Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180414

Investigating Stock Market Volatility and other Volatility Sources using Stochastic Volatility Models

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Received: 21 Dec. 2023, Revised: 13 Mar. 2024, Accepted: 21 Mar. 2024
Published online: 1 Jul. 2024

Abstract: The stock market is exposed to risk due to the movements in market variables as volatility. Economic recession, monetary policy, pandemic etc can induce volatility in the market. The Heston and Black Scholes models are used in predicting the stock prices with some studies deriving a closed form solution. However, this current study focuses on predicting the values of portfolio with a combination of recession Free State and recession non-free state based on the prevailing economic condition. A comparison was made between the models having volatility with a recession free state and a recession non free state. Result show that the Heston model captures volatility better than the Black -Scholes model. Therefore, forecasting future stock price volatility provides vital information to the investors and enables decision making. Numerical illustrations were shown in concrete setting.

Keywords: Volatility, Heston, Black- Scholes, Stochastic volatility model

1 Introduction

The financial market is affected by volatility, that is, the rate at which the price of assets rise and fall given a particular set of returns. Also the stock market is exposed to risk, this risk which is the risk of losses in position, brought about the movements on market variables like prices and volatility. Emenyonu, Osu and Olunkwa [1].

A number of countries are presently undergoing economic recession, largely due to the pandemic. Being able to forecast prices under conditions of recession has become imperative, to aid investment decision of investors, who still invest under these conditions. Several financial instruments have been used in forecasting stock prices, while employing the Heston and Black Scholes models. The Heston model allows volatility to be arbitrary, while the Black Scholes model takes volatility to be deterministic. It has been long known that volatility varies randomly. Stochastic volatility models seem reasonable. Thus several models have been created after years of academic investigation, such as the Hull-White Model, the Scott model and the Heston model etc. see Jiang [2].

The Heston model has been investigated and applied in several studies and used in predicting the stock prices, as in the work of Shehzad, Anwar and Razzaq [3] They explained that the Heston model is a stochastic volatility model, that takes the level of volatility and the correlation between stock price and volatility into account. Also Heston [4], in his work, suggested that the Heston model allows arbitrary correlation between volatility and spot asset returns. On the other hand, another study by Kittisak and Patcharee [5] confirmed that the Heston model is used widely in the real world, with applications in the Energy industry etc.The Black-Scholes model on the other hand is used in forecasting stock prices and has been applied in several studies which includes the work of Fisher and Myron [6] suggested a model for option pricing. Their approach was used for the first time in option pricing. Black-Scholes model is a pricing model used as the benchmark in generating fair price for a call and put option. See Molintas [7].

Several studies have applied these two Models; the

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Heston model is seen to be more flexible and incorporates volatility and correlation, whereas Black-Scholes model is straightforward and assumes constant volatility. In stock price prediction, the Heston model has been seen to be a better model due to the fact that it gives accurate results just as in the study done by Shehzad et al [3].

Against this backdrop, and with some insight through given Literature on the use of these we proceed with explaining the current study. The work focuses on predicting the values of portfolio, with a combination of recession -free state and recession non-free state based on the prevailing economic condition. The recession non-free state comprises of different factors that can affect the market volatility such as monetary policy, Economic recession, pandemic etc. This is different from the study by Suresh et al [8] which established a recession free state. Also here, we will compare results of the Black-Scholes prices for a recession free state and a recession non-free state, with that of a stochastic volatility Model (SVM) for both recession free-state and recession non-free state.

2 Basic Notion and Foundational Concepts: The Models

This section considers important definitions and presents the models. Some theorems and Lemmas required for the realization of this paper are specified.

2.1 The Heston Model

The Heston model is defined as;

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t \, dW_{1t} \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{2t} \end{aligned} \tag{1}$$

Where the parameters of the system are defined as follows: S_t is the asset price at time t, r is risk-free interest rate, $\sqrt{V_t}$ is the Volatility (standard deviation) of the asset price, σ denotes the Volatility of the volatility $\sqrt{V_t}$, θ is long term price variance, k is the rate of reversion to long term price variance θ, dt denotes indefinitely small positive time increment, W_{1t} denotes the Brownian motion of the asset price while W_{2t} is Brownian motion of the asset's price variance. ρ is the correlation coefficient for W_{1t} and W_{2t}.

The Brownian motion is a random process W_t, $t \in [0;T]$, with the following properties,
$W_0 = 0$, W_t has independent increments, W_t is continuous in t, the increments $W_t - W_s$ are normally distributed with mean zero and variance $|t-s|$. Robin et al [9].

Usman et al [10] explained that with movement in market variables like prices and volatility, the assumption of constant variance (homoscedasticity) is unsuitable since the stock market demonstrates fluctuations in variance over time. On the other hand, Pushpa et al [11] in their study examined volatility prediction as a crucial instrument in financial economics, since knowledge of future volatility can help investors reduce losses. In their study, Wen et al [12] revealed that volatility is widely identified as a strong representation for risk, and that accurate volatility prediction plays an important role in financial study. The Heston's model which is a stochastic volatility model accounts for non-log normal distribution of the asset returns, leverage effect, important mean-reverting property of volatility and is analytically tractable. See Mikhailov and Nögel [13].

