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Bivariate Lomax Distribution Based on Clayton Copula: Estimation and Prediction

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Abstract: Lomax distribution has been studied by many statisticians due to its important role in reliability modeling and lifetime testing. The bivariate Lomax distribution is an important lifetime distribution in survival analysis. In this article, a bivariate Lomax distribution is constructed based on Clayton copula. The joint probability density function and the joint cumulative distribution function are derived in closed forms. Some properties of this bivariate distribution are discussed. The maximum likelihood and Bayes estimators for the unknown parameters are derived. Also, the maximum likelihood and Bayesian two-sample prediction for the future observations are obtained. The performance of the proposed bivariate distribution is examined using a simulation study. Finally, one real data set under the proposed distribution is considered to illustrate its flexibility and applicability for real-life applications.

Keywords: Bivariate Lomax Distribution, Clayton Copula, Maximum Likelihood Estimation, Maximum Likelihood Prediction, Bayesian Estimation, Bayesian Prediction, Monte Carlo Simulation.

1 Introduction

In many situations, the data under study is bivariate in nature. For instance, in survival analysis, the researchers are interested in studying the lifetimes of matched human organs such as eyes and kidneys, and also the double recurrence of certain diseases. In reliability studies, the bivariate data may consist of survival times for a system whose duration times rely upon the durability of two components, for example, the damage of lifetimes of motors in twin-engine airplanes (see Louzada et al. [13]). In actual studies, an annuity is a retirement payment for both retiree and his or her spouse which is guaranteed to be paid if both or either of them are living (see Denuit and Cornet [7]).

Pareto Type II distribution is called Lomax distribution which is introduced and studied by Lomax [12]. This distribution is commonly used in reliability, many lifetime testing studies and is used to analyze business data. This distribution is useful in analyzing ordered contingency tables in which the two dimensions can be regarded as alternative measures of the same thing: the injuries to the two drivers in a road accident or the severity of a lesion present in a patient as assessed by two physicians, for instance.

The bivariate Pareto Type II distribution was introduced by Mardia [14] using two dependent gamma variables. Later, Lindley and Singpurwalla [11] presented bivariate Pareto Type II distribution which has simple joint survival function with Pareto Type II marginals. Sankaran and Nair [21] proposed bivariate Pareto distribution which also has Pareto marginals, and it has Lindley-Singpurwalla's bivariate Pareto model as a special case and Sankaran and Kundu [22] discussed several properties of bivariate Pareto distribution. For various other bivariate Pareto distributions and its generalizations, one may refer to Arnold [2], [3] and Kotz et al. [10]. Paul et al. [18] studied Bayesian analysis of singular Marshall-Olkin bivariate Pareto distribution. They derived three parameter singular Marshall-Olkin bivariate Pareto distribution; considering two types of priors, truncated normal prior and gamma prior. They obtained the Bayes estimators for the parameters. Baharith and Alzahrani [4] constructed two bivariate Pareto Type II distributions; one is derived from copula and the other is based on mixture and copula. A bivariate model based on adaptive progressive hybrid censored has been introduced by El-Sherpieny

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et al. [25]. Bivariate Chen distribution based on *Farlie-Gumbel-Morgenstern* (FGM) copula has been obtained by El-Sherpieny et al. [26]. The bivariate models based on copula function with application of accelerated life testing, has been suggested by El-Sherpieny et al. [27].

Some references in the field of the bivariate distributions based on copulas include Almetwally et al. [28] who proposed bivariate distributions called the *Farlie-Gumbel-Morgenstern* (FGM) bivariate Fréchet and Ali-Mikhail-Haq bivariate Fréchet distributions using FGM, Ali-Mikhail-Haq copulas and univariate Fréchet distributions. El-Sherpieny et al. [29] proposed the bivariate FGM Weibull- G family, which is a flexible bivariate generalized family of distributions based on the FGM copula. Muhammed et al. [30] presented the bivariate inverted Topp-Leone distribution, which is derived from FGM copula, Ali-Mikhail-Haq, Plackett and Clayton copulas. Abulebda et al. [31] constructed a bivariate XGamma distribution and investigated its statistical properties through examination of real data. Hassan and Chesneau [32] suggested the bivariate generalized half-logistic distribution using the FGM copula to assess household financial affordability in the Kingdom of Saudi Arabia. Zhao et al. [33] introduced a three-parameter extension of the Lomax distribution via using a class of claim distributions. Furthermore, a bivariate extension of the proposed model called the FGM bivariate Lomax-Claim distribution is also presented and different shapes for the *probability density function* (pdf) are plotted. Qura et al. [34] proposed univariate power Lomax distribution and bivariate power Lomax distribution based on FGM copula, which is referred to as a bivariate power Lomax distribution based on FGM. The statistical properties of the proposed distribution are discussed. The parameters of the model can be estimated by applying *maximum likelihood* (ML) and Bayesian approaches.

This paper is devoted to construct a *bivariate Lomax* (BLO) distribution; based on Clayton copula, some properties of BLO distribution are studied. The rest of this paper is organized as follows: in Section 2, a construction of the BLO distribution with Clayton copula is introduced and some properties are studied. ML estimation and prediction are considered in Section 3. In Section 4, numerical illustration is given; through applying the results of the ML estimation and prediction to the BLO distribution based on simulation and a real data set. In Section 5, Bayesian estimation and prediction is presented for the BLO. Finally in Section 6, numerical illustration for Bayesian estimation and prediction; based on simulation and a real data set are introduced, applying the results to the BLO distribution.

2. Bivariate Lomax Distribution Based on Clayton Copula

Copulas are used to describe the dependence between random variables. Sklar [23] described the functions that join one-dimensional distribution functions to form multivariate distribution functions and named it copula; the name "copula" was chosen to emphasize the way a copula couples a joint distribution function to its univariate margins.

Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions, i.e., copulas are multivariate distribution functions whose one-dimensional margins are uniform on the interval (0, 1) (see Nelsen [16] and Joe [9]).

Copulas are of interest to statisticians for two main reasons: firstly, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions, sometimes with a view to simulation.

