^n \delta^n (-1-\alpha w)^{n-1}}{(\beta\alpha)^{n-1}w^{n-1}(-1+\gamma\delta)^n}. \tag{3}$$

Now, it is evident from Eq. (1) that:

$$Q_{4n-2} = \frac{Q_{4n-3}R_{4n-5}}{R_{4n-4}(-1 - Q_{4n-3}R_{4n-5})}.$$

Uisng Eqs. (3) and (3) to get:

$$Q_{4n-2} = \frac{\left(\frac{\alpha w}{(-1-\alpha w)}\right)}{\left(\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\right)\left(-1-\frac{\alpha}{(-1-\alpha w)}\right)} = \frac{\alpha w}{\left(\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\right)(1-\alpha w+\alpha w)} = \frac{\alpha^{n}w^{n}}{\gamma^{n-1}\delta^{n}},$$

$$R_{4n-2} = \frac{R_{4n-3}Q_{4n-5}}{Q_{4n-4}\left(-1+R_{4n-3}Q_{4n-5}\right)} = \frac{\left(\frac{\gamma\delta}{(-1+\gamma\delta)}\right)}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}d^{n-1}}\right)\left(-1+\left(\frac{\gamma\delta}{(-1+\gamma\delta)}\right)\right)} = \frac{\gamma\delta}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}d^{n-1}}\right)(1-\gamma\delta+\gamma\delta)} = \frac{\gamma^{n}\delta^{n}}{\alpha^{n}w^{n-1}}.$$

Additionally, Eq. (1) shows that:

$$Q_{4n-1} = \frac{Q_{4n-2}R_{4n-4}}{R_{4n-3}\left(-1-Q_{4n-2}R_{4n-4}\right)} = \frac{\alpha w}{\left(\frac{\gamma^{n}\delta^{n}(-1-\alpha w)^{n-1}}{(\beta\alpha)^{n-1}w^{n-1}(-1+\gamma\delta)^{n}}\right)(-1-\alpha w)} = \frac{\alpha w(\beta\alpha)^{n-1}w^{n-1}(-1+\gamma\delta)^{n}}{\gamma^{n}\delta^{n}(-1-\alpha w)^{n-1}\left(-1-\alpha w\right)} = \frac{\beta\alpha^{n}w^{n}(-1+\gamma\delta)^{n}}{\gamma^{n}\delta^{n}(-1-\alpha w)^{n}},$$

$$R_{4n-1} = \frac{R_{4n-2}Q_{4n-4}}{Q_{4n-3}\left(-1+R_{4n-2}Q_{4n-4}\right)} = \frac{\gamma\delta}{\left(\frac{\alpha^{n}w^{n}(-1+\gamma\delta)^{n-1}}{e\gamma^{n-1}\delta^{n-1}(-1-\alpha w)^{n}}\right)(-1+\gamma\delta)} = \frac{\gamma\delta e\gamma^{n-1}\delta^{n-1}(-1-\alpha w)^{n}}{\alpha^{n}w^{n}(-1+\gamma\delta)^{n-1}\left(-1+\gamma\delta\right)} = \frac{e\gamma^{n}\delta^{n}(-1-\alpha w)^{n}}{\alpha^{n}w^{n}(-1+\gamma\delta)^{n}},$$

and so,

$$Q_{4n} = \frac{Q_{4n-1}R_{4n-3}}{R_{4n-2}\left(-1-Q_{4n-1}R_{4n-3}\right)} = \frac{\frac{\alpha w}{(-1-\alpha w)}}{\left(\frac{\gamma^{n}\delta^{n}}{\alpha^{n}w^{n-1}}\right)\left(-1-\frac{\alpha w}{(-1-\alpha w)}\right)} = \frac{\alpha^{n}w^{n-1}\alpha w}{\gamma^{n}\delta^{n}\left(1-\alpha w+\alpha w\right)} = \frac{\alpha^{n+1}w^{n}}{\gamma^{n}\delta^{n}},$$

$$R_{4n} = \frac{R_{4n-1}Q_{4n-3}}{Q_{4n-2}\left(-1+R_{4n-1}Q_{4n-3}\right)} = \frac{\frac{\gamma\delta}{(-1+\gamma\delta)}}{\left(\frac{\alpha^{n}w^{n}}{\gamma^{n-1}\delta^{n}}\right)\left(-1+\frac{\gamma\delta}{(-1+\gamma\delta)}\right)} = \frac{\gamma^{n-1}\delta^{n}\gamma\delta}{\alpha^{n}w^{n}\left(1-\gamma\delta+\gamma\delta\right)} = \frac{\gamma^{n}d^{n+1}}{\alpha^{n}w^{n}}.$$

Finally, Eq. (1) demonstrates that:

$$Q_{4n+1} = \frac{Q_{4n}R_{4n-2}}{R_{4n-1}\left(-1-Q_{4n}R_{4n-2}\right)} = \frac{\alpha w}{\left(\frac{e\gamma^{n}\delta^{n}}{\alpha^{n}w^{n}}\frac{(-1-\alpha w)^{n}}{(-1+\gamma\delta)^{n}}\right)(-1-\alpha w)} = \frac{\alpha^{n}w^{n}\alpha w(-1+\gamma\delta)^{n}}{e\gamma^{n}\delta^{n}(-1-\alpha w)^{n}\left(-1-\alpha w\right)} = \frac{\alpha^{n+1}w^{n+1}(-1+\gamma\delta)^{n}}{e\gamma^{n}\delta^{n}(-1-\alpha w)^{n+1}},$$

$$R_{4n+1} = \frac{R_{4n}Q_{4n-2}}{Q_{4n-1}\left(-1+R_{4n}Q_{4n-2}\right)} = \frac{\gamma\delta}{\left(\frac{\beta\alpha^{n}w^{n}}{\gamma^{n}\delta^{n}}\frac{(-1+\gamma\delta)^{n}}{(-1-\alpha w)^{n}}\right)(-1+\gamma\delta)} = \frac{\gamma^{n}\delta^{n}\gamma\delta(-1-\alpha w)^{n}}{\beta\alpha^{n}w^{n}(-1+\gamma\delta)^{n}\left(-1+\gamma\delta\right)} = \frac{\gamma^{n+1}\delta^{n+1}(-1-\alpha w)^{n}}{\beta\alpha^{n}w^{n}(-1+\gamma\delta)^{n+1}}.$$

The evidence is now complete.