He, J. [14] in his study examined the classical and modified forms of the Black-Scholes model. He applied both analytical and numerical methods in solving each of the Black-Scholes model types. Al Saedi and Tularam [15] evaluated Black-Scholes model using numerical and analytical methods by examining the historical aspects, various applications and their efficiency.Thus revealing that the model is able to examine different types of Valuations of derivatives in different market.

2.2 Black- Scholes Model

The Black-Scholes model is a Mathematical model used for option pricing and the asset price follows a Geometric Brownian motion together with a constant volatility assumption. Franz [16]. The Model calculates a theoretical value of the option, which is considered a bench mark value.

The Black-Scholes Model equation has explicit solution, given for call option, as

$$\begin{aligned} C &= S_0 N(d_1) - ke^{-rT} N(d_2) \\ d_1 &= \frac{ln(s/k) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \tag{2}$$

where C is the price of European Call option, S is current stock price, K, strike price of the option, r is the risk - free interest rate, μ the drift rate, σ is volatility, T is the time period.

Theorem 1

Let $u = e^{\varphi\sqrt{\Delta t}}$, $d = e^{-\varphi\sqrt{\Delta t}}$ and $p = \frac{1}{2} + \frac{\mu}{2\varphi}\sqrt{\Delta t}$

then the price of option in the Black- Scholes framework with recession non-free state is expressed as:

$$C(0) = e^{-\rho T}\int_{\omega>\omega_1}(S(0)e^{\omega}-K)\frac{1}{\varphi\sqrt{2nT}} \cdot e^{-\frac{1}{2}\left(\frac{\omega-\left(\rho-\frac{\varphi^2}{2}\right)T}{\varphi\sqrt{T}}\right)^2} d\omega$$
$$= S(0)\Phi\left(\varphi\sqrt{T}-\frac{1}{\varphi\sqrt{T}}\left(ln\left(\frac{K}{S(0)}\right)-\left(\rho-\frac{\varphi^2}{2}\right)T\right)\right) = S(0)\Phi(d_1) \quad (3)$$

where;

$$d_1 = \frac{ln\left(\frac{S(0)}{K}\right)-\left(\rho-\frac{\varphi^2}{2}\right)T}{\varphi\sqrt{T}}, \ \varphi = \sigma\pm\beta\pm\lambda$$

and $\rho = r+\alpha$

where
α : Initial drift coefficient, φ = the stock market volatility, σ is Volatility, β is Economic recession, and λ is Monetary policy

Lemma 1
To proof theorem 1, we establish lemma 1 following the mathematical expression of Suresh et al [8] to define a counter on uptick and downtick movements on stock price at time $k\Delta t$, $(k = 1,...,n)$, as a Bernoulli random variable

$$Y_k = \begin{cases} 1 \text{ with probability } p, \text{ if stock goes up,} \\ 0 \text{ with probability } 1-p, \text{ if stock goes down.} \end{cases}$$

$$S(n\Delta t) = S(T)$$

$$S(T) = S(0)\mu\sum_{k=1}^{n}Y_k d(n-\sum_{k=1}^{n}Y_k) = d^n S(0)\left(\frac{\mu}{d}\right)\sum_{k=1}^{n}Y_k,$$

giving

$$\frac{S(T)}{S(0)} = d^{\frac{T}{\Delta t}}\left(\frac{\mu}{d}\right)\sum_{k=1}^{n}Y_k.$$

Observe that $Y = \sum_{k=1}^{\frac{T}{\Delta t}}Y_k$ is a simple random walk with $E(Y) = p\left(\frac{T}{\Delta t}\right)$ and

$$Var(Y) = p(1-p)\left(\frac{T}{\Delta t}\right) \quad (4)$$

Proof of Lemma 1
Using the value of (d) obtained from (2), we have

$$\left(\frac{S(T)}{S(0)}\right) = \frac{-T\varphi}{\sqrt{\Delta t}} + 2\varphi\sqrt{\Delta t}\sum_{k=1}^{\frac{T}{\Delta t}}Y_k.$$

Hence,

$$E\left(\left(\frac{S(T)}{S(0)}\right)\right) = \frac{-T\varphi}{\sqrt{\Delta t}} + 2\varphi\sqrt{\Delta t}\sum_{k=1}^{\frac{T}{\Delta t}}EY_k = \frac{-T\varphi}{\sqrt{\Delta t}} + 2\varphi\sqrt{\Delta t}\frac{T}{\Delta t}p = \mu T,$$

where the last equality follows from using the value of p from (3). Also,

$$Var\left(\left(\frac{S(T)}{S(0)}\right)\right) = 4\varphi^2\Delta t\sum_{k=1}^{\frac{T}{\Delta t}}Var(Y_k) = 4\varphi^2 T p(1-p)$$

$\rightarrow \ \varphi^2 T$ as $p \rightarrow \frac{1}{2}$ when $n \rightarrow \infty$
Furthermore, by application of the central limit theorem, we can assume that Y_k follows a normal distribution when time steps approach zero. Summarizing the above discussion, we can conclude that

$$\left(\frac{S(T)}{S(0)}\right) \sim N(\mu T, \varphi^2 T).$$

Observe that

$$u = e^{\varphi\sqrt{T/n}} \cong 1+\varphi\sqrt{\frac{T}{n}}+\frac{\varphi^2 T}{2n} \quad (5)$$
$$d = e^{-\varphi\sqrt{T/n}} \cong 1-\varphi\sqrt{\frac{T}{n}}+\frac{\varphi^2 T}{2n}. \quad (6)$$