Archimedean copulas are an associative class of copulas. Archimedean copulas are popular because they allow modeling dependence in arbitrarily high dimensions with only one parameter, governing the strength of dependence. There are three Archimedean copulas in common use: the Clayton, Frank and Gumbel. The Clayton copula is an asymmetric Archimedean copula, exhibiting greater dependence in the negative tail than in the positive. The first application of copula models in bivariate survival data was considered by Clayton [6] as he studied the bivariate life tables of fathers and sons. Also, he pointed out the random effects interpretation the model which was subsequently developed by Oakes [17]. In this section, the BLO is constructed using Clayton copula and some properties are studied.

2.1 Construction of the bivariate Lomax distribution

Sklar [24] introduced the joint pdf and the joint *cumulative distribution function* (cdf) for two-dimension copula. Considering the two random variables T_1 and T_2, with distribution functions $F(t_1)$ and $F(t_2)$, respectively, then the joint cdf and pdf for bivariate copula, respectively, are

$$F(t_1, t_2) = C[F(t_1), F(t_2)],$$

and

$$f(t_1, t_2) = f(t_1)f(t_2)c[F(t_1), F(t_2)].$$

where C is the cdf of the copula and c is the pdf of the copula. Nadarajah et al. [15] discussed Clayton in general form as

$$C(u_1, u_2, \dots, u_p) = \left[\sum_{i=1}^{p} u_i^{-\theta} - p + 1\right]^{-\frac{1}{\theta}}, \quad \theta > 0,$$

where p is a number of variables. This paper considered the 2-dimension of Clayton copula, which was considered by Clayton [6], it is an asymmetric Archimedean copula, exhibiting greater dependence in the negative tail than in the positive tail. The cdf and pdf of the Clayton copula are defined, respectively, as follows:

$$C(u\,,v;\theta) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \qquad \theta > 0, \tag{2.1}$$

and

$$c(u\,,v;\theta) = (1+\theta)(uv)^{-\theta-1}(u^{-\theta} + v^{-\theta} - 1)^{-2-\frac{1}{\theta}}\grave{u}\grave{v}, \tag{2.2}$$

where θ is the measure of dependence and it is known as an association parameter, $u = F(t_1), v = F(t_2), \grave{u} = f(t_1)$ and $\grave{v} = f(t_2)$.
The measure of dependence can take on many different values depending on the copula, whereas measures of association (see Flores [8]).
To construct the BLO distribution, suppose that T_1 follows LO distribution with the following cdf and pdf, respectively,

$$F(t_1, \alpha_1) = 1 - (1 + t_1)^{-\alpha_1}, \qquad t_1 > 0, \alpha_1 > 0, \tag{2.3}$$

and

$$f(t_1, \alpha_1) = \alpha_1 (1 + t_1)^{-(\alpha_1+1)}, \quad t_1 > 0, \alpha_1 > 0. \tag{2.4}$$

Similarly, if T_2 follows LO distribution with the following cdf and pdf, respectively,

$$F(t_2, \alpha_2) = 1 - (1 + t_2)^{-\alpha_2}, \qquad t_2 > 0, \alpha_2 > 0, \tag{2.5}$$

and

$$f(t_2, \alpha_2) = \alpha_2 (1 + t_2)^{-(\alpha_2+1)}, \quad t_2 > 0, \alpha_2 > 0. \tag{2.6}$$

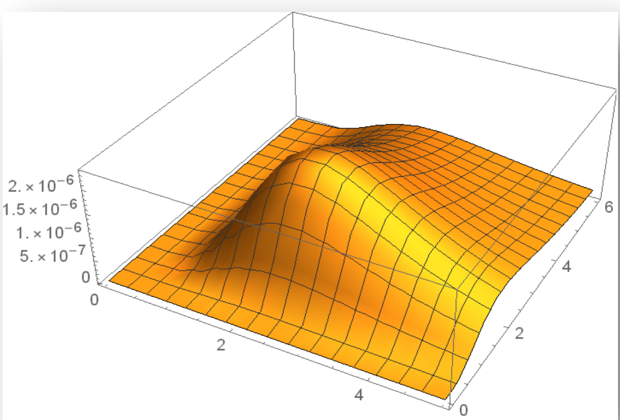
If $\theta = \alpha, u = F(t_1, \alpha_1)$ and $v = F(t_2, \alpha_2)$, then substituting (2.3) and (2.5) in (2.1), hence the joint cdf is

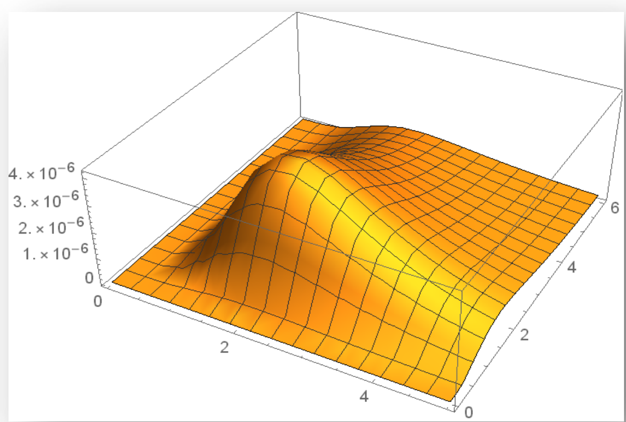
$$\begin{aligned} F(t_1, t_2, \underline{\omega}) &= C\big[[1 - (1 + t_1)^{-\alpha_1}], [1 - (1 + t_2)^{-\alpha_2}]\big] \\ &= \{[1 - (1 + t_1)^{-\alpha_1}]^{-\alpha} + [1 - (1 + t_2)^{-\alpha_2}]^{-\alpha} - 1\}^{-\frac{1}{\alpha}}, \\ &\underline{t} > \underline{0}, \quad \underline{\omega} > \underline{0}, \quad \underline{\omega} = (\alpha, \alpha_1, \alpha_2), \quad \underline{t} = (t_1, t_2) \end{aligned} \tag{2.7}$$

If $u = F(t_1, \alpha_1)$, $v = F(t_2, \alpha_2), \grave{u} = f(t_1, \alpha_1)$, $\grave{v} = f(t_2, \alpha_2)$, and substituting (2.4) and (2.6) in (2.2), then the joint pdf of BLO distribution is

