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The New Extended-X Exponentiated Inverted Weibull Distribution: Statistical Inference and Application to Carbon Data

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Abstract: Developing novel probability distributions holds significant importance in contemporary society across various domains. In this study, we introduce a distinctive continuous lifespan model characterized by three parameters, achieved through integrating the Extended-X Exponentiated (NEX) family core with the foundational Exponentiated Inverted Weibull (EIW) distribution. This amalgamation yields a novel distribution, termed the New Extended-X Exponentiated Inverted Weibull (NEEIW) distribution. Notably, the NEEIW distribution exhibits favorable attributes facilitated by its straightforward linear representation of hazard rate function, moments, and moment-generating function, alongside the provision of stress-strength reliability in concise closed forms. Parameter estimation for the NEEIW model is conducted via conventional methodologies such as maximum likelihood estimation (MLE) and maximum product of spacing (MPS), supplemented by exploring non-classical Bayesian analytical approaches. The empirical validation of the proposed distribution is conducted using two distinct carbon datasets, substantiating its superiority and applicability in modeling real-world data.

Keywords: New Extended-X Exponentiated Inverted Weibull (NEEIW) distribution, quantile function, moments, moment generating function, estimation of parameters

1 Introduction

Modeling real-world occurrences and natural phenomena using probability distributions is a critical procedure in statistics and probability, particularly when these phenomena are complicated and risky. Scientists have sought to develop new probability distributions for these reasons, while old probability distributions continue to fail to correctly characterize data produced from natural events [1–4]. These contributions to probability distribution generalization and modification are substantial as a result. Generalized probability distributions have arisen for the regular allowance of adding additional parameters. By adding a specific parameter or parameters to an existing probability distribution, we may increase the quality and suitability of data derived from natural occurrences, as well as the representation of the tail shape of the distribution. Also, merging the two will provide the same results [5–7].

Numerous research initiatives have been done in recent years to create new distributions by establishing new families and classes and altering the baseline distribution by adding additional shape parameters. Numerous well-known distribution classes have been reported in the literature. For examples, see references [6–11].

The main objective and motivation of this paper is to present a new distribution that fits some specific data sets more than other distributions can do. This distribution will be called the New Extended-X Exponentiated Inverted Weibull distribution.

To study the efficiency of any distribution, we need to estimate the distribution parameters either by a classical or a non-classical approach. In this paper, we made point estimations according to two classical methods, the MLE and the MPS. We also applied the non-classical Bayesian methods for estimating the unknown parameters.

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The exponential distribution is one of the most well-known lifespan distributions and, as such, has garnered substantial attention from statisticians. Zichuan et al. [12] presented a better innovative distribution family termed the new extended-X (NEX) family. It is capable of modeling data using a variety of various hazard functions, including growing, decaying, and bathtub. The NEX family's cumulative distribution function (CDF) and probability density function (PDF) are defined as

$$F(x;\Theta) = 1 - \left\{ \frac{1-[G(x;\Omega)]^2}{1-(1-\theta)[G(x;\Omega)]^2} \right\}^{\theta};x>0,\theta>0, \quad (1)$$

and

$$f(x;\Theta) = 2\theta^2 g(x;\Omega)G(x;\Omega) \frac{\left\{1-[G(x;\Omega)]^2\right\}^{\theta-1}}{\left\{1-(1-\theta)[G(x;\Omega)]^2\right\}^{\theta+1}}; \quad x>0,\theta>0, \quad (2)$$

in which Θ indicates the vector of parameters (Ω,θ) for the family's baseline distribution and the addited shape parameter, accordingly.

The exponentiated inverted Weibull (EIW) distribution is often used to characterize variables connected with several events such as cash flow, rainfall, and hurricanes. EIW was initially introduced by Flaih et al. [13]. Its CDF and PDF are included below:

$$G(x;\Omega) = \left(e^{-x^{-\beta}}\right)^{\alpha};x>0,\alpha,\beta>0, \quad (3)$$

and

$$g(x;\Omega) = \alpha\beta x^{-\beta-1}\left(e^{-x^{-\beta}}\right)^{\alpha};x>0,\alpha,\beta>0, \quad (4)$$

in which the vector of parameters $\Omega = (\alpha,\beta)$ contains the shape and scale parameters for the EIW distribution, accordingly.

The layout of the rest of this article is as follows. In Section 2, we construct the NEEIW distribution and the graphical depiction of its PDF and Hazard Rate (HR) functions. For Section 3, we examined the statistical features of the NEEIW distribution, where we provided several essential mathematical aspects of the proposed distribution and explained their mathematical derivations. Whereas Section 4 covers three-point estimate techniques. Then, the confidence interval for the parameters is estimated in Section 5. In Section 6, we conducted a numerical simulation to validate the accuracy of the estimate approaches used in this research. In Section 7, we analyzed two sets of Carbon data chosen to illustrate the superiority of the NEEIW distribution over alternative distributions. Ultimately, Section 8 describes the conclusions that can be drawn from the paper and the major findings illustrated by this work.

2 The NEEIW Distribution

The NEEIW distribution was constructed using the NEX family and the EIW baseline distribution. Utilization of the distribution and the two equations (1) and (2), We may easily get the CDF and PDF versions for the NEEIW distribution by providing the following information:

$$F(x;\Theta) = 1 - \left\{ \frac{1-\left(e^{-x^{-\beta}}\right)^{2\alpha}}{1-(1-\theta)\left(e^{-x^{-\beta}}\right)^{2\alpha}} \right\}^{\theta};x>0,\alpha,\beta,\theta>0, \quad (5)$$

and

$$f(x;\Theta) = 2\theta^2\alpha\beta x^{-\beta-1}\left(e^{-x^{-\beta}}\right)^{2\alpha} \frac{\left\{1-\left(e^{-x^{-\beta}}\right)^{2\alpha}\right\}^{\theta-1}}{\left\{1-(1-\theta)\left(e^{-x^{-\beta}}\right)^{2\alpha}\right\}^{\theta+1}}; \quad x>0,\alpha,\beta,\theta>0, \quad (6)$$

A random variable X is distributed according to the NEEIW distribution with PDF (6) denoted by $X \sim$NEEIW(α,β,θ).
Special case of the NEEIW model:
For $\alpha = 1$, it represents the new extended standard inverted weibull distribution.
For $\beta = 1$, we can get obviously the the new extended exponentiated standard inverted exponential distribution.
For $\beta = 2$, the new extended exponentiated standard inverted Rayleigh distribution will appear.

The NEEIW distribution's HR function described in the following:

$$h(x;\Theta) = 2\theta^2\alpha\beta x^{-\beta-1}\left(e^{-x^{-\beta}}\right)^{2\alpha} \frac{\left\{1-\left(e^{-x^{-\beta}}\right)^{2\alpha}\right\}^{-1}}{1-(1-\theta)\left(e^{-x^{-\beta}}\right)^{2\alpha}}. \quad (7)$$

To properly investigate the distribution, we varied its parameters and produced the PDF and HR functions graphed in Figure 1. As shown in these pictures, the behavior of the NEEIW's PDF curve may take on various forms. It may be skewed (either to the left or to the right) or having a symmetric or declining form, while the NEEIW's HR curves may be constant, diminishing, or inverted. It indicates that the suggested model is an appealing lifespan model. As shown in the application section, the NEEIW distribution has considerable versatility in its capacity to simulate skewed data. As a result, it is widely used in various domains such as health, bio-medicine, durability, and mortality studies.