So, the risk neutral probability measure (RNPM) is given by

$$\hat{p} = \frac{R-d}{u-d} = \frac{1+\frac{\rho T}{n}-d}{u-d} \cong \frac{\frac{\rho T}{n}+\varphi\sqrt{\frac{T}{n}}-\frac{\varphi^2 T}{2n}}{2\varphi\sqrt{\frac{T}{n}}},$$

where $\rho = r\pm\alpha$.
Thus,

$$\hat{p} = \frac{1}{2}+\frac{2\rho-\varphi^2}{4\varphi}\sqrt{\frac{T}{n}}. \quad (7)$$

We have seen that the European call or put options are simple to price. We therefore concentrate on pricing the

European call option using the scheme described above. As established previously, the European call option price for an n-period binomial lattice model is described as follows.

$$C(0) = \left(1+\frac{\rho T}{n}\right)^{-n} E_{\hat{p}}((S(T)-K)_+)$$
$$= \left(1+\frac{\rho T}{n}\right)^{-n} E_{\hat{p}}((S(0)u^Y d^{n-Y}-K)_+)$$
$$= \left(1+\frac{\rho T}{n}\right)^{-n} E_{\hat{p}}\left(\left(S(0)\left(\frac{u}{d}\right)^Y d^n - K\right)_+\right)$$

It follows from (5) and (6), that

$$C(0) = \left(1+\frac{\rho T}{n}\right)^{-n} E\hat{p}((S(0)e^{\omega}-K)_+), \qquad (8)$$

where

$$\omega = 2\varphi\sqrt{\frac{T}{n}}Y - \varphi\sqrt{nT}.$$

It is to be noted here that

$$E(\omega) = 2\varphi\sqrt{\frac{T}{n}}E(Y) - \varphi\sqrt{nT}$$
$$= 2\varphi\sqrt{\frac{T}{n}}np - \varphi\sqrt{nT}$$
$$= (2p-1)\varphi\sqrt{nT}$$
$$= \mu\sqrt{\Delta t}\sqrt{nT}$$
$$= \mu T,$$

Where the second last equality follows from the first relation in (5), (4) and the value of $U = \varphi\sqrt{\Delta t}$. Also, using $Var(Y) = p(1-p)(T/\Delta t)$ and $n\Delta t = T$, we have

$$Var(\omega) = 4\varphi^2\frac{T}{n}Var(Y)$$
$$= 4p(1-p)\varphi^2 T$$
$$\rightarrow \varphi^2 T,$$

where the last relation follows in the limiting case, when $n \rightarrow \infty$(or $\Delta t \rightarrow 0$), and $p \rightarrow \frac{1}{2}$ from (4) and (6). It is important to note that

$$E_{\hat{p}}(\omega) = 2\varphi\sqrt{\frac{T}{n}}E_{\hat{p}}(Y) - \varphi\sqrt{nT}$$
$$= 2\varphi\sqrt{\frac{T}{n}}np^* - \varphi\sqrt{nT}$$
$$= 2\varphi\sqrt{nT}\left(\frac{1}{2}+\frac{2\rho-\varphi^2}{4\varphi}\sqrt{\frac{T}{n}}\right) - \varphi\sqrt{nT}$$
(using (2), and (7))
$$= \left(\rho-\frac{\varphi^2}{2}\right)T$$

and

$$Var(\omega)_{\hat{p}} = 4\varphi^2\hat{p}(1-\hat{p})T$$
$$\rightarrow \varphi^2 T \qquad (9)$$

(using the limiting case when $n \rightarrow \infty$ and $p \rightarrow \frac{1}{2}$ in (7)).

Proof of Theorem 1

As a consequence of the proved Lemma 1, the above relations of (7) and (8) gives

$$C(0) = e^{-\rho T}\int_{-\infty}^{\infty}(S(0)e^{\omega}-K)_+ \cdot \frac{1}{\varphi\sqrt{2nT}} e^{-\frac{1}{2}\left(\frac{\omega-\left(\rho-\frac{\varphi^2}{2}\right)T}{\varphi\sqrt{T}}\right)^2} d\omega.$$

Now $S(0)e^{\omega} > K$, implies $\omega > ln\left(\frac{K}{S(0)}\right) = \omega_1$. Therefore,

$$C(0) = e^{-\rho T}\int_{\omega>\omega_1}(S(0)e^{\omega}-K) \cdot \frac{1}{\varphi\sqrt{2nT}} e^{-\frac{1}{2}\left(\frac{\omega-\left(\rho-\frac{\varphi^2}{2}\right)T}{\varphi\sqrt{T}}\right)^2} d\omega.$$

To evaluate this integral, substitute

$$y = \frac{\omega-\left(\rho-\frac{\varphi^2}{2}\right)T}{\varphi\sqrt{T}}$$

Then,

$$\omega = y\varphi\sqrt{T} + \left(\rho-\frac{\varphi^2}{2}\right)T, \quad \text{so} \quad d\omega = \varphi\sqrt{T}dy.$$

Moreover, $\omega > \omega_1$ gives $y > y_1$, where

$$y_1 = \frac{1}{\varphi\sqrt{T}}\left(\left(\frac{K}{S(0)}\right) - \left(\rho-\frac{\varphi^2}{2}\right)T\right)$$