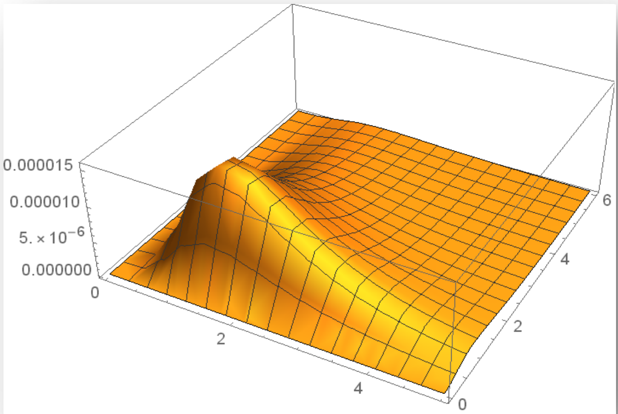
$$\begin{aligned} f(t_1, t_2, \underline{\omega}) &= (\alpha + 1)\alpha_1\alpha_2(1 + t_1)^{-(\alpha_1+1)}(1 + t_2)^{-(\alpha_2+1)}[1 - (1 + t_1)^{-\alpha_1}]^{-(\alpha+1)} \\ &\times [1 - (1 + t_2)^{-\alpha_2}]^{-(\alpha+1)}\{[1 - (1 + t_1)^{-\alpha_1}]^{-\alpha} + [1 - (1 + t_2)^{-\alpha_2}]^{-\alpha} - 1\}^{-\left(\frac{1+2\alpha}{\alpha}\right)}, \\ &(t_1, t_2) > 0, (\alpha, \alpha_1, \alpha_2) > 0. \end{aligned} \tag{2.8}$$

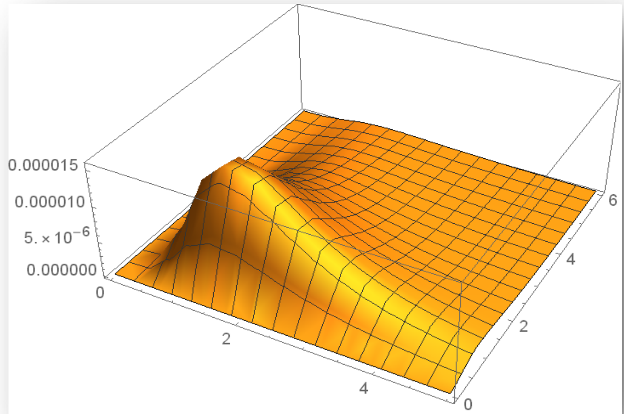
The contour plots of the joint pdf of the BLO distribution for different parameter values are presented in Figure 1





(1.a) (1.b)





(1.c) (1.d)

***Fig. 1**: The contour plots of the joint pdf of the BLO distribution for different parameter values:*
(1.a) $(\alpha = 1, \alpha_1 = 2, \alpha_2 = 3)$, *(1.b)* $(\alpha = 1, \alpha_1 = 2, \alpha_2 = 4)$, *(1.c)* $(\alpha = 1, \alpha_1 = 1, \alpha_2 = 1)$ *and (1.d)* $(\alpha = 1, \alpha_1 = 1, \alpha_2 = 4)$.

From Figure 1, one can observe that the joint pdf of BLO distribution can take different shapes depending on the values of its parameters.

2.2. The joint reliability and joint hazard rate functions

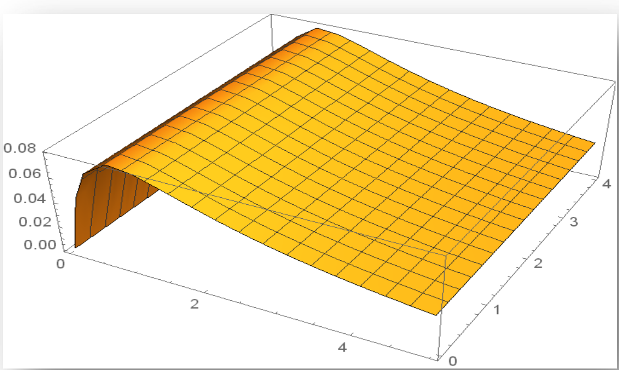
The joint *reliability function* (rf) of BLO distribution is given by

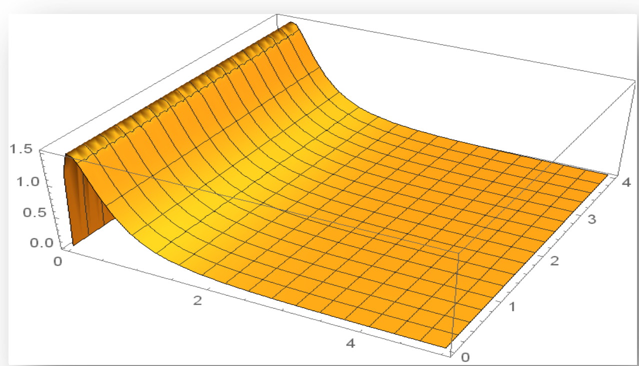
$$R(t_1, t_2, \underline{\omega}) = P(T_1 > t_1, T_2 > t_2)$$
$$= 1 - F(t_1) - F(t_2) + F(t_1, t_2)$$
$$= 1 - [1 - (1 + t_1)^{-\alpha_1}] - [1 - (1 + t_2)^{-\alpha_2}]$$
$$+ \{[1 - (1 + t_1)^{-\alpha_1}]^{-\alpha} + [1 - (1 + t_2)^{-\alpha_2}]^{-\alpha} - 1\}^{-\frac{1}{\alpha}}. \; (t_1, t_2) > 0, (\alpha, \alpha_1, \alpha_2) > 0. \quad (2.9)$$

The joint *hazard rate function* (hrf) is defined by

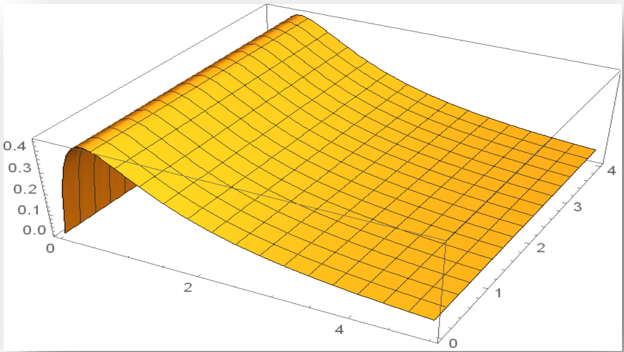
$$h(t_1, t_2) = \frac{f(t_1, t_2, \underline{\omega})}{R(t_1, t_2, \underline{\omega})}, \qquad (t_1, t_2) > 0, (\alpha, \alpha_1, \alpha_2) > 0 \; . \quad (2.10)$$

The contour plots for the joint hrf of the BLO distribution for different parameter values are presented in Figure 2.