Lemma 1. *With the exception of the following scenario, system (1)'s solution is unlimited.*

Theorem 2. *If $\gamma\delta = 2$, $\alpha w = -2$, system (1) has an eight-period then the periodic solution and it has a subsequent form:*

$$\{Q_n\} = \left\{\gamma,\beta,\alpha,\frac{\alpha w}{e},-\gamma,-\beta,-\alpha,-\tfrac{\alpha w}{e},\gamma,\beta,\alpha,...\right\},$$
$$\{R_n\} = \left\{w,e,\gamma,\frac{\gamma\delta}{\beta},-w,-e,-\delta,-\frac{\gamma\delta}{\beta},w,e,\delta,...\right\}.$$

Proof. Let's assume that there is a prime period eight answer first.

$$\{Q_n\} = \left\{\gamma,\beta,\alpha,\frac{\alpha w}{e},-\gamma,-\beta,-\alpha,-\frac{\alpha w}{e},\gamma,\beta,\alpha,...\right\},$$
$$\{R_n\} = \left\{w,e,d,\frac{\gamma\delta}{\beta},-w,-e,-\delta,-\frac{\gamma\delta}{b},w,e,\delta,...\right\},$$

of the system (1), we see from the form of the solution of system (1) that:

$$Q_{4n-2} = \pm\gamma = \frac{\alpha^n w^n}{\gamma^{n-1}\delta^n},$$
$$Q_{4n-1} = \pm\beta = \frac{\beta\alpha^n w^n}{\gamma^n\delta^n}\frac{(-1+\gamma\delta)^n}{(-1-\alpha w)^n},$$
$$Q_{4n} = \pm\alpha = \frac{\alpha^{n+1}w^n}{\gamma^n\delta^n},$$
$$Q_{4n+1} = \pm\frac{\alpha w}{e} = \frac{\alpha^{n+1}w^{n+1}(-1+\gamma\delta)^n}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}},$$

and

$$R_{4n-2} = \pm w = \frac{\gamma^n\delta^n}{\alpha^n w^{n-1}},$$
$$R_{4n-1} = \pm e = \frac{e\gamma^n\delta^n}{\alpha^n w^n}\frac{(-1-\alpha w)^n}{(-1+\gamma\delta)^n},$$
$$R_{4n} = \pm\delta = \frac{\gamma^n\delta^{n+1}}{\alpha^n w^n},$$
$$R_{4n+1} = \pm\frac{\gamma\delta}{\beta} = \frac{\gamma^{n+1}d\delta^{n+1}(-1-\alpha w)^n}{\beta\alpha^n w^n(-1+\gamma\delta)^{n+1}}.$$

Then we get:

$$\gamma\delta = -\alpha w, \quad -1+\gamma\delta = -1-\alpha w = 1$$

Thus

$$\gamma\delta = 2, \quad \alpha w = -2.$$

Second, enumerate $\gamma\delta = 2, \ \alpha w = -2$. The design of the System (1) solution demonstrates this to us”

$$Q_{4n-2} = \frac{\alpha^n w^n}{\gamma^{n-1}\delta^n} = (-1)^n c,$$
$$Q_{4n-1} = \frac{\beta\alpha^n w^n}{\gamma^n\delta^n}\frac{(-1+\gamma\delta)^n}{(-1-\alpha w)^n} = (-1)^n\beta,$$
$$Q_{4n} = \frac{\alpha^{n+1}w^n}{\gamma^n\delta^n} = (-1)^n\alpha,$$
$$Q_{4n+1} = \frac{\alpha^{n+1}w^{n+1}(-1+\gamma\delta)^n}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}} = (-1)^n\frac{\alpha w}{e},$$

and

$$R_{4n-2} = \frac{\gamma^n\delta^n}{\alpha^n w^{n-1}} = (-1)^n w,$$
$$R_{4n-1} = \frac{e\gamma^n\delta^n}{\alpha^n w^n}\frac{(-1-\alpha w)^n}{(-1+\gamma\delta)^n} = (-1)^n e,$$
$$R_{4n} = \frac{\gamma^n\delta^{n+1}}{\alpha^n w^n} = (-1)^n\gamma,$$
$$R_{4n+1} = \frac{\gamma^{n+1}\delta^{n+1}(-1-\alpha w)^n}{\beta\alpha^n w^n(-1+\gamma\delta)^{n+1}} = (-1)^n\frac{\gamma\delta}{\beta},$$

As a result, the proof is finished, and we get an eight-period periodic answer.

We believe that the systems in the next paragraphs are sound, and the reader is therefore left to prove the following theorems.

$$Q_{n+1} = \frac{Q_n R_{n-2}}{R_{n-1}(-1-Q_n R_{n-2})}, \qquad R_{n+1} = \frac{R_n Q_{n-2}}{Q_{n-1}(-1-R_n Q_{n-2})}. \tag{4}$$

The definitions of the form of the solutions to system (4) are the focus of the subsequent theorem $Q_{-2} = c, \ Q_{-1} = b, \ Q_0 = a, \ R_{-2} = f, \ R_{-1} = e$ *and* $R_0 = d$.

Theorem 3. *Assume that Q_n and R_n are the solutions to system (4), in which case $Q_0R_{-2}, \ Q_{-2}R_0 \neq -1$. Then for $n = 0, 1, 2, \cdots$,*

$$Q_{4n-2} = \frac{\alpha^n w^n}{\gamma^{n-1}\delta^n}, \quad Q_{4n-1} = \frac{\beta\alpha^n w^n}{\gamma^n\delta^n}\frac{(-1-\gamma\delta)^n}{(-1-\alpha w)^n},$$
$$Q_{4n} = \frac{\alpha^{n+1}w^n}{\gamma^n\delta^n}, \quad Q_{4n+1} = \frac{\alpha^{n+1}w^{n+1}(-1-\gamma\delta)^n}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}},$$

and

$$R_{4n-2} = \frac{\gamma^n\delta^n}{\alpha^n w^{n-1}}, \quad R_{4n-1} = \frac{e\gamma^n\delta^n}{\alpha^n w^n}\frac{(-1-\alpha w)^n}{(-1-\gamma\delta)^n},$$
$$R_{4n} = \frac{\gamma^n\delta^{n+1}}{\alpha^n w^n}, \quad R_{4n+1} = \frac{\gamma^{n+1}\delta^{n+1}(-1-\alpha w)^n}{\beta\alpha^n w^n(-1-\gamma\delta)^{n+1}}.$$

Lemma 2. *With the exception of the following scenario, system (4)'s solution is unlimited.*

Theorem 4. *If $\gamma\delta = \alpha w = -2$, the system (2) has a periodical solution with a period of four, and will appear as depicted below.* $\{Q_n\} = \left\{\gamma, \beta, \alpha, \frac{\alpha w}{e}, \gamma, \beta, \alpha, ...\right\}$, $\{R_n\} = \left\{w, e, \delta, \frac{\gamma\delta}{b}, w, e, \delta, ...\right\}$.

Proof. Let's start by assuming that there is a prime period four answer.

$$\{Q_n\} = \left\{\gamma, \beta, \alpha, \frac{\alpha w}{e}, \gamma, \beta, \alpha, ...\right\},$$
$$7\{R_n\} = \left\{w, e, \delta, \frac{\gamma\delta}{\beta}, w, e, \delta, ...\right\}.$$

Considering system (4), we note from the structure of its solution that:

$$Q_{4n-2} = \gamma = \frac{\alpha^n w^n}{\gamma^{n-1}\delta^n},$$
$$Q_{4n-1} = \beta = \frac{\beta\alpha^n w^n}{\gamma^n\delta^n}\frac{(-1-\gamma\delta)^n}{(-1-\alpha w)^n},$$
$$Q_{4n} = \alpha = \frac{\alpha^{n+1}w^n}{\gamma^n\delta^n},$$
$$Q_{4n+1} = \frac{\alpha w}{e} = \frac{\alpha^{n+1}w^{n+1}(-1-\gamma\delta)^n}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}},$$

and

$$R_{4n-2} = w = \frac{\gamma^n \delta^n}{\alpha^n w^{n-1}},$$
$$R_{4n-1} = e = \frac{e\gamma^n \delta^n}{\alpha^n w^n} \frac{(-1-\alpha w)^n}{(-1-\gamma\delta)^n},$$
$$R_{4n} = \delta = \frac{\gamma^n \delta^{n+1}}{\alpha^n w^n},$$
$$R_{4n+1} = \frac{\gamma\delta}{\beta} = \frac{\gamma^{n+1}\delta^{n+1}(-1-\alpha w)^n}{\beta\alpha^n w^n(-1-\gamma\delta)^{n+1}}.$$