Subsequently,

$$C(0) = \frac{e^{-\rho T}}{\sqrt{2\pi}}\int_{y>y_1}\left(S(0)e^{y\varphi\sqrt{T}+\left(\rho-\frac{\varphi^2}{2}\right)T} - K\right)e^{-\frac{y^2}{2}}dy$$
$$= \frac{e^{-\rho T}}{\sqrt{2\pi}}\int_{y>y_1}S(0)e^{y\varphi\sqrt{T}+\left(\rho-\frac{\varphi^2}{2}\right)T} e^{-\frac{y^2}{2}}dy$$
$$- \frac{1}{\sqrt{2\pi}}Ke^{-\rho T}\int_{y_1}^{\infty}e^{-\frac{y^2}{2}}dy$$

$$\equiv I - Ke^{-\rho T}\Phi(-y_1), \quad (10)$$

where, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}dz$ is the distribution function of a standard normal random variable and

$$I = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{y>y_1} S(0)e^{y\varphi\sqrt{T}+\left(\rho-\frac{\varphi^2}{2}\right)T} e^{-\frac{y^2}{2}}dy$$
$$= \frac{S(0)}{\sqrt{2\pi}}\int_{y>y_1} e^{-\frac{(y-\varphi\sqrt{T})^2}{2}}dy.$$

To evaluate I we define a new variable $s = y - \varphi\sqrt{T}$. Then for $y > y_1$, we have $s > s_1 = y_1 - \varphi\sqrt{T}$.
Thus,

$$I = \frac{S(0)}{\sqrt{2\pi}}\int_{s>s_1} e^{-\frac{(s)^2}{2}}ds$$
$$= S(0)\Phi(-s_1)$$
$$= S(0)\Phi\left(\varphi\sqrt{T} - \frac{1}{\varphi\sqrt{T}}\left(ln\left(\frac{K}{S(0)}\right) - \left(\rho - \frac{\varphi^2}{2}\right)T\right)\right)$$
$$= S(0)\Phi(d_1),$$

where

$$d_1 = \frac{ln\left(\frac{S(0)}{K}\right) - \left(r + \alpha - \frac{(\sigma+\beta+\gamma)^2}{2}\right)T}{(\sigma+\beta+\gamma)\sqrt{T}} \quad (11)$$

Summarizing the discussion, from (10), we have

$$C(0) = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2), \quad (12)$$

where $d_2 = -y_1 = d_1 - \varphi\sqrt{T}$ and d_1 is as in (11).
Where φ is the stock market volatility
The expression in (12) is known as the Black-Scholes formula for computing the European call option price under the set of assumptions described; where r is the risk-neutral interest rate, q is the dividend yield rate and $\lambda(t)$, is the stochastic intensity parameter. $\rho = r - q - \lambda(t)$. During the period of economic recession, the volatility of the stock is given as $\varphi = \sigma + \beta + \lambda$, due to the effect of economy recession on the market price but during recession-free state, the stock market volatility $\varphi = \sigma$, that is the volatility from other sources, since $\sigma = 0$. The stochastic volatility $\varphi(t)$;

$$\varphi = \begin{cases} \sigma + \beta + \lambda, & \textit{when the economy under recession} \\ \sigma, & \textit{when the economy is in expansion state} \end{cases}$$

2.3 Joint Distribution of Two Independent Asset Returns (One with recession free- state and the other with recession Non-free state).

We now derive the joint distribution of two independent asset returns $R_i(t)$ and $R_j(t)$ with density functions as $(i, j = 1, 2, ..., n)$.

Theorem 2: Let $R_i(t)$ be the asset returns with recession free -state and probability density function

$$f_s(R_i(t)) = \frac{R_i(t)}{\varphi\sqrt{2\pi}} \exp\left\{-1/2\left(\frac{R_i(t)-\lambda_1}{\varphi}\right)^2\right\}$$

And $R_j(t)$ the asset returns with recession Non-free state and probability density function

$$f_h(R_j(t)) = \frac{\gamma}{\beta}\left(\frac{R_j(t)-\lambda}{\beta}\right)^{\gamma-1} exp\left\{-1/2\left(\frac{R_j(t)-\lambda}{\beta}\right)^2\right\}$$

Then
$f_s(R_i(t))f_h(R_j(t))$

$$= \frac{\gamma R_i(t)}{\varphi\beta\sqrt{\pi}}\left(\frac{R_j(t)-\lambda_2}{\beta}\right)^{\gamma-1}$$
$$\cdot \exp\left\{-1/2\left(\frac{R_i(t)-\lambda_1}{\varphi}\right)^2\right\} exp\left\{-\left(\frac{R_j(t)-\lambda_2}{\beta}\right)^{\gamma}\right\}. \quad (13)$$