3 The Properties of the proposed distribution

3.1 Linear Representation

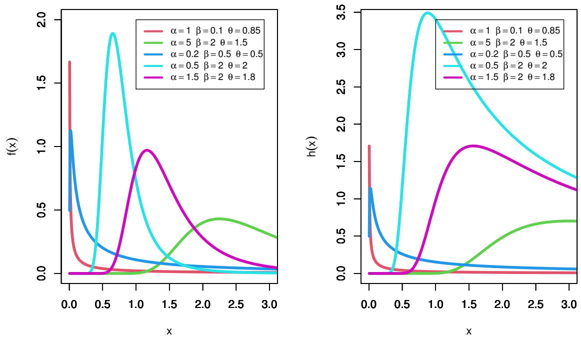

Fig. 1: PDF and HR plots of NEEIW distribution

the linear representation for the CDF of NEX family (1) is given by

$$F(x;\Omega) = 1 - \sum_{i,j=0}^{\infty} \binom{i+\theta-1}{\theta-1} \binom{\theta}{j} (-1)^j (1-\theta)^i G(x;\zeta)^{2(i+j)}. \tag{8}$$

We obtain the expanded CDF of the NEEIW distribution as follows by using the prior expansions to the EIW distribution:

$$F(x) = 1 - \sum_{i,j=0}^{\infty} \binom{i+\theta-1}{\theta-1} \binom{\theta}{j} (-1)^j (1-\theta)^i \left(e^{-x^{-\beta}}\right)^{2(i+j)\alpha}.$$

By taking the derivative with respect to x for the previous equation,we can easily obtain the PDF of the NEEIW distribution in its extended version.

$$\begin{aligned} f(x) &= \alpha\beta \sum_{i,j=0}^{\infty} \binom{i+\theta-1}{\theta-1} \binom{\theta}{j} (-1)^j (1-\theta)^i 2(i+j) x^{-\beta-1} \left(e^{-x^{-\beta}}\right)^{2(i+j)\alpha} \\ &= \sum_{i,j=0}^{\infty} \Phi_{i,j}(\theta)\, g_(x;2(i+j)\alpha,\beta), \end{aligned}$$

We can say that $\Phi_{i,j}(\theta) = \binom{i+\theta-1}{\theta-1}\binom{\theta}{j}(-1)^j(1-\theta)^i$ and $g_(x;2(i+j)\alpha,\beta)$ is the EIW density function having $2(i+j)\alpha$ as scale and β as shape parameters of EIW distribution.

3.2 Quantile Function (QF)

The QF of the NEEIW distribution may be expressed and described as an inverse for the CDF mentioned in Equation (5).

The three quartiles of the NEEIW distribution are obtained by providing particular values for p; hence, when we set $p = 0.25, 0.5$, in equation (9), we get the first and second quartiles, respectively, while setting and $p = 0.75$ yields the third quartile.

$$x_i = \left\{ -\ln\left(\left\{ \theta \left[(1-p)^{\frac{-1}{\theta}} - 1 \right]^{-1} + 1 \right\}^{\frac{-1}{2\alpha}} \right) \right\}^{\frac{-1}{\beta}}, \quad i = 1,2,\ldots,n. \tag{9}$$

3.3 The Moments

The most critical aspect of any distribution is to locate its moments easily. Thus, as shown in the following stages, we may determine the rth moments of the proposed NEEIW distribution:

$$\begin{aligned} \mu_r' = E(X^r) &= \int_0^{\infty} x^r f(x)dx \\ &= \sum_{i,j=0}^{\infty} \Phi_{i,j}(\theta) \int_0^{\infty} x^r g_(x;2(i+j)\alpha,\beta)dx \\ &= \sum_{i,j=0}^{\infty} \Phi_{i,j}(\theta)\, [2(i+j)\alpha]^{\frac{r}{\beta}}\, \Gamma\left(1-\frac{r}{\beta}\right), \quad r<\beta. \end{aligned}$$

We may derive the moments about the origin from the previous equation by selecting $r = 1,2,3$, and 4.

3.4 Moment Generating function (MGF)

The following equation may be used to express the MGF of the NEEIW distribution.

$$\begin{aligned} M(t) &= \int_0^{\infty} e^{tx} f(x)dx \\ &= \sum_{i,j=0}^{\infty} \Phi_{i,j}(\theta) \int_0^{\infty} e^{tx} g_(x;2(i+j)\alpha,\beta)dx \\ &= \sum_{i,j,k=0}^{\infty} \frac{t^k}{k!} \Phi_{i,j}(\theta) \int_0^{\infty} x^k g_(x;2(i+j)\alpha,\beta)dx \\ &= \sum_{i,j,k=0}^{\infty} \frac{t^k}{k!} \Phi_{i,j}(\theta)\, [2(i+j)\alpha]^{\frac{k}{\beta}}\, \Gamma\left(1-\frac{k}{\beta}\right), \quad k<\beta, \end{aligned}$$

4 Methods Used To Estimate The Parameters

Here, we explain three estimation strategies to calculate the values of the estimated NEEIW parameters in this study. The MLE, which is the most well-known methodology for assessing parameters, the MPS approach and the Bayesian estimation method that depends on a function called the squared error loss function. All are among the estimation strategies that we will use.

4.1 The MLE Method

By getting a random sample say it $x_1,\cdots,x_n$ from the NEEIW distribution, this would be the probability function for the NEEIW distribution when vectors parameter $\Theta=(\alpha,\beta,\theta)$ are included

$$L(\Theta)=2^n\theta^{2n}\alpha^n\beta^n e^{-2\alpha\sum_{i=1}^n x_i^{-\beta}} \prod_{i=1}^n x_i^{-\beta-1}\frac{\left\{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}^{\theta-1}}{\left\{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}^{\theta+1}}. \tag{10}$$

The log-likelihood function of NEEIW takes the form

$$\ell(\Theta)\propto 2n\ln(\theta)+n\ln(\alpha)-2\alpha\sum_{i=1}^n x_i^{-\beta}-(\beta+1)\sum_{i=1}^n\ln(x_i)+ (\theta-1)\sum_{i=1}^n\ln\left\{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}+n\ln(\beta)- (\theta+1)\sum_{i=1}^n\ln\left\{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}. \tag{11}$$