Then we get:

$$\gamma\delta = \alpha w, \quad -1-\gamma\delta = -1-\alpha w = 1$$

Thus

$$\gamma\delta = \alpha w = -2.$$

Second, assume that $\gamma\delta = \alpha w = -2$. Then we see from the form of the solution of system (4) that Secondly, suppose $\gamma\delta = \alpha w = -2$.. The shape of the solution to System (4) shows us that:

$$Q_{4n-2} = \frac{\alpha^n w^n}{\gamma^{n-1}\delta^n} = \gamma,$$
$$Q_{4n-1} = \frac{\beta\alpha^n w^n}{\gamma^n \delta^n} \frac{(-1-\gamma\delta)^n}{(-1-\alpha w)^n} = \beta,$$
$$Q_{4n} = \frac{\alpha^{n+1} w^n}{\gamma^n \delta^n} = \alpha,$$
$$Q_{4n+1} = \frac{\alpha^{n+1} w^{n+1}(-1-\gamma\delta)^n}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}} = \frac{\alpha w}{e},$$

and

$$R_{4n-2} = \frac{\gamma^n \delta^n}{\alpha^n w^{n-1}} = w,$$
$$R_{4n-1} = \frac{e\gamma^n\delta^n}{\alpha^n w^n} \frac{(-1-\alpha w)^n}{(-1-\gamma\delta)^n} = e,$$
$$R_{4n}7 = frac\gamma^n\delta^{n+1}\alpha^n w^n = \gamma,$$
$$R_{4n+1} = \frac{\gamma^{n+1}\delta^{n+1}(-1-\alpha w)^n}{\beta\alpha^n w^n(-1-\gamma\delta)^{n+1}} = \frac{\gamma\delta}{\beta}.$$

Since we currently have a periodical solution for period 4, the proof is finished.

3 The Second System:

The solutions to the set of difference equations are given in this section.

$$Q_{n+1} = \frac{Q_n R_{n-2}}{R_{n-1}(-1+Q_n R_{n-2})},$$
$$R_{n+1} = \frac{R_n Q_{n-2}}{Q_{n-1}(1+R_n Q_{n-2})}, \tag{5}$$

in which $n = 0, 1, 2, \cdots$ and the initial circumstances $Q_{-2}, Q_{-1}, Q_0, R_{-2}, R_{-1}$ *and* R_0 are random nonzero real values that have $Q_0 R_{-2} \neq 1$.

Theorem 5.*If* $\{Q_n, R_n\}$ *are answers to the system of three difference equations. Then for* $n = 0, 1, 2, \cdots$,

$$Q_{4n-2} = \frac{\alpha^n w^n}{\gamma^n \delta^{n-1}} \prod_{j=0}^{n-1} (1+(4)j\gamma\delta),$$
$$Q_{4n-1} = \frac{\beta\alpha^n w^n}{\gamma^n\delta^n} \frac{\prod_{j=0}^{n-1}(1+(4j+1)\gamma\delta)}{(-1+\alpha w)^n},$$
$$Q_{4n} = \frac{\alpha^{n+1} w^n}{\gamma^n\delta^n} \prod_{j=0}^{n-1}(1+(4j+2)\gamma\delta),$$
$$Q_{4n+1} = \frac{\alpha^{n+1} w^{n+1}}{e\gamma^n\delta^n(-1+\alpha w)^{n+1}} \prod_{j=0}^{n-1}(1+(4j+3)\gamma\delta),$$

and

$$R_{4n-2} = \frac{\gamma^n\delta^n}{\alpha^n w^{n-1}} \prod_{j=0}^{n-1} \frac{1}{(1+(4j+2)\gamma\delta)},$$
$$R_{4n-1} = \frac{e\gamma^n\delta^n}{\alpha^n w^n} \frac{(-1+\alpha w)^n}{\prod_{j=0}^{n-1}(1+(4j+3)\gamma\delta)},$$
$$R_{4n} = \frac{\gamma^n\delta^{n+1}}{\alpha^n w^n} \prod_{j=0}^{n-1} \frac{1}{(1+(4j+4)\gamma\delta)},$$
$$R_{4n+1} = \frac{\gamma^{n+1} d\delta^{n+1}}{\beta\alpha^n w^n(1+\gamma\delta)} \frac{(-1+\alpha w)^n}{\prod_{j=0}^{n-1}(1+(4j+5)\gamma\delta)},$$

where $Q_{-2} = \gamma,\ Q_{-1} = \beta,\ Q_0 = \alpha,\ R_{-2} = w,\ R_{-1} = e$ *and* $R_0 = \delta$.

*Proof.*Since $n = 0$, the conclusion is valid. Let's now assume that $n > 0$ and that $n-1$ are consistent with our assumption. meaning,

$$Q_{4n-6} = \frac{\alpha^{n-1} w^{n-1}}{\gamma^{n-1}\delta^{n-2}} \prod_{j=0}^{n-2}(1+(4j)\gamma\delta),$$
$$Q_{4n-5} = \frac{(\beta\alpha)^{n-1} w^{n-1}}{\gamma^{n-1}\delta^{n-1}} \frac{\prod_{j=0}^{n-2}(1+(4j+1)\gamma\delta)}{(-1+\alpha w)^{n-1}},$$
$$Q_{4n-4} = \frac{\alpha^n w^{n-1}}{\gamma^{n-1}\delta^{n-1}} \prod_{j=0}^{n-2}(1+(4j+2)\gamma\delta),$$
$$Q_{4n-3} = \frac{\alpha^n w^n}{e\gamma^{n-1}\delta^{n-1}} \frac{\prod_{j=0}^{n-2}(1+(4j+3)\gamma\delta)}{(-1+\alpha w)^n},$$

and

$$R_{4n-6}=\frac{\gamma^{n-1}\delta^{n-1}}{\alpha^{n-1}w^{n-2}}\prod_{j=0}^{n-2}\frac{1}{(1+(4j+2)\gamma\delta)},$$
$$R_{4n-5}=\frac{e\gamma^{n-1}\delta^{n-1}}{\alpha^{n-1}w^{n-1}}\frac{(-1+\alpha w)^{n-1}}{\prod_{j=0}^{n-2}(1+(4j+3)\gamma\delta)},$$
$$R_{4n-4}=\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\prod_{j=0}^{n-2}\frac{1}{(1+(4j+4)\gamma\delta)},$$
$$R_{4n-3}=\frac{\gamma^{n}\delta^{n}}{(\beta\alpha)^{n-1}w^{n-1}(1+\gamma\delta)}\frac{(-1+\alpha w)^{n-1}}{\prod_{j=0}^{n-2}(1+(4j+5)\gamma\delta)}.$$