Proof:
Let

$$U = R_i + R_j \qquad (\text{where } R = R(t)),$$

and

$$V = R_i - R_j,$$

then

$$R_i = \frac{U+V}{2}$$

and

$$R_j = \frac{U-V}{2}.$$

Equation (13) now becomes
$f_{s,h}(U,V)$

$$= \frac{\gamma(U+V)}{2\varphi\beta\sqrt{2\pi}}\left(\frac{\frac{U-V}{2}-\lambda_2}{\beta}\right)^{\gamma-1} exp\left[-\frac{1}{2}\left\{\left(\frac{\frac{U+V}{2}-\lambda_1}{\varphi}\right)^2 + 2\left(\frac{\frac{U-V}{2}-\lambda_2}{\beta}\right)^{\gamma}\right\}\right]$$
$$= \frac{(U+V)}{\varphi^2\beta\sqrt{2\pi}}\left(\frac{U-V}{2}-\lambda_1\right) exp\left[-\frac{1}{2\varphi^2}\left(\frac{U+V}{2}-\lambda_1\right)^2\right] \cdot \exp\left[-\frac{1}{\beta^2}\left(\frac{U-V}{2}-\lambda_2\right)^2\right]$$
$$= \frac{(U+V)}{\varphi^2\beta\sqrt{2\pi}}\left(\frac{U-V}{2}-\lambda_1\right) \exp\left[-\frac{1}{2\varphi^2}\left(\frac{U^2+V^2+2UV}{4} - (U+V)\lambda_1 + \lambda_1^2\right)\right]$$
$$\times exp\left[-\frac{1}{\beta^2}\left(\frac{U^2+V^2-2UV}{4} - (U-V)\lambda_2 + \lambda_2^2\right)\right]$$

$$= \frac{R_i(R_j-\lambda)}{\varphi^2\beta\sqrt{2\pi}} exp\left[-\frac{1}{2\varphi^2}(R_i^2-2R_i\lambda_1+\lambda_1^2)\right] \cdot \exp\left[-\frac{1}{\beta^2}(R_j^2-2R_j\lambda_1+\lambda_2^2)\right]$$

$$= \frac{R_i(R_j-\lambda)}{\varphi^2\beta\sqrt{2\pi}} exp\left[-\frac{1}{2}\left\{-\frac{1}{\varphi^2}(R_i-\lambda_1)^2+\frac{2}{\beta^2}(R_j-\lambda_2)\right\}\right] \quad (14)$$

Proposition 1: Given (14), then the behaviour of the portfolio having two independent asset returns $R_i(t)$ and $R_j(t)$ is then;

$$H\{V(Z,v)\} = \frac{R_i^2}{2\varphi^2\sqrt{2}} exp\left\{-\frac{1}{2}(\varphi^{-2}(R_i-\lambda_1)^2)\right\} \quad (15)$$

Proof of Proposition 1:
Given (15), let $x = R_i(R_j-\lambda_2) \Rightarrow R_j-\lambda_2 = xR_i^{-1}$. We now have

$$H\{V(R_j,v)\} = \frac{x}{\varphi^2\beta\sqrt{2\pi}} exp\left[-\frac{1}{2}(\varphi^{-2}(R_i-\lambda_1)^2) + 2\beta^{-2}(xR_i^{-1})^2\right] dR_i$$

Then,

$$\frac{dx}{dR_i}(R_j-\lambda_2) \Rightarrow dR_i = \frac{dx}{(R_j-\lambda_2)}$$

So that,
$H\{V(R_j,v)\}$

$$= \frac{R_i}{\varphi^2\beta\sqrt{2\pi}} exp\left\{-\frac{1}{2}\varphi^{-2}(R_i-\lambda_1)^2\right\} \cdot \int_0^\infty exp\{-\beta^{-2}(xR_i^{-1})^2\}dx$$
$$= \frac{R_i}{\varphi^2\beta\sqrt{2\pi}} exp\left\{-\frac{1}{2}\varphi^{-2}(R_i-\lambda_1)^2\right\} \frac{\beta R_i}{2}\sqrt{\pi}$$
$$= \frac{R_i^2}{2\varphi^2\sqrt{2}} exp\left\{-\frac{1}{2}\varphi^{-2}(R_i-\lambda_1)^2\right\}$$

as required
If γ is fractional,
$f_{s,h}(R_i,R_j)$

$$= \frac{\gamma R_i}{\varphi\beta\sqrt{2\pi}}\left(\frac{R_j-\lambda_2}{\beta}\right)^{\gamma-1} \cdot \exp\left\{-\left(\frac{R_j(t)-\lambda_2}{\beta}\right)^{\gamma}\right\} exp\left\{-\left(\frac{R_i-\lambda_1}{\varphi}\right)^2\right\}$$

$$= \frac{\gamma R_i\left(\frac{R_j}{\beta_2}\right)^{\gamma-1}}{\varphi\beta\sqrt{2\pi}}\left(1-\frac{\lambda_2}{R_j}\right)^{\gamma-1} \cdot \exp\left\{-\left(\frac{R_j}{\beta}\right)^{\gamma}\left(1-\frac{\lambda_2}{R_j}\right)^{\gamma}\right\} exp\left\{-\frac{1}{2}\left(\frac{R_i-\lambda_1}{\varphi}\right)^2\right\}$$

$$= \frac{\gamma R_i R_j^{\gamma}}{\varphi\beta^{\gamma}R_{2,j}\sqrt{2\pi}}\left(1-\frac{\lambda_2}{R_j}\right)^{-1} \cdot \exp\left\{log\left(1-\frac{\lambda_2}{R_j}\right)^{\gamma}\right\} exp\left\{-\left(\frac{R_j}{\beta}\right)^{\gamma} \exp\left(log\left(1-\frac{\lambda_2}{R_j}\right)\right)\right\} \times \exp\left\{-\frac{1}{2\varphi}(R_i-\lambda_1)\right\},$$