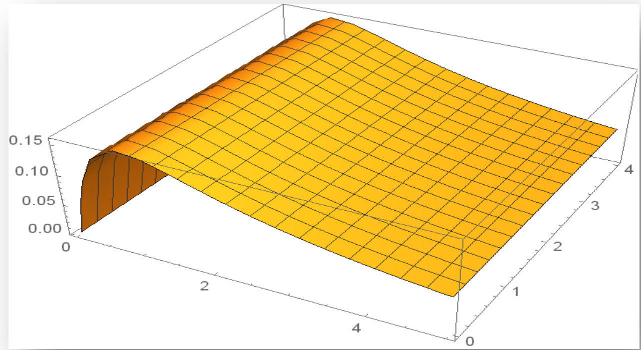




(2.a) (2.b)



(2.c)



(2.d)

Fig.2: *The contour plots of the joint hrf of the BLO distribution for different parameter values:*
(2.a) $(\alpha = 6, \alpha_1 = 3, \alpha_2 = 2.5)$, (2.b) $(\alpha = 2.5, \alpha_1 = 0.5, \alpha_2 = 1.5)$,
(2.c) $(\alpha = 0.2, \alpha_1 = 0.5, \alpha_2 = 3)$ and (2.d) $(\alpha = 0.8, \alpha_1 = 1, \alpha_2 = 3)$.

2.3. The conditional functions

The conditional pdf and conditional cdf of $T_1|T_2$ are given, respectively, by

$$f(t_1|t_2) = (\alpha+1)\alpha_1(1+t_1)^{-(\alpha_1+1)}[1-(1+t_1)^{-\alpha_1}]^{-(\alpha+1)}[1-(1+t_2)^{-\alpha_2}]^{-(\alpha+1)}$$
$$\times\{[1-(1+t_1)^{-\alpha_1}]^{-\alpha}+[1-(1+t_2)^{-\alpha_2}]^{-\alpha}-1\}^{-\left(\frac{1+2\alpha}{\alpha}\right)}, \quad (2.11)$$

and

$$F(t_1|t_2) = \int_0^{t_1} f(t_1|t_2)dt_1$$
$$= [1-(1+t_2)^{-\alpha_2}]^{-(\alpha+1)}\{[1-(1+t_1)^{-\alpha_1}]^{-\alpha}+[1-(1+t_2)^{-\alpha_2}]^{-\alpha}-1\}^{-\left(\frac{1+\alpha}{\alpha}\right)}, \quad (2.12)$$

where $f(t_1|t_2)$ is given by (2.11).

3. Maximum Likelihood Estimation and Prediction

In this section, the ML estimators of the vector of the parameters $\underline{\omega} = (\alpha, \alpha_1, \alpha_2)$, rf and hrf of the BLO distribution are derived. Also, the two-sample ML prediction is obtained for new observations from a future sample from the same distribution.

3.1. Maximum likelihood estimation

Using (2.8), the likelihood function can be written as

$$L(\underline{\omega}|\underline{t}) = \prod_{i=1}^{n} f(t_{1i}, t_{2i}),$$
$$= (\alpha+1)^n\alpha_1^n\alpha_2^n \prod_{i=1}^{n}(1+t_{1i})^{-(\alpha_1+1)}(1+t_{2i})^{-(\alpha_2+1)}[1-(1+t_{1i})^{-\alpha_1}]^{-(\alpha+1)}$$
$$\times[1-(1+t_{2i})^{-\alpha_2}]^{-(\alpha+1)}[[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+t_{2i})^{-\alpha_2}]^{-\alpha}-1]^{-\left(\frac{1+2\alpha}{\alpha}\right)}, \quad (3.1)$$

where $\underline{\omega} = (\alpha, \alpha_1, \alpha_2)$.

Hence, the log likelihood function is given by

$$\ell \equiv lnL(\underline{\omega}|\underline{t})$$
$$= nln(\alpha+1) + nln\alpha_1 + nln\alpha_2 - (\alpha_1+1)\sum_{i=1}^{n}\ln(1+t_{1i}) - (\alpha_2+1)\sum_{i=1}^{n}\ln(1+t_{2i})$$
$$-(\alpha+1)\sum_{i=1}^{n}\ln[1-(1+t_{1i})^{-\alpha_1}] - (\alpha+1)\sum_{i=1}^{n}\ln[1-(1+t_{2i})^{-\alpha_2}]$$

$$-\left(\frac{1+2\alpha}{\alpha}\right)\sum_{i=1}^{n}\ln\{[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+t_{2i})^{-\alpha_2}]^{-\alpha}-1\}. \tag{3.2}$$

The ML estimators of the parameters can be obtained by differentiating (3.2) with respect to α, α_1 and α_2 and setting to zeros then solving the resulting non-linear system of likelihood equations.

Hence

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{(\alpha+1)}-\sum_{i=1}^{n}\ln[1-(1+t_{1i})^{-\alpha_1}]-\sum_{i=1}^{n}\ln[1-(1+t_{2i})^{-\alpha_2}]+\left\{\left(\frac{1+2\alpha}{\alpha}\right)\right.$$
$$\left.\times\sum_{i=1}^{n}\ln\frac{[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}\ln[1-(1+t_{1i})^{-\alpha_1}]+[1-(1+t_{2i})^{-\alpha_1}]^{-\alpha}ln[1-(1+t_{2i})^{-\alpha_2}]}{[[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+t_{2i})^{-\alpha_2}]^{-\alpha}-1]}\right\}$$
$$+\left\{\sum_{i=1}^{n}\ln[[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+t_{2i})^{-\alpha_2}]^{-\alpha}-1]\right\}\left(\frac{1}{\alpha^2}\right), \tag{3.3}$$