Now, Eq. (5) indicates that:

$$\begin{aligned}Q_{4n-2}&=\frac{Q_{4n-3}R_{4n-5}}{R_{4n-4}\left(-1+Q_{4n-3}R_{4n-5}\right)}\\&=\frac{\left(\frac{\alpha w}{(-1+\alpha w)}\right)}{\left(\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\prod_{j=0}^{n-2}\frac{1}{(1+(4j+4)\gamma\delta)}\right)}\times\frac{1}{\left(-1+\left(\frac{\alpha w}{(-1+\alpha w)}\right)\right)}\\&=\frac{\alpha w}{\left(\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\prod_{j=0}^{n-2}\frac{1}{(1+(4j+4)\gamma\delta)}\right)(1-\alpha w+\alpha w)}\\&=\frac{\alpha^{n-1}w^{n-1}\alpha w}{\gamma^{n-1}\delta^{n}}\prod_{j=0}^{n-2}(1+(4j+4)\gamma\delta)\\&=\frac{\alpha^{n}w^{n}}{\gamma^{n-1}\delta^{n}}\prod_{j=0}^{n-1}(1+(4j)\gamma\delta),\end{aligned}$$

$$\begin{aligned}R_{4n-2}&=\frac{R_{4n-3}Q_{4n-5}}{Q_{4n-4}\left(1+R_{4n-3}Q_{4n-5}\right)}\\&=\frac{\left(\frac{\gamma\delta}{(1+\gamma\delta)}\frac{\prod_{j=0}^{n-2}(1+(4j+1)\gamma\delta)}{\prod_{j=0}^{n-2}(1+(4j+5)\gamma\delta)}\right)}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}\delta^{n-1}}\prod_{j=0}^{n-2}(1+(4j+2)\gamma\delta)\right)}\times\frac{1}{\left(1+\left(\frac{\gamma\delta}{(1+\gamma\delta)}\frac{\prod_{j=0}^{n-2}(1+(4j+1)\gamma\delta)}{\prod_{j=0}^{n-2}(1+(4j+5)\gamma\delta)}\right)\right)}\\&=\frac{\frac{\gamma\delta}{(1+(4n-3)\gamma\delta)}}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}\delta^{n-1}}\prod_{j=0}^{n-2}(1+(4j+2)\gamma\delta)\right)\left(1+\left(\frac{\gamma\delta}{(1+(4n-3)\gamma\delta)}\right)\right)}\\&=\frac{\gamma\delta}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}\delta^{n-1}}\prod_{j=0}^{n-2}(1+(4j+2)\gamma\delta)\right)(1+(4n-3)\gamma\delta+\gamma\delta)}\\&=\frac{\gamma^{n-1}\delta^{n-1}\gamma\delta}{\left(\alpha^{n}w^{n-1}\prod_{j=0}^{n-2}(1+(4j+2)\gamma\delta)\right)(1+(4n-2)\gamma\delta)}\\&=\frac{\gamma^{n}\delta^{n}}{\alpha^{n}w^{n-1}\prod_{j=0}^{n-1}(1+(4j+2)\gamma\delta)}.\end{aligned}$$

Moreover, Eq. (5) illustrates that:

$$\begin{aligned}Q_{4n-1}&=\frac{Q_{4n-2}R_{4n-4}}{R_{4n-3}\left(-1+Q_{4n-2}R_{4n-4}\right)}\\&=\frac{\alpha w}{\left(\frac{\gamma^{n}\delta^{n}}{(\beta\alpha)^{n-1}w^{n-1}(1+\gamma\delta)}\frac{(-1+\alpha w)^{n-1}}{\prod_{j=0}^{n-2}(1+(4j+5)\gamma\delta)}\right)(-1+\alpha w)}\\&=\frac{(\beta\alpha)^{n-1}w^{n-1}(1+\gamma\delta)\alpha w\prod_{j=0}^{n-2}(1+(4j+5)\gamma\delta)}{\gamma^{n}\delta^{n}(-1+\alpha w)^{n-1}(-1+\alpha w)}\\&=\frac{\beta\alpha^{n}w^{n}\prod_{j=0}^{n-1}(1+(4j+1)\gamma\delta)}{\gamma^{n}\delta^{n}(-1+\alpha w)^{n}},\end{aligned}$$

$$\begin{aligned}R_{4n-1}&=\frac{R_{4n-2}Q_{4n-4}}{Q_{4n-3}\left(1+R_{4n-2}Q_{4n-4}\right)}\\&=\frac{\left(\frac{\gamma\delta}{(1+(4n-2)\gamma\delta)}\right)}{\left(\frac{\alpha^{n}w^{n}}{e\gamma^{n-1}\delta^{n-1}}\frac{\prod_{j=0}^{n-2}(1+(4j+3)\gamma\delta)}{(-1+\alpha w)^{n}}\right)}\times\frac{1}{\left(1+\left(\frac{\gamma\delta}{(1+(4n-2)\gamma\delta)}\right)\right)}\end{aligned}$$

$$= \frac{\gamma\delta}{\left(\frac{\alpha^n w^n}{e\gamma^{n-1}\delta^{n-1}}\frac{\prod_{j=0}^{n-2}(1+(4j+3)\gamma\delta)}{(-1+\alpha w)^n}\right)} \times \frac{1}{(1+(4n-2)\gamma\delta+\gamma\delta)}$$

$$= \frac{e\gamma^{n-1}\delta^{n-1}\gamma\delta(-1+\alpha w)^n}{\alpha^n w^n \prod_{j=0}^{n-2}(1+(4j+3)\gamma\delta)\,(1+(4n-1)\gamma\delta)}$$

$$= \frac{e\gamma^n\delta^n(-1+\alpha w)^n}{\alpha^n w^n \prod_{j=0}^{n-1}(1+(4j+3)\gamma\delta)}.$$

We can also demonstrate the opposite formula. The evidence is conclusive.

4 The Third System:

We look into the answers to a set of two difference equations in this part:

$$Q_{n+1} = \frac{Q_n R_{n-2}}{R_{n-1}\left(1 - Q_n R_{n-2}\right)}, \qquad R_{n+1} = \frac{R_n Q_{n-2}}{Q_{n-1}\left(1 + R_n Q_{n-2}\right)}, \tag{6}$$

in which $n \in \mathbb{N}_0$ and the initial circumstances $Q_{-2}, Q_{-1}, Q_0, R_{-2}, R_{-1}$ *and* R_0 are randomly chosen nonzero real numbers.

Theorem 6. *Assuming that the solutions to system (6) are $\{Q_n, R_n\}$. Then we observe that the subsequent formula gives all of the solutions to the system (6) for $n = 0, 1, 2, \cdots$:*

$$Q_{4n-2} = \frac{\alpha^n w^n}{\gamma^n \delta^{n-1}} \prod_{j=0}^{n-1} \frac{(1+(4j)\gamma\delta)}{(1-(4j+2)\alpha w)},$$

$$Q_{4n-1} = \frac{\beta\alpha^n w^n}{\gamma^n \delta^n} \prod_{j=0}^{n-1} \frac{(1+(4j+1)\gamma\delta)}{(1-(4j+3)\alpha w)},$$

$$Q_{4n} = \frac{\alpha^{n+1} w^n}{\gamma^n \delta^n} \prod_{j=0}^{n-1} \frac{(1+(4j+2)\gamma\delta)}{(1-(4j+4)\alpha w)},$$

$$Q_{4n+1} = \frac{\alpha^{n+1} w^{n+1}}{e\gamma^n \delta^n (1-\alpha w)} \prod_{j=0}^{n-1} \frac{(1+(4j+3)\gamma\delta)}{(1-(4j+5)\alpha w)},$$