We will get the first partial derivative of equation (11) for every parameter partially, so the equations resulting are

$$\frac{\partial\ell(\Theta)}{\partial\alpha}=\frac{n}{\alpha}-2\sum_{i=1}^n x_i^{-\beta}+(\theta-1)\sum_{i=1}^n\frac{2x_i^{-\beta}\left(e^{-2\alpha x_i^{-\beta}}\right)}{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}}+ 2(\theta^2-1)\sum_{i=1}^n\frac{x_i^{-\beta}\left(e^{-2\alpha x_i^{-\beta}}\right)}{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}}, \tag{12}$$

$$\frac{\partial\ell(\Theta)}{\partial\beta}=\frac{n}{\beta}+2\alpha\sum_{i=1}^n x_i^{-\beta}\ln(x_i)-\sum_{i=1}^n\ln(x_i)- 2(\theta-1)\alpha\sum_{i=1}^n\frac{x_i^{-\beta}\ln(x_i)e^{-2\alpha x_i^{-\beta}}}{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}} -2\alpha(\theta^2-1)\sum_{i=1}^n\frac{x_i^{-\beta}\ln(x_i)e^{-2\alpha x_i^{-\beta}}}{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}}, \tag{13}$$

and

$$\frac{\partial\ell(\Theta)}{\partial\theta}=\frac{2n}{\theta}+\sum_{i=1}^n\ln\left\{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}- \sum_{i=1}^n\ln\left\{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}- (\theta+1)\sum_{i=1}^n\frac{e^{-2\alpha x_i^{-\beta}}}{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}}. \tag{14}$$

Since the equations in (12), (13), and (14) are not to be solved easily, the Newton-Raphson method will be employed to address these problems.

4.2 The MPS Method

We investigated one of the most well-known classical estimation strategy in this section, which is the MPS methodology, known as the first competitive approach for MLE method. For more information and refernces about this approch see Ng et al. [14], Almetwally et al. [15] and Alshenawy et al. [16]. The log-MPS estimates for the NEEIW distribution takes the form

$$lS(\Theta)\propto\ln\left(1-\left\{\frac{1-\left(e^{-x_1^{-\beta}}\right)^{2\alpha}}{1-(1-\theta)\left(e^{-x_1^{-\beta}}\right)^{2\alpha}}\right\}^{\theta}\right)+ \theta\ln\left\{\frac{1-\left(e^{-x_n^{-\beta}}\right)^{2\alpha}}{1-(1-\theta)\left(e^{-x_n^{-\beta}}\right)^{2\alpha}}\right\}+ \sum_{i=2}^r\ln\left(\left\{\frac{1-\left(e^{-x_{i-1}^{-\beta}}\right)^{2\alpha}}{1-(1-\theta)\left(e^{-x_{i-1}^{-\beta}}\right)^{2\alpha}}\right\}- \left\{\frac{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}}{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}}\right\}\right). \tag{15}$$

The basic steps may be used to calculate the MPS estimates of distribution parameters:

1. To begin, construct the log-product equation. (15) .
2. Calculate the partial derivative of the given equation (15), with regard to every current parameter.
3. We are well aware that these equations are incredibly difficult to solve. Hence, we will use a nonlinear optimization strategies like the Newton–Raphson algorithm to solve these types of problems.

4.3 Bayesian Estimation

On the basis of the squared error (SE) loss function of the NEEIW distribution parameters, this part gives Bayesian parameter α,β and λ estimates for the NEEIW distribution parameters. In this case, the previous distributions of the parameters have been selected as gamma distributions. Thus

$$\pi_1(\alpha)\propto\alpha^{b_1-1}e^{-\alpha d_1},\quad \alpha>0,\ b_1,d_1>0,$$

$$\pi_2(\beta)\propto\beta^{b_2-1}e^{-\beta d_2},\quad \beta>0,\ b_2,d_2>0,$$

and

$$\pi_3(\theta)\propto\theta_3^{b_3-1}e^{-\theta d_3},\quad \theta>0,\ b_3,d_3>0.$$

Assuming the suggested model parameters are independent, we may get and construct the joint PDF of the priors as follows.

$$\pi(\alpha,\beta,\theta) \propto \alpha^{b_1-1}\beta^{b_2-1}\theta^{b_3-1}e^{-(\alpha d_1+\beta d_2+\theta d_3)}. \quad (16)$$

Therefore, the posterior function of the parameters for the proposed distribution may be determined using equation (10) and also equation (16) in the following manner:

$$\begin{aligned}\pi^*(\Theta|\mathbf{x}) &\propto L(\alpha,\beta,\theta)\pi(\alpha,\beta,\theta)\\ &\propto \theta^{2n+b_3-1}\alpha^{n+b_1-1}\beta^{n+b_2-1}\\ &\quad e^{-\alpha\left(d_1+2\sum_{i=1}^{n}x_i^{-\beta}\right)}e^{-\beta\left[d_2+\sum_{i=1}^{n}\ln(x_i)\right]}e^{-\theta d_3}\\ &\quad \prod_{i=1}^{n}\frac{\left\{1-\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}^{\theta-1}}{\left\{1-(1-\theta)\left(e^{-x_i^{-\beta}}\right)^{2\alpha}\right\}^{\theta+1}}.\end{aligned} \quad (17)$$

The SE loss function is used to estimate the Bayesian parameters of the NEEIW distribution, and the results are provided by

$$\tilde{\alpha}_{SE} = \int_0^\infty \alpha \int_0^\infty \int_0^\infty \pi^*(\Theta|\mathbf{x})\, d\beta\, d\theta\, d\alpha, \quad (18)$$

$$\tilde{\beta}_{SE} = \int_0^\infty \beta \int_0^\infty \int_0^\infty \pi^*(\Theta|\mathbf{x})\, d\theta\, d\alpha\, d\beta, \quad (19)$$

and

$$\tilde{\theta}_{SE} = \int_0^\infty \theta \int_0^\infty \int_0^\infty \pi^*(\Theta|\mathbf{x})\, d\alpha\, d\beta\, d\theta. \quad (20)$$

It should be obvious that the integrals in equations (18),(19), and (20) are difficult. As a result, approximations for these integrals are obtained using the Markov Chain Monte Carlo (MCMC) and Metropolis-Hastings (MH) algorithms.

4.3.1 Markov Chain Monte Carlo

Since we all understand that numerous integrals are hard to solve analytically or even statistically by hand, this is especially true for complex problems. We must employ the MCMC approach to get a set of estimates for these integrals. It is critical to understand that the MH algorithm, more generally referred to as the random walk algorithm, is a necessary component of the MCMC technique. It is relatively similar to how samples are accepted and rejected in the estimate steps setup.

5 Estimation of the Parameters Intervals

This section of the article was dedicated to interval estimation, as we estimated the distribution parameters using two methods: asymptotic confidence intervals and credible confidence intervals.

5.1 Asymptotic confidence intervals

Asymptotic confidence intervals (CI) seem to be the most commonly used strategy for getting approximate confidence bounds for parameters. This method makes use of the MLEs to produce the Fisher information matrix, which is the most frequently used approach.
We can establish the $100(1-\gamma)\%$ asymptotic CI for parameters α, β and θ as follows:

$$\left(\hat{\Theta}_l,\, \hat{\Theta}_u\right) = \hat{\Theta} \pm Z_{1-\gamma/2}\sqrt{V(\hat{\Theta})}, \quad (21)$$

where Θ is α, θ or β, and Z_q is the $100q-th$ that is well known as the percentile of the standard normal distribution .