$$\frac{\partial \ell}{\partial \alpha_1}=\frac{n}{\alpha_1}-\sum_{i=1}^{n}\ln(1+t_{1i})-(\alpha+1)\sum_{i=1}^{n}\frac{\ln(1+t_{1i})}{(1+t_{1i})^{\alpha_1}-1}$$
$$+\left(\frac{1+2\alpha}{\alpha}\right)\sum_{i=1}^{n}\frac{\alpha[1-(1+t_{1i})^{-\alpha_1}]^{-(\alpha+1)}(1+t_{1i})^{-\alpha_1}\ln(1+t_{1i})}{[[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+t_{2i})^{-\alpha_2}]^{-\alpha}-1]}, \tag{3.4}$$

and

$$\frac{\partial \ell}{\partial \alpha_2}=\frac{n}{\alpha_2}-\sum_{i=1}^{n}\ln(1+t_{2i})-(\alpha+1)\sum_{i=1}^{n}\frac{\ln(1+t_{2i})}{(1+t_{2i})^{\alpha_2}-1}$$
$$+\left(\frac{1+2\alpha}{\alpha}\right)\sum_{i=1}^{n}\frac{\alpha[1-(1+t_{2i})^{-\alpha_2}]^{-(\alpha+1)}(1+t_{2i})^{-\alpha_2}\ln(1+t_{2i})}{[[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+t_{2i})^{-\alpha_2}]^{-\alpha}-1]}. \tag{3.5}$$

The invariance property of the ML estimators can be applied to obtain the ML estimators of the rf and hrf by replacing the parameters by their ML estimators in (2.9) and (2.10).

Then

$$\hat{R}(t_1,t_2)=1-\left[1-(1+t_1)^{-\hat{\alpha}_1}\right]-\left[1-(1+t_2)^{-\hat{\alpha}_2}\right]$$
$$+\left\{\left[1-(1+t_1)^{-\hat{\alpha}_1}\right]^{-\hat{\alpha}}+\left[1-(1+t_2)^{-\hat{\alpha}_2}\right]^{-\hat{\alpha}}-1\right\}^{-\frac{1}{\hat{\alpha}}}, \tag{3.6}$$

and

$$\hat{h}(t_1,t_2)=\frac{f\left(t_1,t_2,\underline{\hat{\omega}}\right)}{R\left(t_1,t_2,\underline{\hat{\omega}}\right)}, \tag{3.7}$$

$\hat{R}(t_1,t_2)$ and $\hat{h}(t_1,t_2)$ can be evaluated numerically. The $(1-\delta)100\%$ *confidence intervals* (CIs) of the parameters $\underline{\omega}=(\alpha,\alpha_1,\alpha_2)$, rf and hrf can be derived by using variance covariance matrix as follows:

$$\hat{\omega}_i \pm Z_{1-\frac{\delta}{2}}\sqrt{\widehat{var}(\hat{\omega}_i)}, \qquad i=1,2,3,$$

where $Z_{1-\frac{\delta}{2}}$ is the standard normal variate and δ is the confidence coefficient.

3.2. Maximum likelihood prediction

In two-sample prediction, the two samples are assumed to be independent and drawn from the same distribution. In bivariate Type II censoring where the first variable in the vector is the ordered observation and the second variable is its concomitants, therefore the joint pdf of ordered observations and the concomitants is needed to obtain the joint predictive density function of future ordered observations and their concomitants.

For a future bivariate censored sample of size m, the joint pdf of the future *s-th* ordered observation and its *s-th* concomitant denoted by $\left(y_{1(s:m)}, y_{2(s:m)}\right), s=1,2,\ldots,m$, has the joint pdf which is given by (2.8) after replacing t_1 by $y_{1(s:m)}$ and t_2 by $y_{2(s:m)}$. For simplicity, it can be written as $\left(y_{1(s)}, y_{2(s)}\right)$ for $\left(y_{1(s:m)}, y_{2(s:m)}\right)$ (for more details, see AL-Hossaini [1], Singh et al. [19] and [20]). Then the joint pdf of $\left(y_{1(s)}, y_{2(s)}\right)$ is given by

$$f_{s:m}\left(y_{1(s)}, y_{2(s)};\underline{\omega}\right)=$$
$$\frac{m!}{(s-1)!(m-s)!}f\left(y_{1(s)}, y_{2(s)};\underline{\omega}\right)\left[F\left(y_{1(s)}, y_{2(s)}\right)\right]^{s-1}\left[1-F\left(y_{1(s)}, y_{2(s)}\right)\right]^{m-s}.$$

Using the binomial expansion to simplify the last term in the previous equation, one gets

$$\left[1-F\left(y_{1(s)},y_{2(s)}\right)\right]^{m-s}=\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\left[F\left(y_{1(s)},y_{2(s)}\right)\right]^j$$

Thus, the joint pdf of $\left(y_{1(s)},y_{2(s)}\right)$ is as follows:

$$f_{s:m}\left(y_{1(s)},y_{2(s)};\underline{\omega}\right)$$

$$=\frac{m!}{(s-1)!\,(m-s)!}f\left(y_{1(s)},y_{2(s)};\underline{\omega}\right)\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\left[F\left(y_{1(s)},y_{2(s)}\right)\right]^{s+j-1},$$

$$=f\left(y_{1(s)},y_{2(s)};\underline{\omega}\right)\sum_{j=0}^{m-s}\frac{m!}{(s-1)!\,(m-s-j)!\,(j)!}(-1)^j\left[F\left(y_{1(s)},y_{2(s)}\right)\right]^{s+j-1},$$

$$=f\left(y_{1(s)},y_{2(s)};\underline{\omega}\right)\sum_{j=0}^{m-s}C_{m,s,j}\left[F\left(y_{1(s)},y_{2(s)}\right)\right]^{s+j-1}, \tag{3.8}$$

where

$$C_{m,s,j}=\frac{m!}{(s-1)!\,(m-s-j)!\,(j)!}(-1)^j\,. \tag{3.9}$$

Substituting $F(t_1,t_2)$ and $f(t_1,t_2)$ given, respectively, in (2.7) and (2.8) after replacing t_1 by $y_{1(s)}$ and t_2 by $y_{2(s)}$, then the joint ML predictive density of ordered observations and their concomitants is given by