and

$$R_{4n-2} = \frac{\gamma^n \delta^n}{\alpha^n w^{n-1}} \prod_{j=0}^{n-1} \frac{(1-(4j)\alpha w)}{(1+(4j+2)\gamma\delta)},$$

$$R_{4n-1} = \frac{e\gamma^n \delta^n}{\alpha^n w^n} \prod_{j=0}^{n-1} \frac{(1-(4j+1)\alpha w)}{(1+(4j+3)\gamma\delta)},$$

$$R_{4n} = \frac{\gamma^n \delta^{n+1}}{\alpha^n w^n} \prod_{j=0}^{n-1} \frac{(1-(4j+2)\alpha w)}{(1+(4j+4)\gamma\delta)},$$

$$R_{4n+1} = \frac{c^{n+1} \delta^{n+1}}{\beta\alpha^n w^n (1+\gamma\delta)} \prod_{j=0}^{n-1} \frac{(1-(4j+3)\alpha w)}{(1+(4j+5)\gamma\delta)},$$

where $Q_{-2} = \gamma,\ Q_{-1} = \beta,\ Q_0 = \alpha,\ R_{-2} = w,\ R_{-1} = e$ and $R_0 = \delta$.

Proof. When $n = 0$, the conclusion is valid. Let's now assume that $n > 0$ and that $n - 1$ are consistent with our assumption. meaning,

$$Q_{4n-6} = \frac{\alpha^{n-1} w^{n-1}}{\gamma^{n-1} \delta^{n-2}} \prod_{j=0}^{n-2} \frac{(1+(4j)\gamma\delta)}{(1-(4j+2)\alpha w)},$$

$$Q_{4n-5} = \frac{(\beta\alpha)^{n-1} w^{n-1}}{\gamma^{n-1} \delta^{n-1}} \prod_{j=0}^{n-2} \frac{(1+(4j+1)\gamma\delta)}{(1-(4j+3)\alpha w)},$$

$$Q_{4n-4} = \frac{\alpha^n w^{n-1}}{\gamma^{n-1} \delta^{n-1}} \prod_{j=0}^{n-2} \frac{(1+(4j+2)\gamma\delta)}{(1-(4j+4)\alpha w)},$$

$$Q_{4n-3} = \frac{\alpha^n w^n}{e\gamma^{n-1} \delta^{n-1} (1-\alpha w)} \prod_{j=0}^{n-2} \frac{(1+(4j+3)\gamma\delta)}{(1-(4j+5)\alpha w)},$$

and

$$R_{4n-6} = \frac{\gamma^{n-1} \delta^{n-1}}{\alpha^{n-1} w^{n-2}} \prod_{j=0}^{n-2} \frac{(1-(4j)\alpha w)}{(1+(4j+2)\gamma\delta)},$$

$$R_{4n-5} = \frac{e\gamma^{n-1} \delta^{n-1}}{\alpha^{n-1} w^{n-1}} \prod_{j=0}^{n-2} \frac{(1-(4j+1)\alpha w)}{(1+(4j+3)\gamma\delta)},$$

$$R_{4n-4} = \frac{\gamma^{n-1} \delta^n}{\alpha^{n-1} w^{n-1}} \prod_{j=0}^{n-2} \frac{(1-(4j+2)\alpha w)}{(1+(4j+4)\gamma\delta)},$$

$$R_{4n-3} = \frac{\gamma^n \delta^n}{(\beta\alpha)^{n-1} w^{n-1} (1+\gamma\delta)} \prod_{j=0}^{n-2} \frac{(1-(4j+3)\alpha w)}{(1+(4j+5)\gamma\delta)}.$$

Now, Eq. (6) indicates that:

$$Q_{4n-2} = \frac{Q_{4n-3} R_{4n-5}}{R_{4n-4}\left(1 - Q_{4n-3} R_{4n-5}\right)}$$

$$=\frac{\left(\frac{\alpha w}{(1-\alpha w)}\prod_{j=0}^{n-2}\frac{(1-(4j+1)\alpha w)}{(1-(4j+5)\alpha w)}\right)}{\left(\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\prod_{j=0}^{n-2}\frac{(1-(4j+2)\alpha w)}{(1+(4j+4)\gamma\delta)}\right)}\times\frac{1}{\left(1-\left(\frac{\alpha w}{(1-\alpha w)}\prod_{j=0}^{n-2}\frac{(1-(4j+1)\alpha w)}{(1-(4j+5)\alpha w)}\right)\right)}$$

$$=\frac{\left(\frac{\alpha w}{(1-(4n-3)\alpha w)}\right)}{\left(\frac{\gamma^{n-1}\delta^{n}}{\alpha^{n-1}w^{n-1}}\prod_{j=0}^{n-2}\frac{(1-(4j+2)\alpha w)}{(1+(4j+4)\gamma\delta)}\right)}\frac{1}{\left(1-\frac{\alpha w}{(1+(4n-3)\alpha w)}\right)}$$

$$=\frac{\alpha^{n-1}w^{n-1}\alpha w}{\gamma^{n-1}\delta^{n}\left(1-(4n-3)\alpha w-\alpha w\right)}\prod_{j=0}^{n-2}\frac{(1+(4j+4)\gamma\delta)}{(1-(4j+2)\alpha w)}$$

$$=\frac{\alpha^{n}w^{n}}{\gamma^{n-1}\delta^{n}\left(1-(4n-2)\alpha w\right)}\prod_{j=0}^{n-2}\frac{(1+(4j+4)\gamma\delta)}{(1-(4j+2)\alpha w)}$$

$$=\frac{\alpha^{n}w^{n}}{\gamma^{n-1}\delta^{n}}\prod_{j=0}^{n-1}\frac{(1+(4j)\gamma\delta)}{(1-(4j+2)\alpha w)},$$

$$R_{4n-2}=\frac{R_{4n-3}Q_{4n-5}}{Q_{4n-4}\left(1+R_{4n-3}Q_{4n-5}\right)}$$

$$=\frac{\left(\frac{\gamma\delta}{(1+\gamma\delta)}\prod_{j=0}^{n-2}\frac{(1+(4j+1)\gamma\delta)}{(1+(4j+5)\gamma\delta)}\right)}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}\delta^{n-1}}\prod_{j=0}^{n-2}\frac{(1+(4j+2)\gamma\delta)}{(1-(4j+4)\alpha w)}\right)}\times\frac{1}{\left(1+\left(\frac{\gamma\delta}{(1+\gamma\delta)}\prod_{j=0}^{n-2}\frac{(1+(4j+1)\gamma\delta)}{(1+(4j+5)\gamma\delta)}\right)\right)}$$

$$=\frac{\left(\frac{\gamma\delta}{(1+(4n-3)\gamma\delta)}\right)}{\left(\frac{\alpha^{n}w^{n-1}}{\gamma^{n-1}\delta^{n-1}}\prod_{j=0}^{n-2}\frac{(1+(4j+2)\gamma\delta)}{(1-(4j+4)\alpha w)}\right)}\times\frac{1}{\left(1+\frac{\gamma\delta}{(1+(4n-3)\gamma\delta)}\right)}$$

$$=\frac{\gamma^{n-1}d^{n-1}\gamma\delta}{\alpha^{n}w^{n-1}\left(1+(4n-3)\gamma\delta+\gamma\delta\right)}\prod_{j=0}^{n-2}\frac{(1-(4j+4)\alpha w)}{(1+(4j+2)\gamma\delta)}$$

$$=\frac{\gamma^{n}\delta^{n}}{\alpha^{n}w^{n-1}\left(1+(4n-2)\gamma\delta\right)}\prod_{j=0}^{n-2}\frac{(1-(4j+4)\alpha w)}{(1+(4j+2)\gamma\delta)}$$

$$=\frac{\gamma^{n}\delta^{n}}{\alpha^{n}w^{n-1}}\prod_{j=0}^{n-1}\frac{(1-(4j)\alpha w)}{(1+(4j+2)\gamma\delta)}.$$