for

$$\left(\left(1-\frac{\lambda}{R}\right) = exp\left(\gamma log\left(1-\frac{\lambda}{R}\right)\right)\right)$$
$$\leq \frac{\gamma R_i R_j^{\gamma}}{\varphi\beta^{\gamma}R_2\sqrt{2\pi}} exp\left\{\frac{\lambda_2}{R_j}\right\} exp\left\{\left\{-\gamma\frac{\lambda_2}{R_j}\right\}\right\} \exp\left\{-\frac{R_j}{\beta}\left(exp\left(-\gamma\frac{\lambda_2}{R_j}\right)\right)\right\} \times exp\left\{-\frac{R_i^2}{2\varphi}\left(exp\left(-2\frac{\lambda_1}{R_i}\right)\right)\right\}$$

$$= \frac{\gamma R_i R_j^{\gamma}}{\varphi\beta^{\gamma}R_j\sqrt{2\pi}} exp\left\{\frac{\lambda}{R_j}(\gamma-1)\right\} \exp\left\{-\left[\left(\frac{R_j}{\beta}\right)^{\gamma} exp\left(-\gamma\frac{\lambda}{R_j}\right)+\left(\frac{R_i^2}{2\varphi}\right)\left(exp\left(-2\frac{\lambda_1}{R_i}\right)\right)\right]\right\}.$$

Let

$$V = f(R_i)+f(R_j) \quad (16.1)$$

and

$$U = \frac{f(R_i)}{f(R_j)} \quad (16.2)$$

where $f(R_i) = \left(\frac{R_i^2}{2\varphi}\right) exp\left(-2\frac{\mu}{R_i}\right)$ and $f(R_j) = \left(\frac{R_j}{\beta}\right) exp\left(-\gamma\frac{\lambda}{R_j}\right)$

Then

$f(R_i) = \frac{UV}{U+1}$ and $f(R_j) = \frac{V}{U+1}$

The Jacobian of the transformation from $(f(R_i))$, $(f(R_j))$ to (U,V) is $\frac{V}{(U+1)^2}$

From (16.1), we have $\frac{\partial V}{\partial R_j} = 1+\frac{\lambda_2}{R_j} = 0 \Rightarrow R_j = -\lambda_2$,

Similarly,

$$\frac{\partial V}{\partial R_i} = \frac{R_i}{\lambda_1}+1 = 0 \Rightarrow R_i = -\lambda_1$$

So that

$$f_{s,h}(U,V) = \rho(\gamma)\frac{V}{(1+U)^2}e^{-v}, \qquad 0 < U,\, V < \infty \tag{16}$$

$$\text{Where} \quad \rho(\gamma) = \frac{\gamma\beta^{-\gamma}\lambda_1\lambda_2^{\gamma-1}}{\varphi\sqrt{w\pi}}e^{-(\gamma-1)}$$

3 Application of Results

A comparison between the results of the Black-Scholes prices, for a recession free state and a recession non-free state, with that of a stochastic volatility Model (SVM) for both recession free-state and recession non-free state is made. This is done using an option price index with a maturity of 250 trading days, and each strike price is repeated 20 times.
The results of the Black-Scholes and the Heston (SVM) models based on a recession free- state are generated. Then the results of Black-Scholes with the Heston models on the recession non-free state which consists of factors that induce the volatility market are also obtained. These factors include economic recession, monetary policy etc.

Parameters for the recession free state and recession non-free state:

As stated, before section 2.3, during recession-free state, the stock market volatility $\varphi = \sigma$ which implies that $\varphi - \sigma = 0$, while during recession period, the volatility of the stock is given as $\varphi = \sigma + \beta + \lambda$
Where φ , σ, β, and λ are Stock Price volatility, volatility, Economic recession, and Monetary policy respectively
Economic recession and monetary policy are external factors that can induce volatility in the stock market and are denoted by parameters, β and λ respectively.
Option price for the price index with a maturity of 250 trading days, and with each strike price repeated 20 times; were calculated using the Heston Model and the Black-Schole's model for the two cases, recession free and recession non-free states. The results obtained are shown on the Tables below:

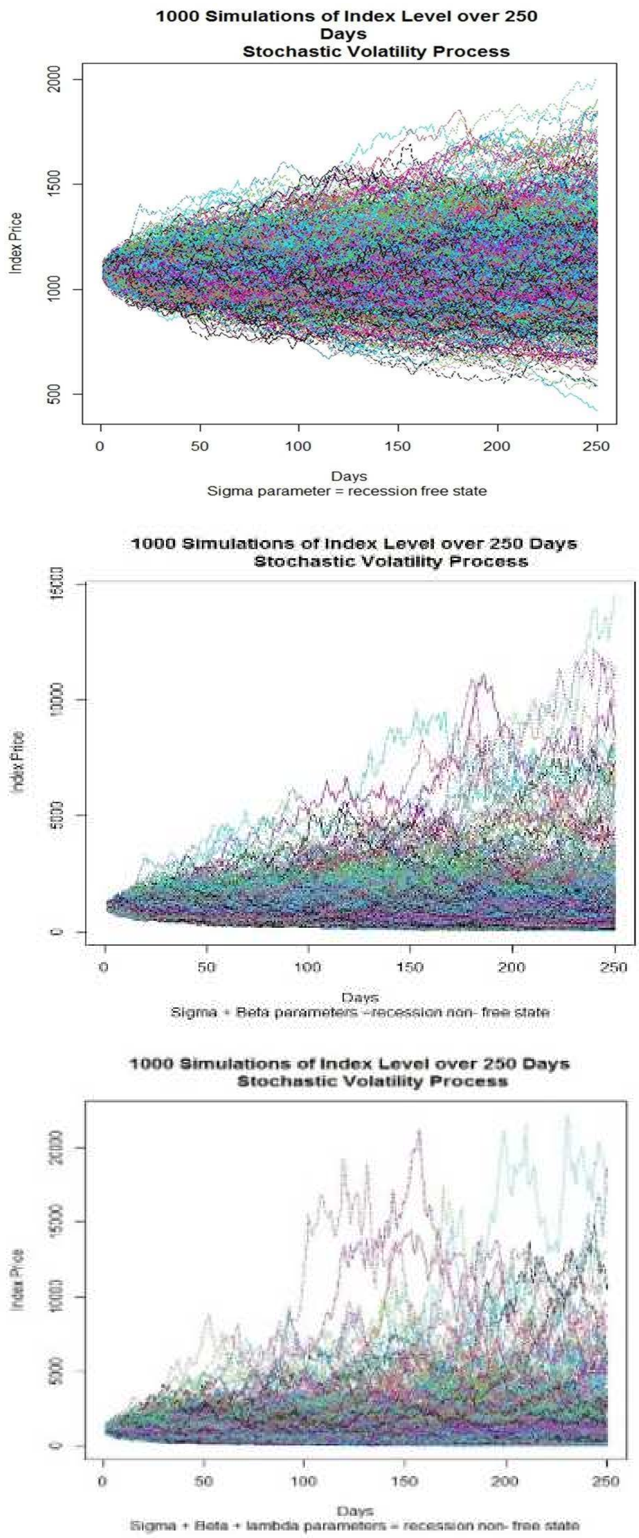


Fig. 1: Plots showing price simulations over a period of 250 days for the recession free state and the recession non-free state following volatility pattern

4 Discussion and Conclusion:

The work focused on predicting the values of portfolio with a combination of recession Free State and recession non-free state based on the prevailing economic condition. The recession non-free state comprises of different factors that can affect the market volatility such as monetary policy and Economic recession. Comparison of the Black-Scholes and the Heston model based on a recession free state was done. The two model prices tend

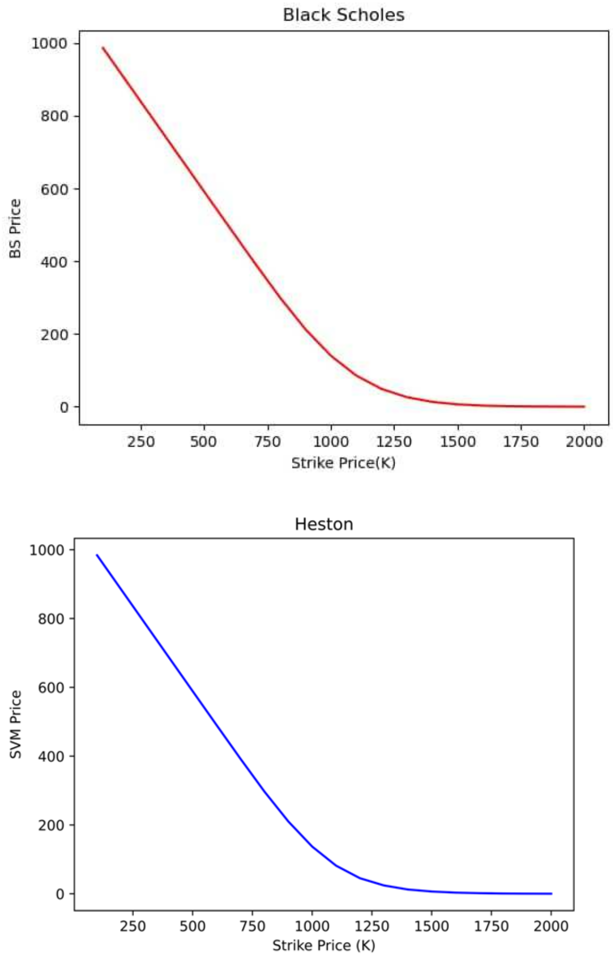
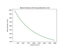


Fig. 2: Plot for the call price over the strike prices for Black-Scholes and Heston

Table 1: Option price comparison of Black-Scholes and Heston models based on a recession free state (σ)

Strike Price (k)	Black-Scholes	Heston
100	986.24	983.88
200	887.48	885.13
300	788.73	786.37
400	689.97	687.61
500	591.21	588.85
600	492.51	490.30
700	394.42	392.46
800	299.37	297.30
900	212.58	210.43
1000	140.19	137.23
1100	85.81	81.75
1200	48.99	45.21
1300	26.28	24.35
1400	13.36	12.49
1500	6.49	6.60
1600	3.04	3.39
1700	1.38	1.74
1800	0.61	0.79
1900	0.27	0.38
2000	0.11	0.21