$$f_{s:m}\left(y_{1(s)},y_{2(s)};\underline{\hat{\omega}}\right)=$$

$$(\hat{\alpha}+1)\hat{\alpha}_1\hat{\alpha}_2\left(1+y_{1(s)}\right)^{-(\hat{\alpha}_1+1)}\left(1+y_{2(s)}\right)^{-(\hat{\alpha}_2+1)}\left[1-\left(1+y_{1(s)}\right)^{-\hat{\alpha}_1}\right]^{-(\hat{\alpha}+1)}$$

$$\times\left[1-\left(1+y_{2(s)}\right)^{-\hat{\alpha}_2}\right]^{-(\hat{\alpha}+1)}\left\{\left[1-\left(1+y_{1(s)}\right)^{-\hat{\alpha}_1}\right]^{-\hat{\alpha}}+\left[1-\left(1+y_{2(s)}\right)^{-\hat{\alpha}_2}\right]^{-\hat{\alpha}}-1\right\}^{-\left(\frac{1+2\hat{\alpha}}{\hat{\alpha}}\right)}$$

$$\times\sum_{j=0}^{m-s}C_{m,s,j}\left\{\left[1-\left(1+y_{1(s)}\right)^{-\hat{\alpha}_1}\right]^{-\hat{\alpha}}+\left[1-\left(1+y_{2(s)}\right)^{-\hat{\alpha}_2}\right]^{-\hat{\alpha}}-1\right\}^{-\left(\frac{s+j-1}{\hat{\alpha}}\right)}.$$

$$\left(y_{1(s)},y_{2(s)}\right)>0,(\hat{\alpha},\hat{\alpha}_1,\hat{\alpha}_2)>0, \tag{3.10}$$

The point predictors of future ordered observation and their concomitants $\left(Y_{1(s)},Y_{2(s)}\right)$, $s=1,2,\ldots,m$, can be obtained as given below

$$\hat{Y}_1=E\left(y_{1(s)};\underline{\hat{\omega}}_{ML}\right)=\int_{y_{1(s)}=0}^{\infty}y_{1(s)}\int_{y_{2(s)}}^{\infty}f\left(y_{1(s)},y_{2(s)};\underline{\hat{\omega}}_{ML}\right)dy_{2(s)}dy_{1(s)}, \tag{3.11}$$

and

$$\hat{Y}_2=E\left(y_{2(s)};\underline{\hat{\omega}}_{ML}\right)=\int_{y_{2(s)}=0}^{\infty}y_{2(s)}\int_{y_{1(s)}}^{\infty}f\left(y_{1(s)},y_{2(s)};\underline{\hat{\omega}}_{ML}\right)dy_{1(s)}dy_{2(s)}. \tag{3.12}$$

The joint point predictors of the future ordered observation is

$$\hat{Y}_1,\hat{Y}_2=E\left(y_{1(s)},y_{2(s)};\underline{\hat{\omega}}_{ML}\right)=\int_0^{\infty}\int_0^{\infty}y_{1(s)}y_{2(s)}f\left(y_{1(s)},y_{2(s)};\underline{\hat{\omega}}_{ML}\right)dy_{2(s)}dy_{1(s)}, \tag{3.13}$$

which can be evaluated numerically.

4. Numerical Illustration for the Maximum Likelihood Results

This section aims to investigate the precision of the theoretical results of ML estimation and prediction based on simulated data and real data set.

4.1. Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on generated data from the BLO distribution. The ML averages of the parameters, rf and hrf are computed. Moreover, *confidence intervals* (CIs) for the parameters, rf and hrf are calculated. Simulation studies are performed using Mathematica 11. Nelsen [16] discussed generating a sample from a specified joint distribution. By conditional distribution method, the joint distribution function is as follows:

$$f(t_1,t_2)=f(t_1)f(t_2|t_1).$$

The steps of the simulation procedure are given below

a) For given values of $\underline{\omega}$ (where $\underline{\omega}=(\alpha,\alpha_1,\alpha_2)$), random samples of size *n* are generated from the BLO distribution.
b) For each sample size, sort the t_{ij}'s, such that $(t_{11},t_{21}),(t_{12},t_{22}),\ldots,(t_{1n},t_{2n})$.

c) Repeat the previous steps N times where N represents a fixed number of simulated samples.
d) Newton-Raphson method can be applied to obtain the ML averages and the *Confidence intervals* (CIs) of the parameters. Also, the rf, hrf and their CIs are calculated using the ML averages of the parameters.
e) Evaluating the performance of the estimates is considered through some measurements of accuracy. To study the precision and variation of the estimates, it is convenient to use the *estimated risk* $(ER) = \frac{\sum_{i=1}^{N}(\text{estimate}-\text{true value})^2}{\text{N}}$.
f) Simulation results of averages of the ML estimates are displayed in Tables 2 and 3, where N = 10000 is the number of repetitions and samples of size (n=30, 50, 100) and the population parameter values are $(\alpha = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3)$ and $(\alpha = 0.1, \alpha_1 = 0.3, \alpha_2 = 0.4)$.
g) Tables 2 and 3 present the ML averages, ERs and CIs of the unknown parameters. While Tables 4 and 5 display the ML averages, ERs and CIs of the rf and hrf for different values of time t_{01}, t_{02}. The ML two-sample predictors are presented in Table 9.

4.2. Application of real data

American Football Data Set: This data set is obtained from the American Football *National Football League* (NFL) League from the matches on three consecutive weekends in (1986). The data was also available in Csorgo and Welsch [5]. They converted the seconds of the data to decimal points. The bivariate data set (T_1, T_2) is presented in Table 1; where the variable T_1 represents the game time to the first point scored by kicking the ball between goal posts and T_2 represents the game time by moving the ball into the end zone.

Table 1: American Football League NFL data

T_1	T_2	T_1	T_2	T_1	T_2
02.05	03.98	05.78	25.98	10.40	14.25
09.05	09.05	13.80	49.75	02.98	02.98
00.85	00.85	07.25	07.25	03.88	06.43
03.43	03.43	04.25	04.25	00.75	00.75
07.78	07.78	01.65	01.65	11.63	17.37
10.57	14.28	06.42	15.08	01.38	01.38
07.05	07.05	04.22	09.48	10.35	10.35
02.58	02.58	15.53	15.53	12.13	12.13
07.23	09.68	02.90	02.90	14.58	14.58
06.85	34.58	07.02	07.02	11.82	11.82
32.45	42.35	06.42	06.42	05.52	1127
08.53	14.57	08.98	08.98	19.65	10.70
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	08.87	08.87	10.85	38.07

Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively, by 0.065 and 0.064. The p values showed that the model fits the data very well.