Additionally, Eqs. (6) show that:

$$Q_{4n-1}=\frac{Q_{4n-2}R_{4n-4}}{R_{4n-3}\left(1-Q_{4n-2}R_{4n-4}\right)}$$

$$=\frac{\left(\frac{\alpha w}{(1-(4n-2)\alpha w)}\right)}{\left(\frac{\gamma^{n}\delta^{n}}{(\beta\alpha)^{n-1}w^{n-1}(1+\gamma\delta)}\prod_{j=0}^{n-2}\frac{(1-(4j+3)\alpha w)}{(1+(4j+5)\gamma\delta)}\right)}\times\frac{1}{\left(1-\left(\frac{\alpha w}{(1-(4n-2)\alpha w)}\right)\right)}$$

$$=\frac{(\beta\alpha)^{n-1}w^{n-1}(1+\gamma\delta)\alpha w}{(\gamma^{n}\delta^{n})\left(1-(4n-2)\alpha w-\alpha w\right)}\prod_{j=0}^{n-2}\frac{(1+(4j+5)\gamma\delta)}{(1-(4j+3)\alpha w)}$$

$$=\frac{\beta\alpha^{n}w^{n}(1+\gamma\delta)}{\gamma^{n}\delta^{n}\left(1-(4n-1)\alpha w\right)}\prod_{j=0}^{n-2}\frac{(1+(4j+5)\gamma\delta)}{(1-(4j+3)\alpha w)}$$

$$=\frac{\beta\alpha^{n}w^{n}}{\gamma^{n}\delta^{n}}\prod_{j=0}^{n-1}\frac{(1+(4j+1)\gamma\delta)}{(1-(4j+3)\alpha w)},$$

$$R_{4n-1}=\frac{R_{4n-2}Q_{4n-4}}{Q_{4n-3}\left(1+R_{4n-2}Q_{4n-4}\right)}$$

$$=\frac{\left(\frac{\gamma\delta}{(1+(4n-2)\gamma\delta)}\right)}{\left(\frac{\alpha^{n}w^{n}}{e\gamma^{n-1}\delta^{n-1}(1-\alpha w)}\prod_{j=0}^{n-2}\frac{(1+(4j+3)\gamma\delta)}{(1-(4j+5)\alpha w)}\right)}\times\frac{1}{\left(1+\frac{\gamma\delta}{(1+(4n-2)\gamma\delta)}\right)}$$

$$=\frac{\gamma\delta}{\left(\frac{\alpha^{n}w^{n}}{e\gamma^{n-1}\delta^{n-1}(1-\alpha w)}\prod_{j=0}^{n-2}\frac{(1+(4j+3)\gamma\delta)}{(1-(4j+5)\alpha w)}\right)}\times\frac{1}{(1+(4n-2)\gamma\delta+\gamma\delta)}$$

$$=\frac{\gamma\delta\gamma^{n-1}\delta^{n-1}(1-\alpha w)}{\alpha^{n}w^{n}\left(1+(4n-1)\gamma\delta\right)}\prod_{j=0}^{n-2}\frac{(1-(4j+5)\alpha w)}{(1+(4j+3)\gamma\delta)}$$

$$=\frac{e\gamma^{n}\delta^{n}}{\alpha^{n}w^{n}}\prod_{j=0}^{n-1}\frac{(1-(4j+1)\alpha w)}{(1+(4j+3)\gamma\delta)}.$$

We may also demonstrate the other relationships. The evidence is conclusive.

5 Other Systems

Since all of the theorems being proved here are similar to those in the systems of difference equations discussed before, they will not all be proved here $Q_{-2}=\gamma,\ Q_{-1}=\beta,\ Q_0=\alpha,\ R_{-2}=w,\ R_{-1}=e$ and $R_0=\delta$.

$$Q_{n+1}=\frac{Q_nR_{n-2}}{R_{n-1}\left(1-Q_nR_{n-2}\right)},\quad R_{n+1}=\frac{R_nQ_{n-2}}{Q_{n-1}\left(1-R_nQ_{n-2}\right)}. \tag{7}$$

$$Q_{n+1}=\frac{Q_nR_{n-2}}{R_{n-1}\left(-1+Q_nR_{n-2}\right)},\quad R_{n+1}=\frac{R_nQ_{n-2}}{Q_{n-1}\left(1-R_nQ_{n-2}\right)}. \tag{8}$$

$$Q_{n+1}=\frac{Q_nR_{n-2}}{R_{n-1}\left(-1-Q_nR_{n-2}\right)},\quad R_{n+1}=\frac{R_nQ_{n-2}}{Q_{n-1}\left(1+R_nQ_{n-2}\right)}. \tag{9}$$

$$Q_{n+1}=\frac{Q_nR_{n-2}}{R_{n-1}\left(-1-Q_nR_{n-2}\right)},\quad R_{n+1}=\frac{R_nQ_{n-2}}{Q_{n-1}\left(1-R_nQ_{n-2}\right)}. \tag{10}$$

Theorem 7. *The formula below provides the solutions of the subsequent system (5) $n=0,1,2,\cdots$:*

$$Q_{4n-2}=\frac{\alpha^nw^n}{\gamma^n\delta^{n-1}}\prod_{j=0}^{n-1}\frac{(1-(4j)\gamma\delta)}{(1-(4j+2)\alpha w)},$$

$$Q_{4n-1}=\frac{\beta\alpha^nw^n}{\gamma^n\delta^n}\prod_{j=0}^{n-1}\frac{(1-(4j+1)\gamma\delta)}{(1-(4j+3)\alpha w)},$$

$$Q_{4n}=\frac{\alpha^{n+1}w^n}{\gamma^n\delta^n}\prod_{j=0}^{n-1}\frac{(1-(4j+2)\gamma\delta)}{(1-(4j+4)\alpha w)},$$

$$Q_{4n+1}=\frac{\alpha^{n+1}w^{n+1}}{e\gamma^n\delta^n(1-\alpha w)}\prod_{j=0}^{n-1}\frac{(1-(4j+3)\gamma\delta)}{(1-(4j+5)\alpha w)},$$

and

$$R_{4n-2}=\frac{\gamma^n\delta^n}{\alpha^nw^{n-1}}\prod_{j=0}^{n-1}\frac{(1-(4j)\alpha w)}{(1-(4j+2)\gamma\delta)},$$

$$R_{4n-1}=\frac{e\gamma^n\delta^n}{\alpha^nw^n}\prod_{j=0}^{n-1}\frac{(1-(4j+1)\alpha w)}{(1-(4j+3)\gamma\delta)},$$

$$R_{4n}=\frac{\gamma^n\delta^{n+1}}{\alpha^nw^n}\prod_{j=0}^{n-1}\frac{(1-(4j+2)\alpha w)}{(1-(4j+4)\gamma\delta)},$$

$$R_{4n+1}=\frac{c^{n+1}\delta^{n+1}}{\beta\alpha^nw^n(1-\gamma\delta)}\prod_{j=0}^{n-1}\frac{(1-(4j+3)\alpha w)}{(1-(4j+5)\gamma\delta)}.$$