5.2 Highest Posterior Density (HPD) interval algorithm

We need to construct interval estimates for the distribution's parameter since Bayesian estimation yields point estimates. The HPD interval is also known as the credible interval. For more informatins on how to get the $(1-\gamma)$ HPD interval for α,β,θ, see [17, 18].

6 Simulation Study

As is publicly recognized, we must review the efficiency of any distribution across a wide range of parameter values, which involves doing a simulation study using both whole and partial distributions. Monte Carlo simulations are used in this section to judge the effectiveness of the approaches presented in the research and to identify the parameters using MLEs, MPS, and Bayesian estimates of the NEEIW parameters using full samples, as stated in the R studio. Using the following parameter combinations, a number of 10,000 randomly selected samples were generated from the NEEIW distribution.

We used several parameter values and combinations for experimental procedures, as detailed in Tables 1, 2.In order to calculate the efficiency and accuracy of the estimators, we calculated the Relative Absolute Bias (RAB), Mean Squared Error (MSE), and Confidence Interval Length (L.CI). Comparing the outcomes of point estimation is depending on the RAB, MSE, and L.CI values. Also, we estimated the coverage probability (CP). Tables 1, 2 depict the varied results of modeling the various point estimation strategies addressed throughout this research.

Table 1: The following table records the results conducted from the estimation methods when the data follows the NEEIW distribution when fixing parameter $\alpha = 0.5$

$\alpha = 0.5$				MLE				MPS				Bayesian		
β	θ	n		RAB	MSE	L.CI	CP	RAB	MSE	L.CI	CP	RAB	MSE	L.CI
0.5	0.5	30	α	0.5558	0.6864	3.0610	94.50%	0.4282	0.6485	3.0447	95.00%	0.1688	0.0693	0.8999
			β	0.3821	0.2798	1.9346	95.00%	0.2980	0.2655	1.9347	95.20%	0.1080	0.0238	0.5442
			θ	0.9644	3.8989	7.5097	96.80%	1.0329	5.5296	8.9974	94.60%	0.0798	0.0323	0.6366
		75	α	0.2375	0.2751	2.0036	94.50%	0.2186	0.3037	2.1185	95.90%	0.1121	0.0335	0.6913
			β	0.1937	0.1085	1.2348	95.50%	0.1372	0.1048	1.2406	95.25%	0.0549	0.0111	0.3944
			θ	0.2904	0.3474	2.2405	94.60%	0.3622	0.4035	2.3878	94.60%	0.0716	0.0174	0.4954
		150	α	0.1830	0.1645	1.5498	94.30%	0.2117	0.1978	1.6940	94.90%	0.1125	0.0256	0.5653
			β	0.0933	0.0534	0.8872	95.40%	0.0415	0.0522	0.8923	96.30%	0.0199	0.0055	0.2942
			θ	0.1781	0.1325	1.3842	93.80%	0.2583	0.1806	1.5879	95.40%	0.0763	0.0144	0.4384
	2	30	α	0.3145	0.6018	2.9793	94.30%	0.0663	0.0316	0.6851	93.50%	0.1201	0.0437	0.7949
			β	0.5887	0.3979	2.1879	95.80%	0.0326	0.0124	0.4314	94.60%	0.0422	0.0126	0.4274
			θ	0.7520	4.7764	8.3635	95.80%	0.0249	0.2408	1.9148	94.00%	0.1248	0.5686	2.6247
		75	α	0.0429	0.1920	1.7164	95.40%	0.0351	0.0161	0.4923	91.70%	0.0815	0.0247	0.6064
			β	0.3142	0.1588	1.4362	93.20%	0.0197	0.0061	0.3039	93.90%	0.0261	0.0063	0.3145
			θ	0.1119	3.4498	7.2314	96.60%	0.0120	0.1209	1.3602	94.00%	0.0786	0.2645	1.8179
		150	α	0.0043	0.0812	1.1176	96.50%	0.0259	0.0089	0.3657	93.30%	0.0435	0.0117	0.4034
			β	0.1543	0.0616	0.9254	93.90%	0.0162	0.0033	0.2224	94.30%	0.0180	0.0034	0.2226
			θ	0.0169	0.7003	3.2794	96.10%	0.0106	0.0622	0.9743	93.30%	0.0412	0.1026	1.1560
2	0.5	30	α	0.6039	0.7045	3.0714	94.20%	0.0184	0.1069	1.2820	95.10%	0.1637	0.0626	0.8858
			β	0.3143	3.7026	7.1325	95.40%	0.0181	0.2894	2.1052	94.90%	0.0996	0.3966	2.3280
			θ	1.0335	3.9400	7.5164	97.10%	0.0877	0.0894	1.1599	95.40%	0.0786	0.0290	0.6481
		75	α	0.2969	0.3034	2.0802	93.70%	0.0166	0.0634	0.9867	94.70%	0.1512	0.0417	0.7296
			β	0.1533	1.4795	4.6164	94.80%	0.0105	0.1762	1.6445	94.50%	0.0409	0.1733	1.6502
			θ	0.3368	0.3900	2.3587	95.30%	0.0530	0.0403	0.7805	94.90%	0.0971	0.0232	0.5555
		150	α	0.2031	0.1685	1.5600	94.30%	0.0111	0.0344	0.7273	94.80%	0.1132	0.0259	0.5886
			β	0.0796	0.7999	3.4517	95.50%	0.0008	0.1184	1.3494	95.60%	0.0229	0.0970	1.2506
			θ	0.1994	0.1403	1.4159	94.70%	0.0315	0.0200	0.5513	94.10%	0.0808	0.0148	0.4466
	2	30	α	0.2671	0.5792	2.9385	93.90%	0.0858	0.0419	0.7851	95.00%	0.1633	0.0558	0.8901
			β	0.5785	5.8660	8.3452	96.20%	0.0526	0.1406	1.4113	95.80%	0.0161	0.1944	1.7234
			θ	0.6990	4.0396	8.1512	95.30%	0.0503	0.3163	2.1702	95.50%	0.1705	0.7668	3.0647
		75	α	0.0433	0.1645	1.5883	95.80%	0.0800	0.0213	0.5503	96.50%	0.0896	0.0248	0.6017
			β	0.2641	2.0146	5.1670	93.60%	0.0437	0.0773	1.0348	96.90%	0.0204	0.1055	1.2703
			θ	0.0822	2.0483	5.5759	96.10%	0.0399	0.1542	1.5080	95.90%	0.0788	0.2512	1.8330
		150	α	0.0198	0.0854	1.1455	96.40%	0.0592	0.0135	0.4399	96.20%	0.0518	0.0130	0.4278
			β	0.1703	1.0349	3.7598	93.50%	0.0309	0.0515	0.8569	97.60%	0.0153	0.0556	0.8815
			θ	0.0069	0.7736	3.4490	96.10%	0.0327	0.1015	1.2230	95.60%	0.0450	0.1160	1.2262

6.1 Observation recorded from the results of the simulation section

The following conclusions may readily be made based on the simulation findings.