Table 6 displays the ML estimates ERs of the unknown parameters for the data set. While Tables 7 and 8 present the ML estimates and *standard errors* (Se) of the rf and hrf for different values of time t_{01}, t_{02} in two states $(t_{01} = 2, t_{02} = 3), (t_{01} = 3, t_{02} = 4)$ respectively. Table 10 provides the ML two-sample predictors for the future sample.

4.3 Concluding remarks

1. It is noticed, from Tables 2 and 3 that the ML averages are very close to the population parameter values as the sample size increases. Also, ERs decrease when the sample size increases as expected, which indicates that the estimates are consistent and approaches the parameter values as the sample size increases.
2. The lengths of the CIs of the parameters become narrower as the sample size increases.
3. The ML averages are very close to the initial values of the rf and hrf as the sample size increases. Also, ERs decrease when the sample size increases.
4. The length of the first future order statistic is smaller than the length of the last future order statistic (Tables 9 and 10).
5. The ML interval includes the estimates (between the *Lower limit* (LL) and *Upper Limit* (UL)).

Table 2: ML averages, estimated risks and 95% confidence intervals
for the parameters ($N = 10000, \alpha = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3$)

n	parameters	Averages	ER	UL	LL	Length
30	α	0.6899	0.5428	1.4855	0.0000	1.4855
	α_1	0.1899	0.0011	0.2507	0.1290	0.1217
	α_2	0.2848	0.0024	0.3761	0.1935	0.1826
50	α	0.7126	0.4686	1.3113	0.1139	1.1973
	α_1	0.1919	0.0006	0.2388	0.1449	0.0938
	α_2	0.2878	0.0014	0.3588	0.2174	0.1408
100	α	0.6393	0.3050	0.8723	0.4063	0.4660
	α_1	0.1908	0.0004	0.2280	0.1536	0.0744
	α_2	0.2862	0.0010	0.3420	0.2304	0.1117

Table 3: ML averages, estimated risks and 95% confidence intervals
for the parameters ($N = 10000, \alpha = 0.1, \alpha_1 = 0.3, \alpha_2 = 0.4$)

n	parameters	Averages	ER	UL	LL	Length
30	α	0.7335	0.6188	1.6474	0.0000	1.6474
	α_1	0.2667	0.0040	0.3728	0.1605	0.2124
	α_2	0.3555	0.0072	0.4971	0.2139	0.2832
50	α	0.6842	0.4614	1.3637	0.0047	1.3590
	α_1	0.2945	0.0011	0.3569	0.02319	0.1249
	α_2	0.3926	0.0019	0.3569	0.3093	0.1667
100	α	0.5899	0.2587	0.8586	0.3212	0.5374
	α_1	0.2736	0.0015	0.3281	0.2190	0.1092
	α_2	0.3648	0.0026	0.4375	0.2920	0.1455

Table 4: ML averages, estimated risks and 95% confidence
intervals of the reliability and hazard rate functions
($N = 10000, \alpha = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3, t_{01} = 2, t_{02} = 3$)

n	rf and hrf	Averages	ER	UL	LL	Length
30	$R(t_{01}, t_{02})$	0.5947	0.0013	0.7317	0.4578	0.2738
	$h(t_{01}, t_{02})$	0.0026	0.0350	0.0037	0.0015	0.0022
50	$R(t_{01}, t_{02})$	0.5412	0.0004	0.6319	0.4507	0.1811
	$h(t_{01}, t_{02})$	0.0028	0.0289	0.0040	0.0020	0.0020
100	$R(t_{01}, t_{02})$	0.5357	0.0003	0.6214	0.4499	0.1715
	$h(t_{01}, t_{02})$	0.0029	0.0284	0.0039	0.0019	0.0019

Table 5: ML averages, estimated risks and 95% confidence intervals of the reliability and hazard rate functions

n	rf and hrf	Averages	ER	UL	LL	Length
30	$R(t_{01}, t_{02})$	0.6168	0.0006	0.7211	0.5126	0.2085
	$h(t_{01}, t_{02})$	0.0047	0.0374	0.0066	0.0029	0.0037
50	$R(t_{01}, t_{02})$	0.6098	0.0005	0.7036	0.5161	0.1875
	$h(t_{01}, t_{02})$	0.0052	0.0366	0.0066	0.0038	0.0028
100	$R(t_{01}, t_{02})$	0.5875	0.0002	0.6979	0.4769	0.2209
	$h(t_{01}, t_{02})$	0.0054	0.0339	0.0075	0.0033	0.0042

$(\mathrm{N} = 10000, \alpha = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3, t_{01} = 3, t_{02} = 4)$

Table 6: ML estimates and standard errors of the parameters for the data set $(\alpha = 1.1, \alpha_1 = 1.5, \alpha_2 = 2.2)$

n	Parameters	Estimates	SE
42	α	1.1621	0.0039
	α_1	1.2659	0.0548
	α_2	1.4091	0.0255

Table 7: ML estimates and standard errors for the reliability and hazard rate functions for the real data set $(\alpha = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3, t_{01} = 2, t_{02} = 3)$

n	rf and hrf	Estimates	SE
42	$R(t_{01}, t_{02})$	0.2009	0.0013
	$h(t_{01}, t_{02})$	0.0319	0.0272

Table 8: ML estimates and standard errors of the reliability and hazard rate functions for the real data set $(\alpha = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3, t_{01} = 3, t_{02} = 4)$

n	rf and hrf	Estimates	SE
42	$R(t_{01}, t_{02})$	0.1466	0.0009
	$h(t_{01}, t_{02})$	0.0194	0.0155

Table 9: ML predictive and bounds of the future observation under two-sample prediction $(\mathrm{N} = 10000, \alpha = 1.1, \alpha_1 = 1.5, \alpha_2 = 2.2)$

n	s	$\hat{y}_{(s)}$	Predictor	UL	LL	Length
30	1	$\hat{y}_{1(s)}$	0.0821	0.2873	0.0140	0.2733
		$\hat{y}_{2(s)}$	0.0532	0.1878	0.0044	0.1834
	2	$\hat{y}_{1(s)}$	0.1681	0.4837	0.0463	0.4374
		$\hat{y}_{2(s)}$	0.1068	0.3087	0.0021	0.3066
	3	$\hat{y}_{1(s)}$	0.3332	0.5289	0.0964	0.4325
		$\hat{y}_{2(s)}$	0.1956	0.3641	0.0651	0.2990