Theorem 8. *If $\{Q_n,R_n\}$ are solutions of the difference equation system (6) where the initial circumstances $Q_{-2},\ Q_{-1},\ Q_0,\ R_{-2},\ R_{-1}$ and R_0 are arbitrarily nonzero real numbers with $Q_{-2}R_0\neq 1$. Then, for $n=0,1,2,\cdots$:*

$$Q_{4n-2}=\frac{\alpha^nw^n}{\gamma^n\delta^{n-1}}\prod_{j=0}^{n-1}(1-(4j)\gamma\delta),$$

$$Q_{4n-1}=\frac{\beta\alpha^nw^n}{\gamma^n\delta^n}\frac{\prod_{j=0}^{n-1}(1-(4j+1)\gamma\delta)}{(-1+\alpha w)^n},$$

$$Q_{4n}=\frac{\alpha^{n+1}w^n}{\gamma^n\delta^n}\prod_{j=0}^{n-1}(1-(4j+2)\gamma\delta),$$

$$Q_{4n+1}=\frac{\alpha^{n+1}w^{n+1}}{e\gamma^n\delta^n(-1+\alpha w)^{n+1}}\prod_{j=0}^{n-1}(1-(4j+3)\gamma\delta),$$

and

$$R_{4n-2}=\frac{\gamma^n\delta^n}{\alpha^nw^{n-1}}\prod_{j=0}^{n-1}\frac{1}{(1-(4j+2)\gamma\delta)},$$

$$R_{4n-1}=\frac{e\gamma^n\delta^n}{\alpha^nw^n}\frac{(-1+\alpha w)^n}{\prod_{j=0}^{n-1}(1-(4j+3)\gamma\delta)},$$

$$R_{4n}=\frac{\gamma^n\delta^{n+1}}{\alpha^nw^n}\prod_{j=0}^{n-1}\frac{1}{(1-(4j+4)\gamma\delta)},$$

$$R_{4n+1}=\frac{\gamma^{n+1}\delta^{n+1}}{\beta\alpha^nw^n(1-\gamma\delta)}\frac{(-1+\alpha w)^n}{\prod_{j=0}^{n-1}(1-(4j+5)\gamma\delta)}.$$

Theorem 9. *If $\{Q_n,R_n\}$ are solutions of the difference equations system (7) where the initial circumstances $Q_{-2},\ Q_{-1},\ Q_0,\ R_{-2},\ R_{-1}$, and R_0 are arbitrarily nonzero real numbers with $Q_{-2}R_0\neq -1$. Then, for $n=0,1,2,\cdots$:*

$$Q_{4n-2}=\frac{\alpha^nw^n}{\gamma^n\delta^{n-1}}\prod_{j=0}^{n-1}(1+(4j)\gamma\delta),$$

$$Q_{4n-1}=\frac{\beta\alpha^nw^n}{\gamma^n\delta^n}\frac{\prod_{j=0}^{n-1}(1+(4j+1)\gamma\delta)}{(-1-\alpha w)^n},$$

$$Q_{4n}=\frac{\alpha^{n+1}w^n}{\gamma^n\delta^n}\prod_{j=0}^{n-1}(1+(4j+2)\gamma\delta),$$

$$Q_{4n+1}=\frac{\alpha^{n+1}w^{n+1}}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}}\prod_{j=0}^{n-1}(1+(4j+3)\gamma\delta),$$

and

$$R_{4n-2}=\frac{\gamma^n\delta^n}{\alpha^n w^{n-1}}\prod_{j=0}^{n-1}\frac{1}{(1+(4j+2)\gamma\delta)},$$
$$R_{4n-1}=\frac{e\gamma^n\delta^n}{\alpha^n w^n}\frac{(-1-\alpha w)^n}{\prod_{j=0}^{n-1}(1+(4j+3)\gamma\delta)},$$
$$R_{4n}=\frac{\gamma^n\delta^{n+1}}{\alpha^n w^n}\prod_{j=0}^{n-1}\frac{1}{(1+(4j+4)\gamma\delta)},$$
$$R_{4n+1}=\frac{\gamma^{n+1}\delta^{n+1}}{\beta\alpha^n w^n(1+\gamma\delta)}\frac{(-1-\alpha w)^n}{\prod_{j=0}^{n-1}(1+(4j+5)\gamma\delta)}.$$

Theorem 10.*Consider that $\{Q_n, R_n\}$ are solutions of the system (8) with the initial circumstances Q_{-2}, Q_{-1}, Q_0, R_{-2}, R_{-1}, and R_0 are arbitrarily nonzero real numbers with $Q_{-2}R_0 \neq -1$. Then, for $n = 0, 1, 2, \cdots$:*

$$Q_{4n-2}=\frac{\alpha^n w^n}{\gamma^n\delta^{n-1}}\prod_{j=0}^{n-1}(1-(4j)\gamma\delta),$$
$$Q_{4n-1}=\frac{\beta\alpha^n w^n}{\gamma^n\delta^n}\frac{\prod_{j=0}^{n-1}(1-(4j+1)\gamma\delta)}{(-1-\alpha w)^n},$$
$$Q_{4n}=\frac{\alpha^{n+1}w^n}{\gamma^n\delta^n}\prod_{j=0}^{n-1}(1-(4j+2)\gamma\delta),$$
$$Q_{4n+1}=\frac{\alpha^{n+1}w^{n+1}}{e\gamma^n\delta^n(-1-\alpha w)^{n+1}}\prod_{j=0}^{n-1}(1-(4j+3)\gamma\delta),$$

and

$$R_{4n-2}=\frac{\gamma^n\delta^n}{\alpha^n w^{n-1}}\prod_{j=0}^{n-1}\frac{1}{(1-(4j+2)\gamma\delta)},$$
$$R_{4n-1}=\frac{e\gamma^n\delta^n}{\alpha^n w^n}\frac{(-1-\alpha w)^n}{\prod_{j=0}^{n-1}(1-(4j+3)\gamma\delta)},$$
$$R_{4n}=\frac{\gamma^n\delta^{n+1}}{\alpha^n w^n}\prod_{j=0}^{n-1}\frac{1}{(1-(4j+4)\gamma\delta)},$$
$$R_{4n+1}=\frac{\gamma^{n+1}\delta^{n+1}}{\beta\alpha^n w^n(1-\gamma\delta)}\frac{(-1-\alpha w)^n}{\prod_{j=0}^{n-1}(1-(4j+5)\gamma\delta)}.$$

6 Numerical Examples

In this section, we take a look at several fascinating numerical examples in order to support our theoretical talks and to explain the findings of the other sections. These illustrations show several qualitative behaviors of nonlinear difference equation solutions.

*Example 1.*We take into account a numerical example of the difference system (1) with the initial values $Q_{-2} = 0.5$, $Q_{-1} = 4$, $Q_0 = 0.2$, $R_{-2} = 0.7$, $R_{-1} = 8$, and $R_0 = 0.2$, (See Fig. 1).