1. The MSE, RAB, and L.CI lengths of each of the parameters drop as the sample size grows, which is known as the consistency property.
2. When the sample is constructed using a complete sample, we found that the credible intervals have the shortest length of all the L.CIs.
3. In most circumstances, increased θ causes the MSE, RAB, and L.CI for the parameter α to decrease while the length for β and θ to increase.
4. In most circumstances, increasing β causes the MSE, RAB, and L.CI for all parameters to decrease.
5. In most circumstances, increasing α causes the MSE, RAB, and L.CI for the most parameters to decrease.
6. In most parameter measures, MPS is a good alternative to MLE.
7. The Bayesian estimation is the best estimation method.
8. The credible intervals HPD are the shortest L.CI.
9. The CP is very high in most runs, meaning the estimator falls between the lower and upper bounds.

7 Modeling carbon real data sets

In this part, we demonstrate the adaptability and capability of our proposed model to fit two carbon fiber real data sets. The first data set consists of 56 observations, which Nichols and Padgett [20] introduced. The second data set consists of 63 observations and can be founded in Mahmoud and Mandouh [19].

The carbon data sets are used to show the flexibility of the proposed model compared to some well-known distributions. The compared models, along with their CDFs, are defined as the follows

– Kumaraswamy Fréchet (KF) distribution [21]

$$F(x) = 1 - \left(1 - e^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{b}.$$

Table 2: The following table records the results conducted from the estimation methods when the data follows the NEEIW distribution when fixing parameter $\alpha = 3$

$\alpha=3$				MLE				MPS				Bayesian		
β	θ	n		RAB	MSE	L.CI	CP	RAB	MSE	L.CI	CP	RAB	MSE	L.CI
0.5	0.5	30	α	0.1325	0.5051	2.3965	96.2%	0.0276	0.0906	1.1806	94.2%	0.0979	0.4145	1.8163
			β	0.5177	0.1465	1.1469	95.0%	0.4638	0.1464	1.2417	94.0%	0.3291	0.0605	0.6297
			θ	0.1638	0.1014	1.2519	94.7%	0.1501	0.1060	1.2926	93.9%	0.1073	0.0303	0.6809
		75	α	0.0086	0.0957	1.2120	96.0%	0.0393	0.1060	1.1930	96.9%	0.0184	0.0913	1.1792
			β	0.0518	0.0293	0.6646	94.9%	0.0109	0.0403	0.7883	95.4%	0.0909	0.0176	0.4862
			θ	0.0658	0.0417	0.7924	94.1%	0.1563	0.0744	1.0269	94.0%	0.1027	0.0399	0.6520
		150	α	0.0402	0.1864	1.6340	95.4%	0.0176	0.0354	0.7115	95.9%	0.0564	0.1581	1.3018
			β	0.0021	0.0232	0.6006	96.1%	0.0308	0.0167	0.5052	97.3%	0.0312	0.0078	0.3290
			θ	0.1569	0.0597	0.9117	95.1%	0.1370	0.0451	0.7924	94.4%	0.1239	0.0294	0.4794
	2	30	α	0.3861	5.5493	7.7831	97.1%	0.9446	4.8334	6.8178	99.6%	0.0932	0.4840	2.1863
			β	0.9466	1.1493	3.7726	94.1%	0.8247	1.3507	4.2615	94.7%	0.2221	0.0538	0.6911
			θ	0.7167	2.8384	4.5959	96.1%	0.6289	2.2194	5.3694	97.7%	0.0416	0.6073	2.6849
		75	α	0.0611	1.7724	5.1717	99.5%	0.0024	4.2910	8.1242	99.8%	0.0250	0.0808	1.0638
			β	0.3549	0.2606	1.8772	94.6%	0.2391	0.2361	1.8469	94.5%	0.1240	0.0215	0.4661
			θ	0.0957	2.3493	3.9643	96.7%	0.1247	0.9103	3.6022	96.8%	0.0222	0.2605	1.9791
		150	α	0.0194	0.1226	1.3544	95.6%	0.0112	0.1047	1.2624	95.6%	0.0115	0.0314	0.6790
			β	0.1536	0.0698	0.9911	94.7%	0.0628	0.0512	0.8787	95.3%	0.0715	0.0085	0.3193
			θ	0.0335	0.8165	3.5341	96.0%	0.0744	0.7387	3.3200	95.4%	0.0122	0.1219	1.3914
2	0.5	30	α	0.1505	1.3380	4.1768	95.0%	0.1087	0.7725	3.2012	94.8%	0.2014	0.9898	2.8301
			β	0.0651	0.4770	2.6602	98.4%	0.0458	0.4721	2.6707	98.4%	0.0612	0.4203	2.4132
			θ	0.5404	0.3876	2.1997	92.3%	0.3348	0.2914	2.0127	91.8%	0.0982	0.0432	0.6913
		75	α	0.0580	0.5326	2.7798	96.1%	0.0805	0.3837	2.2370	95.1%	0.0510	0.4617	2.0525
			β	0.0303	0.3020	2.1420	94.4%	0.0419	0.3015	2.1282	94.8%	0.0733	0.2085	1.6541
			θ	0.0829	0.0393	0.7607	96.7%	0.1997	0.1303	1.3607	93.7%	0.0836	0.0278	0.5598
		150	α	0.0405	0.2663	1.9671	95.2%	0.0553	0.2106	1.6782	95.0%	0.0617	0.2277	1.6955
			β	0.0099	0.2045	1.7719	94.0%	0.0264	0.1885	1.6900	95.1%	0.0438	0.1224	1.3268
			θ	0.1358	0.0685	0.9910	94.7%	0.1034	0.0528	0.8780	94.3%	0.0748	0.0177	0.4550
	2	30	α	0.0979	0.8910	3.5182	95.7%	0.0447	0.4075	2.4477	95.0%	0.0739	0.4743	2.4499
			β	0.2759	1.3721	4.0526	97.7%	0.0549	0.5558	2.8919	96.6%	0.2975	1.2510	3.4432
			θ	0.0012	1.2297	4.3492	97.5%	0.0280	0.7340	3.3529	96.7%	0.0103	0.7012	3.0012
		75	α	0.0414	0.2355	1.8396	95.1%	0.0199	0.1411	1.4545	95.2%	0.0252	0.0948	1.0833
			β	0.1738	0.9807	3.6368	95.7%	0.0064	0.3021	2.1551	94.7%	0.1561	0.4716	2.1287
			θ	0.0441	0.9885	3.8839	95.0%	0.0606	0.4638	2.6283	96.7%	0.0090	0.2681	2.0468
		150	α	0.0114	0.0807	1.1059	94.2%	0.0167	0.0651	0.9808	94.6%	0.0075	0.0258	0.6317
			β	0.0913	0.3722	2.2830	95.3%	0.0054	0.1681	1.6076	95.3%	0.0822	0.1453	1.2201
			θ	0.0043	0.3702	2.3861	97.9%	0.0418	0.2740	2.0265	95.9%	0.0014	0.1194	1.3095