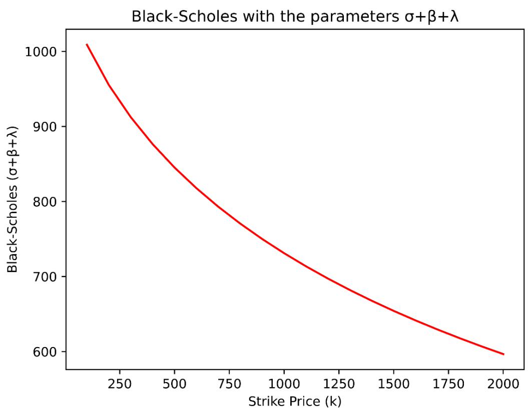


Fig. 3: Plots for call price over strike prices for Black Scholes model with recession non-free parameters

to be greater at lower strike prices especially as the strike prices decrease in relation to the spot price. On the other hand, the Black-Scholes prices are higher compared to the Heston prices (SVM). This shows that Heston model captures volatility better than the Black-Scholes model. Consequently, the work shows that price of the Black-Scholes and the Heston models based on a recession non- free state reveals that the added factors influenced volatility, especially for the Heston model which exhibited lower prices with every additional factor compared to the recession free state.

These parameters were introduced to the models to ascertain their impact on the models. It was found that the parameters actually impact on the economy. Unlike the Black-Scholes model, the prices continue to increase with every additional parameter. This is to say that volatility is induced by these factors. Also, volatility increases as stock prices decline and decreased volatility is accompanied with stock price rise as seen in the result. That is greater volatility corresponds to a greater probability of a declining market while lower volatility corresponds to higher probability of rising market. The price simulation was done for a period of one year that is 250 days during which 1000 simulations were generated for the Heston model, which is a stochastic volatility model, based on the recession free state and the recession non free state which is seen in Figure 1. Figure 2 shows the call price over the strike price for the Black –Sholes

Table 2: **Option price comparison of Black-Scholes and Heston models based on a recession non-free state** $(\sigma+\beta+\lambda)$

Strike Price (k)	**Black-Scholes** with $\sigma+\beta$	**Heston** with $\sigma+\beta$	**Black-Scholes** with $\sigma+\beta+\lambda$	**Heston** with $\sigma+\beta+\lambda$
100	987.05	981.85	1008.90	962.07
200	895.13	888.21	955.29	881.33
300	812.51	802.98	912.46	812.33
400	739.37	726.06	876.44	752.50
500	674.87	657.75	845.23	699.60
600	617.93	597.29	817.65	652.22
700	567.54	544.40	792.91	610.10
800	522.77	497.31	770.49	571.95
900	482.86	454.75	749.98	537.77
1000	447.14	417.16	731.09	506.55
1100	415.06	383.85	713.58	478.20
1200	386.14	354.35	697.28	452.46
1300	359.99	327.33	682.03	429.08
1400	336.27	303.11	667.71	407.82
1500	314.69	280.77	654.22	388.39
1600	295.00	260.22	641.48	370.40
1700	276.99	241.64	629.41	353.89
1800	260.48	225.16	617.95	338.44
1900	245.30	210.01	607.04	323.87
2000	231.33	195.79	596.64	310.11

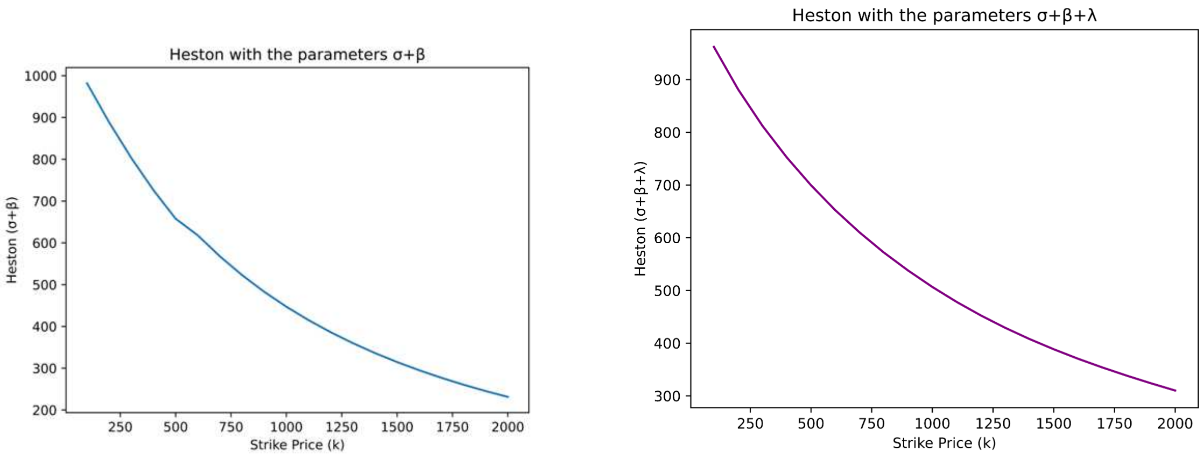


Fig. 4: Plots for the call price over the strike prices for Heston model with recession non-free parameters

and the Heston model. The simulation was done via monte carlo simulation method and this Monte Carlo method is a forecasting tool used for studying analytical intractable problem. Milos and Panagiota [17]. These models were applied on an European call option index for a period of 250 trading days. Forecasting future stock price volatility provides vital information to the investors and enables decision making, under Economic recession condition.

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflict of interest

There is no conflict of interest.

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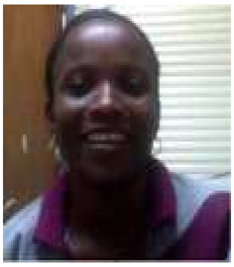

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