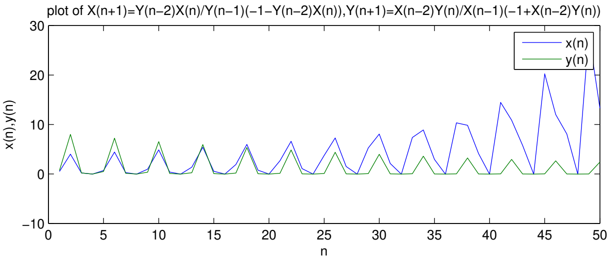

Fig. 1: Example 1.

*Example 2.*We consider a fascinating example for the difference system (1) with the beginning circumstances. $Q_{-2} = 5$, $Q_{-1} = -4$, $Q_0 = -2/7$, $R_{-2} = 7$, $R_{-1} = 8$, and $R_0 = 0.4$, (See Fig. 2).

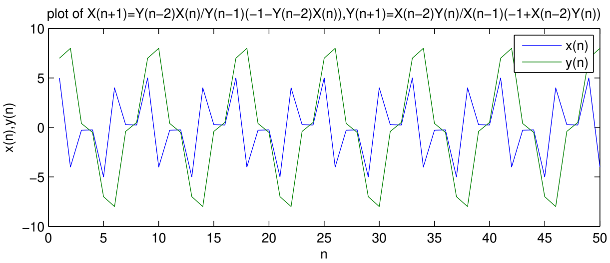

Fig. 2: Example 2.

*Example 3.*With the initial circumstances, we investigate an intriguing numerical example for the difference system (5), $Q_{-2} = 0.15$, $Q_{-1} = -0.24$, $Q_0 = 0.13$, $R_{-2} = 0.17$, $R_{-1} = 0.18$, and $R_0 = 0.2$, see Fig. 3.

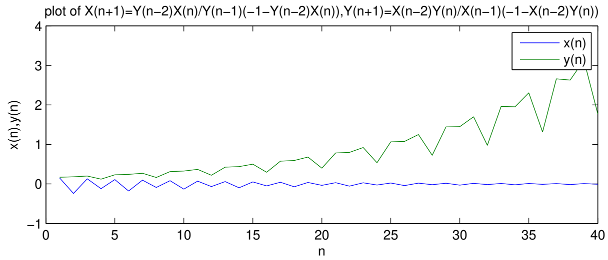

Fig. 3: Example 3.

*Example 4.*If we consider system (2) with the initial conditions: $Q_{-2} = 5$, $Q_{-1} = -4$, $Q_0 = -2/7$, $R_{-2} = 7$, $R_{-1} = 8$, and $R_0 = -0.4$, we get Fig. 4.

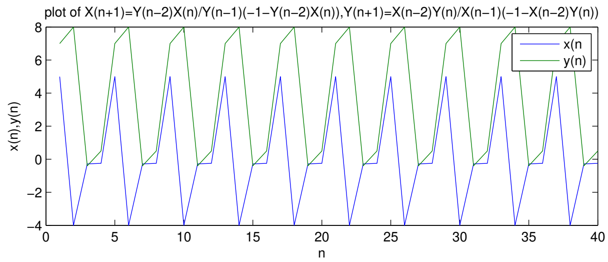

Fig. 4: Example 3.

*Example 5.*When we input the beginning conditions: $Q_{-2} = 0.15$, $Q_{-1} = -0.24$, $Q_0 = 0.13$, $R_{-2} = 0.17$, $R_{-1} = 0.18$, and $R_0 = 0.2$. We presume the difference equations system (6), see Fig. 5.

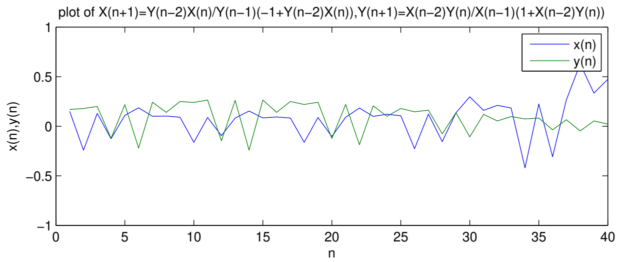

Fig. 5: Example 5

*Example 6.*Fig. 6 depicts the behavior of the difference system's solution under its initial conditions: $Q_{-2} = 0.11$, $Q_{-1} = -0.14$, $Q_0 = -0.13$, $R_{-2} = 0.14$, $R_{-1} = -0.128$, and $R_0 = 0.22$.

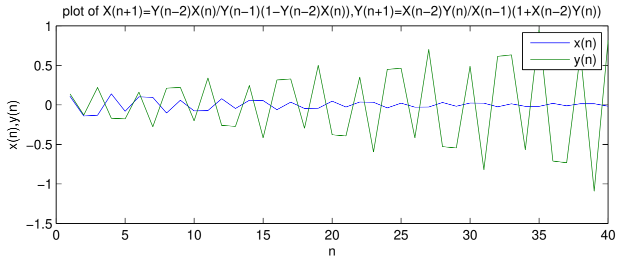

Fig. 6: Example 6

*Example 7.*When we set the initial circumstances $Q_{-2} = 0.41$, $Q_{-1} = 0.6$, $Q_0 = 0.13$, $R_{-2} = 0.14$, $R_{-1} = 0.38$, and $R_0 = -0.22$, we take the system of difference equations (7) into consideration, see Fig. 7.

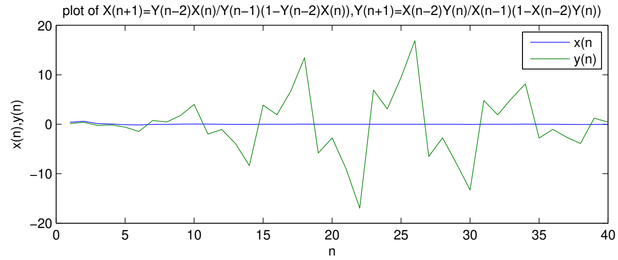

Fig. 7: Example 7.

*Example 8.*Fig. 8 depicts the behaviour of the difference system's solution under the initial conditions: $Q_{-2} = 1.9$, $Q_{-1} = -0.6$, $Q_0 = -0.3$, $R_{-2} = -0.4$, $R_{-1} = -0.3$ and $R_0 = 0.2$.

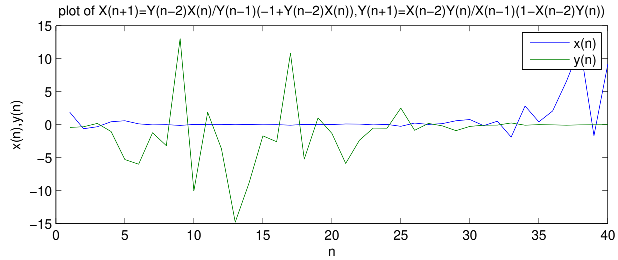

Fig. 8: Example 8.

7 Conclusion

In this article, the solvability of systems of third-order rational difference equations has been discussed. Consequently, according to their nonzero initial conditions, several systems are introduced concerning their periodicity. In addition, each system is examined and presented as associated with its theoretical existence and uniqueness. Finally, several numerical examples are depicted to ensure the theoretical studies of the presented and developed systems of third-order rational difference equations.

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