–Exponentiated Fréchet (EF) distribution [22]

$$F(x) = 1-\left(1-e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}.$$

–Marshall-Olkin Fréchet (MOF) distribution [23]

$$F(x) = \frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\left(-\frac{be^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{(1-\alpha)e^{-\left(\frac{\alpha}{x}\right)^{\beta}}+\alpha}+b+1\right)}{(1-\alpha)e^{-\left(\frac{\alpha}{x}\right)^{\beta}}+\alpha}.$$

–Modified Fréchet (MF) distribution [24]

$$F(x) = e^{e^{-\lambda x}\left(-\left(\frac{\alpha}{x}\right)^{\beta}\right)}.$$

–Modified Kies–Fréchet (MKF) distribution [25]

$$F(x) = 1-e^{-\left(\frac{1}{e^{\alpha x^{-\lambda}}-1}\right)^{\beta}}.$$

–odd Lindely Fréchet (OLF) distribution [26]

$$F(x) = 1-\frac{\left(\lambda+\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right)\right)e^{-\frac{\lambda e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}}}{(\lambda+1)\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right)}.$$

–Fréchet-Weibull mixture exponential (FWME) distribution [27]

$$F(x) = \frac{a}{a+\left(\frac{x}{\lambda}\right)^{\alpha(-k)}}.$$

–Fréchet-Weibull (FW) distribution [28]

$$F(x) = e^{-\beta^{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha k}}.$$

In comparing between models, we applied widely used and essential measures like the Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), the Bayesian information criterion (BIC), and the Hannan information criterion (HQIC). Also, we used some statistics such as Cramér–Von Mises (W), Anderson–Darling (A), and Kolmogorov–Smirnov (K-S) statistics with its corresponding p-value.

Tables 3 and 4 introduce the numerical results obtained by analyzing the carbon data sets, respectively. The values in these tables show that the NEEIW distribution is a close fit to the modeled data sets compared to other competing distributions. Estimated PDF, CDF, SF, and P-P plots for our proposed model for the two real data sets are presented in Figures 4 and 5.

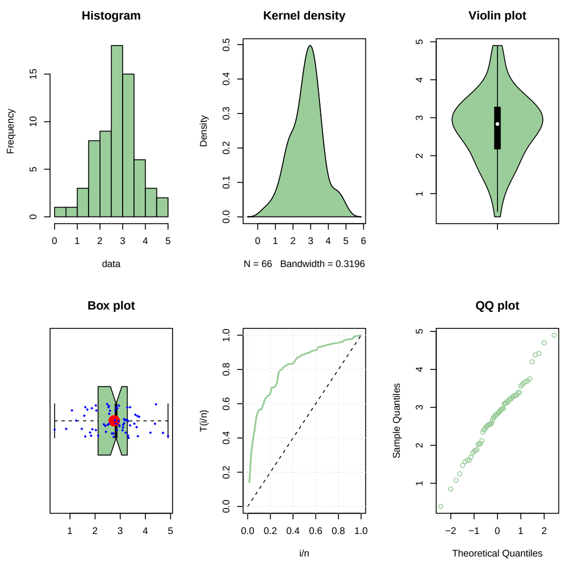

Fig. 2: Graphical representation for some plots of the first real data set.

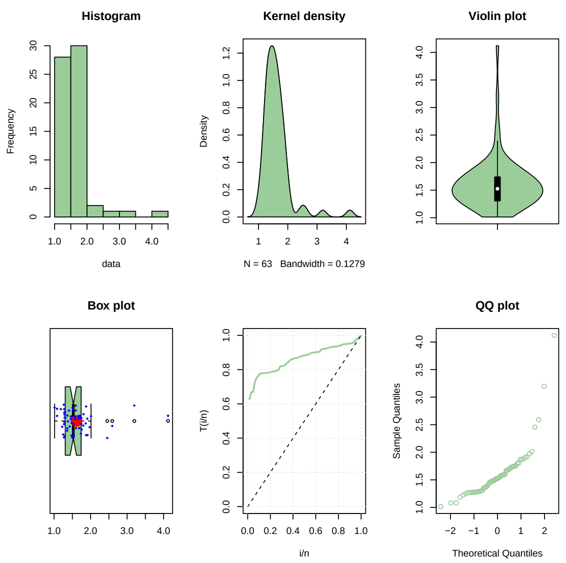

Fig. 3: Graphical representation for some plots of the second real data set.

Some graphical representations of the two real data sets are presented in Figures 2 and 2. The results in these tables were obtained using Mathematica Wolfram Software version 12.0. Also, we used the NMaximize function to get the estimated parameters of all models; this function attempts to find a global maximum solution. Again, we provided Figures 6 and 7, which show the behavior of the log-likelihood function with the estimated parameters of the NEEIW model for the carbon data sets, respectively, which give a maximum value of the function with the values of the estimated parameters.

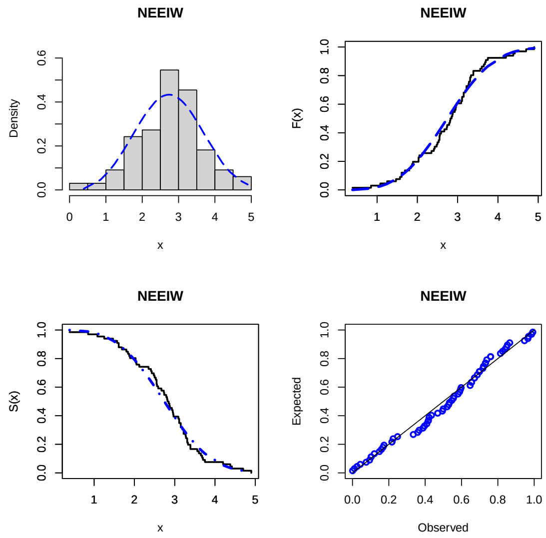

Fig. 4: Histogram of first data set with the estimated NEEIW PDF, CDF, SF and P-P plot.

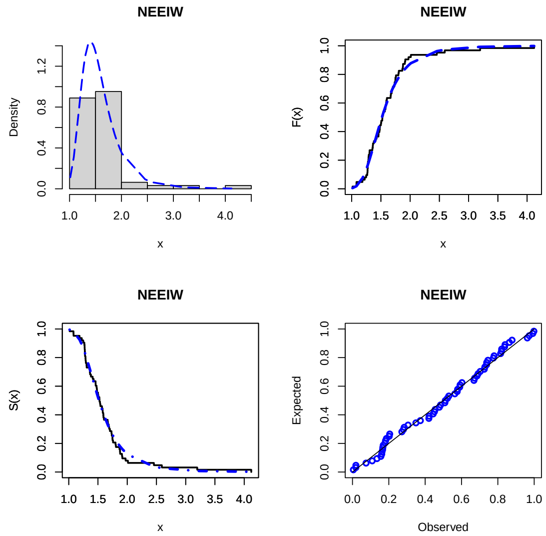

Fig. 5: Histogram of second data set with the estimated NEEIW PDF, CDF, SF and P-P plot.

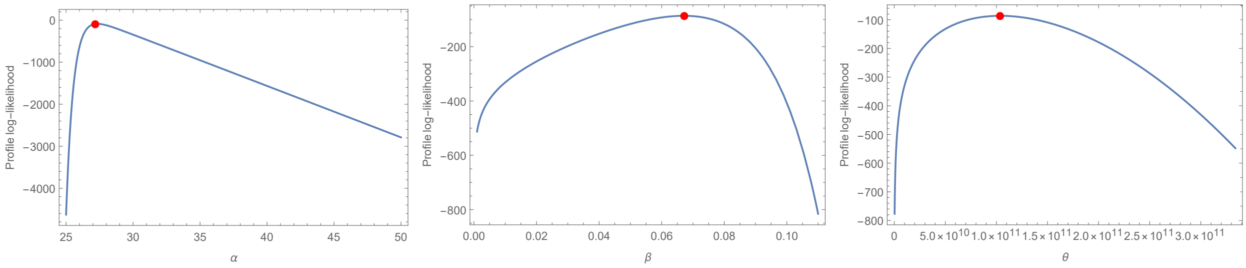

Fig. 6: Behavior log-likelihood function with estimated parameters of NEEIW model for the first data set.

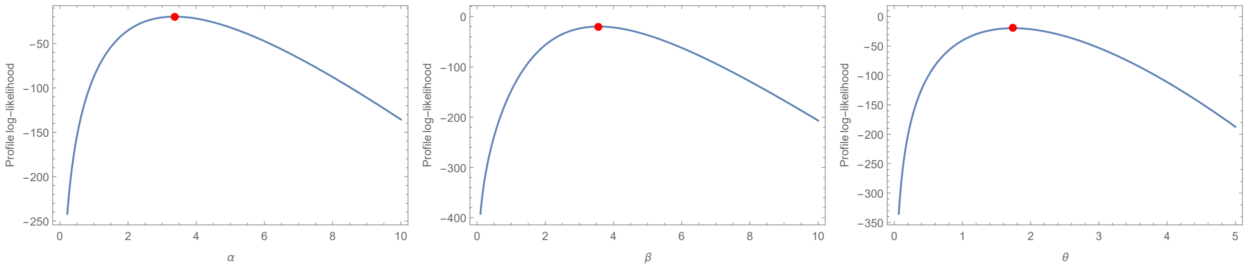

Fig. 7: Behavior log-likelihood function with estimated parameters of NEEIW model for the second data set.

Table 3: Numerical Values by analyzing first data set for all compared models.

Distribution	AIC	CAIC	BIC	HQIC	A	W	K-S (stat)	K-S (p-value)	Estimates
NEEIW	178.688	179.075	185.257	181.284	0.521042	0.0915272	0.0862444	0.710149	$\hat{\alpha} = 27.3376$ (2.77759) $\hat{\beta} = 0.0670846$ (0.00965365) $\hat{\theta} = 1.03816 \times 10^{11}$ (2.87865×10^{11})
KF	183.763	184.419	192.522	187.224	0.751943	0.140918	0.104905	0.461909	$\hat{\alpha} = 28.7403$ (10.92168) $\hat{\beta} = 0.286123$ (0.0520963) $\hat{a} = 5.9227$ (3.94303) $\hat{b} = 80103.8$ (154201)
EF	187.142	187.529	193.711	189.738	1.22991	0.232108	0.130307	0.212375	$\hat{\lambda} = 0.484053$ (0.149947) $\hat{\alpha} = 389.71$ (462.779) $\hat{\beta} = 119.92$ (190.227)
MOF	189.29	189.678	195.859	191.886	1.1527	0.157235	0.0938397	0.606408	$\hat{\alpha} = 888660$ (1.45627×10^{8}) $\hat{\beta} = 4.89582$ (0.513683) $\hat{a} = 0.165217$ (5.52977)
MF	192.97	193.357	199.539	195.566	1.82109	0.320396	0.143032	0.134309	$\hat{\alpha} = 2.92362 \times 10^{7}$ (1.07398×10^{8}) $\hat{\beta} = 0.141525$ (0.0313508) $\hat{\lambda} = 1.01202$ (0.0900319)
MKF	179.803	180.19	186.372	182.399	0.5633	0.104402	0.0933388	0.613233	$\hat{\alpha} = 0.830679$ (0.0936299) $\hat{\beta} = 14.9524$ (9.50064) $\hat{\lambda} = 0.162931$ (0.101981)
OLF	179.99	180.377	186.559	182.586	0.628665	0.112893	0.0930215	0.617562	$\hat{\alpha} = 0.521397$ (0.258958) $\hat{\beta} = 2.29667$ (0.211233) $\hat{\lambda} = 0.0374918$ (0.0434638)
FWME	191.291	191.946	200.049	194.751	1.15257	0.157234	0.0938391	0.606417	$\hat{\alpha} = 1.54959$ (6.07464) $\hat{\lambda} = 2.99225$ (1.832563) $\hat{k} = 3.15944$ (3138.15) $\hat{a} = 1.62149$ (1.63855)
FW	250.39	251.046	259.148	253.851	6.50403	1.15621	0.230263	0.00182595	$\hat{\alpha} = 1.08573$ (570.361) $\hat{\beta} = 1.15779$ (927.169) $\hat{k} = 1.51792$ (797.403) $\hat{\lambda} = 1.84819$ (1012.32)

Table 4: Numerical Values by analyzing second data set for all compared models.

Distribution	AIC	CAIC	BIC	HQIC	A	W	K-S (stat)	K-S (p-value)	Estimates
NEEIW	45.3862	45.793	51.8156	47.9149	0.420494	0.0508308	0.0650183	0.952729	$\hat{\alpha} = 3.36302$ (0.473157) $\hat{\beta} = 3.5687$ (1.40975) $\hat{\theta} = 1.74191$ (0.877988)
KF	47.8663	48.556	56.4389	51.2379	0.4746	0.059182	0.0715026	0.904138	$\hat{\alpha} = 1.50069$ (1818.54) $\hat{\beta} = 1.31395$ (4.77259) $\hat{a} = 0.912467$ (5277.2) $\hat{b} = 1.2828$ (0.639307)
EF	45.8663	46.2731	52.2957	48.395	0.4746	0.059182	0.0715026	0.904138	$\hat{\lambda} = 4.77259$ (1.31395) $\hat{\alpha} = 1.2828$ (0.639307) $\hat{\beta} = 1.47217$ (0.137606)
MOF	46.0295	46.4363	52.459	48.5583	0.482735	0.058807	0.0726242	0.893862	$\hat{\alpha} = 1.93866$ (3.36547) $\hat{\beta} = 6.2198$ (2.08586) $\hat{a} = 1.31646$ (0.225456)
MF	46.1269	46.5337	52.5563	48.6556	0.527521	0.0694782	0.0771355	0.847667	$\hat{\alpha} = 1.42506$ (0.508015) $\hat{\beta} = 5.38126$ (2.04808) $\hat{\lambda} = 0.0375776$ (1.31546)
MKF	57.0627	57.4694	63.4921	59.5914	1.92364	0.326409	0.165964	0.0621973	$\hat{\alpha} = 17.84$ (6.80249) $\hat{\beta} = 0.375057$ (0.0728259) $\hat{\lambda} = 5.93688$ (0.854428)
OLF	65.4658	65.8725	71.8952	67.9945	2.08947	0.321705	0.135566	0.197217	$\hat{\alpha} = 2.95834$ (0.79311) $\hat{\beta} = 1.55599$ (0.300425) $\hat{\lambda} = 9.96719$ (6.35689)
FWME	52.6507	53.3404	61.2233	56.0223	0.561021	0.0463413	0.0751462	0.868942	$\hat{\alpha} = 3.46384$ (16.0117) $\hat{\lambda} = 5.45667$ (1.88173) $\hat{k} = 2.39563$ (663.762) $\hat{a} = 5.47024$ (3773.9)
FW	48.1277	48.8174	56.7003	51.4993	0.529087	0.0698934	0.0772243	0.846687	$\hat{\alpha} = 1.85751$ (640.099) $\hat{\beta} = 2.89635$ (1180.26) $\hat{k} = 2.92752$ (1008.82) $\hat{\lambda} = 0.981082$ (99.4793)

8 Conclusion

This study suggested the NEEIW distribution, a novel combination of the NEX family and the EIW baseline distributions. We have examined its statistical features and produced a linear representation for its CDF, which efficiently predicted the linear representations of the PDF, moments, moment-generating function, and stress-strength reliability. To get the point estimates for the undetermined NEEIW parameters α, β, and θ, several conventional and non-classical Bayesian estimation techniques were examined. Using the R program, a simulated study was conducted to investigate the efficacy of several estimating techniques. Using the MCMC technique, we concluded that the Bayesian strategy beats all traditional methods investigated. Two distinct Carbon data sets were evaluated. It was determined that NEEIW ”fit” the data ”better” than the majority of competing distributions. The existence and uniqueness of the log-likelihood function have been shown.

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Conflicts of Interest

The author declares that there is no conflict of interest.

References

[1] Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, **31(4)**, 497-512.

[2] Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation*, **81(7)**, 883-898.

[3] Al-Babtain, A. A., Elbatal, I., Al-Mofleh, H., Gemeay, A. M., Afify, A. Z., Sarg, A. M. (2021) The Flexible Burr XG Family: Properties, Inference, and Applications in Engineering Science. Symmetry, 13(3), 474.

[4] Zografos, K., & Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inference. *Statistical methodology*, **6(4)**, 344-362.

[5] Ristić, M. M., & Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. *Journal of statistical computation and simulation*, **82(8)**, 1191-1206.

[6] Amini, M., MirMostafaee, S. M. T. K., & Ahmadi, J. (2014). Log-gamma-generated families of distributions. *Statistics*, **48(4)**, 913-932.

[7] Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, **71(1)**, 63-79.
[8] Alzaghal, A., Famoye, F., & Lee, C. (2013). Exponentiated TX family of distributions with some applications. *International Journal of Statistics and Probability*, **2(3)**, 31.
[9] Abonongo, A. I. L., & Abonongo, J. (2024). Exponentiated Generalized Weibull Exponential Distribution: Properties, Estimation and Applications. *Computational Journal of Mathematical and Statistical Sciences*, **3(1)**, 57-84.
[10] Tahir, M. H., Zubair, M., Mansoor, M., Cordeiro, G. M., ALİZADEHK, M., & Hamedani, G. G. (2016). A new Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics*, **45(2)**, 629-647.
[11] Almetwally, E. M. (2021). The Odd Weibull Inverse Topp–Leone Distribution with Applications to COVID-19 Data. *Annals of Data Science*, 1-20. https://doi.org/10.1007/s40745-021-00329-w.
[12] Zichuan, M., Hussain, S., Iftikhar, A., Ilyas, M., Ahmad, Z., Khan, D. M., & Manzoor, S. (2020). A New Extended-Family of Distributions: Properties and Applications. Computational and mathematical methods in medicine, 2020.
[13] Flaih, A., Elsalloukh, H., Mendi, E., & Milanova, M. (2012). The exponentiated inverted Weibull distribution. Appl. Math. Inf. Sci, 6(2), 167-171.
[14] Ng, H. K. T., Luo, L., Hu, Y., & Duan, F. (2012). Parameter estimation of three-parameter Weibull distribution based on progressively type-II censored samples. *Journal of Statistical Computation and Simulation*, **82(11)**, 1661-1678.
[15] Almetwally, E. M., Almongy, H. M., Rastogi, M. K., & Ibrahim, M. (2020). Maximum product spacing estimation of Weibull distribution under adaptive type-II progressive censoring schemes. *Annals of Data Science*, **7(2)**, 257-279.
[16] Alshenawy, R., Sabry, M. A., Almetwally, E. M., & Almongy, H. M. (2021). Product Spacing of Stress–Strength under Progressive Hybrid Censored for Exponentiated-Gumbel Distribution. *Computers, Materials & Continua*, **66(3)**, 2973-2995.
[17] M.H.Chen, Q.M. Shao, "Monte Carlo estimation of Bayesian credible and HPD intervals", J Comput Graph Stat 8, (1999),pp. 69-92.
[18] Albadawy, A., Ashour, E., EL-Helbawy, A. A., & AL-Dayian, G. R. (2024). Bayesian Estimation and Prediction for Exponentiated Inverted Topp-Leone Distribution. *Computational Journal of Mathematical and Statistical Sciences*, **3(1)**, 33-56.
[19] Mahmoud, M. R. and Mandouh, R. M., 2013, On the transmuted Fréchet distribution. Journal of Applied Sciences Research, 9(10), 5553-5561.
[20] Nichols, M. D., & Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. Quality and reliability engineering international, 22(2), 141-151.
[21] Mead, M. E. A. (2014). A note on Kumaraswamy Fréchet distribution. Australia, 8, 294-300.
[22] Nadarajah, S., Kotz, S.. (2003). The exponentiated Fréchet distribution. Interstat Electronic Journal, 14, 1-7.
[23] Krishna, E., Jose, K. K., Alice, T., & Ristić, M. M. (2013). The Marshall-Olkin Fréchet distribution. Communications in Statistics-Theory and Methods, 42(22), 4091-4107.
[24] Tablada, C. J., & Cordeiro, G. M. (2017). The modified Fréchet distribution and its properties. Communications in Statistics-Theory and Methods, 46(21), 10617-10639.
[25] Al Sobhi, M. M. (2021). The modified Kies–Fréchet distribution: properties, inference and application. AIMS Math, 6, 4691-4714
[26] Mansour, M. M. and Abd Elrazik, E. M. and Altun, E. and Afify, A. Z. and Iqbal, Z., 2018, A New Three-Parameter Fréchet Distribution: Properties And Applications. Pak. J. Statist, 34(6), 441–458.
[27] Teamah, A. E. A., Elbanna, A. A., & Gemeay, A. M. Fréchet-Weibull Mixture Distribution: Properties and Applications. Applied Mathematical Sciences, 14(2), 75-86 (2020).
[28] Teamah, A. E. A., Elbanna, A. A., & Gemeay, A. M. Fréchet-Weibull Distribution With Applications to Earthquakes Data Sets. Pakistan Journal of Statistics, 36(2) (2020).