Table 10: ML predictive and bounds of the future observation under two-sample prediction for the real data set $(\alpha = 1.1, \alpha_1 = 1.5, \alpha_2 = 2.2)$

n	s	$\hat{y}_{(s)}$	predictor	UL	LL	Length
42	1	$\hat{y}_{1(s)}$	0.4691	0.6131	0.2402	0.3729
		$\hat{y}_{2(s)}$	0.3533	0.3550	0.0025	0.3525
	2	$\hat{y}_{1(s)}$	0.9254	0.9271	0.9226	0.0045
		$\hat{y}_{2(s)}$	0.6844	0.6867	0.6803	0.0064
	3	$\hat{y}_{1(s)}$	1.3699	1.6225	0.7307	0.8918
		$\hat{y}_{2(s)}$	0.9989	1.1429	0.4899	0.6530

5. Bayesian Estimation and Prediction

In this section Bayesian estimation and two-sample prediction of the vector of parameters $\underline{\omega} = (\alpha, \alpha_1, \alpha_2)$ for the BLO distribution will be introduced.

5.1. Bayesian estimation

Assuming that the parameters are independent, and each one has an informative prior. The joint prior distribution of the three unknown parameters is

$$\pi(\underline{\omega}) = \prod_{\iota=1}^{3} \pi(\omega_\iota), \iota = 1,2,3, \tag{5.1}$$

where $\omega_\iota \sim$ gamma (a_ι, b_ι), where a_ι, b_ι are the hyper-parameters of the prior distribution, a_ι is the shape parameter and b_ι is the scale parameter. Hence, ω_ι has the following pdf

$$\pi(\omega_\iota) \propto \omega_\iota^{a_\iota - 1} e^{-\frac{\omega_\iota}{b_\iota}}, \omega_\iota > 0, (a_\iota, b_\iota) > 0, \tag{5.2}$$

where $\omega_1 = \alpha,\ \omega_2 = \alpha_1,\ \omega_3 = \alpha_2$.

Then, the joint prior distribution of $\underline{\omega}$ is

$$\pi(\underline{\omega}) \propto \alpha^{a_1 - 1} \alpha_1^{a_2 - 1} \alpha_2^{a_3 - 1} e^{-\left(\frac{\alpha}{b_1} + \frac{\alpha_1}{b_2} + \frac{\alpha_2}{b_3}\right)}, (a_1, a_2, a_3) > 0, (b_1, b_2, b_3) > 0. \tag{5.3}$$

The joint posterior density of $\underline{\omega}$ is given by

$$\pi(\underline{\omega}|t_1, t_2) \propto \pi(\underline{\omega}) L(\underline{\omega}|t_1, t_2). \tag{5.4}$$

Using (3.1) and (5.3), one can obtain the joint posterior density

$$\pi(\underline{\omega}|t_1, t_2) \propto (\alpha + 1)^n \alpha^{a_1 - 1} \alpha_1^{n + a_2 - 1} \alpha_2^{n + a_3 - 1} e^{-\left(\frac{\alpha}{b_1} + \frac{\alpha_1}{b_2} + \frac{\alpha_2}{b_3}\right)} e^{-\sum_{i=1}^{n} (\alpha_1+1)\ln(1+t_{1i}) + (\alpha_2+1)\ln(1+t_{2i})}$$
$$\times e^{-\sum_{i=1}^{n} (\alpha+1)\ln[1-(1+t_{1i})^{-\alpha_1}] + (\alpha+1)\ln[1-(1+t_{2i})^{-\alpha_2}] + \left(\frac{1+2\alpha}{\alpha}\right)\ln[[1-(1+t_{1i})^{-\alpha_1}]^{-\alpha} + [1-(1+t_{2i})^{-\alpha_2}]^{-\alpha} - 1]}. \tag{5.5}$$

The Bayes estimator for the rf and hrf under *squared error loss* (SEL) function can be derived using (2.9), (2.10) and (5.5), respectively, as given below

$$R_{SE}^{*}(t_1, t_2) = E\big(R(t_1, t_2)|\underline{\omega}\big) = \int_{\underline{\omega}} R(t_1, t_2)\, \pi(\underline{\omega}|t_1, t_2) d\underline{\omega}, \tag{5.6}$$

and

$$h_{SE}^{*}(t_1, t_2) = E\big(h(t_1, t_2)|\underline{\omega}\big) = \int_{\underline{\omega}} h(t_1, t_2)\, \pi(\underline{\omega}|t_1, t_2) d\underline{\omega}. \tag{5.7}$$

Equations (5.6) and (5.7) can be calculated numerically to obtain the Bayes estimators of the rf and hrf based on SEL function.

5.2. Bayesian prediction

In this subsection, Bayesian two-sample prediction for future ordered observations and their concomitants is considered; based on BLO distribution for future bivariate samples of size m. For given parameters the joint pdf of ordered observations and their concomitants is given by (3.8).

From (2.7) and (2.8) after replacing t_1, t_2 by $y_{1(s)}, y_{2(s)}$, one gets

$$f_{s:m}(y_{1(s)}, y_{2(s)}; \underline{\omega}) = (\alpha + 1)\alpha_1\alpha_2 (1 + y_{1(s)})^{-(\alpha_1+1)} (1 + y_{2(s)})^{-(\alpha_2+1)} [1 - (1 + y_{1(s)})^{-\alpha_1}]^{-(\alpha+1)}$$
$$\times [1 - (1 + y_{2(s)})^{-\alpha_2}]^{-(\alpha+1)} \left\{[1 - (1 + y_{1(s)})^{-\alpha_1}]^{-\alpha} + [1 - (1 + y_{2(s)})^{-\alpha_2}]^{-\hat{\alpha}} - 1\right\}^{-\left(\frac{1+2\alpha}{\alpha}\right)}$$

$$\times \sum_{j=0}^{m-s} C_{m,s,j} \left\{\left[1-\left(1+y_{1(s)}\right)^{-\alpha_1}\right]^{-\alpha} + \left[1-\left(1+y_{2(s)}\right)^{-\alpha_2}\right]^{-\alpha} - 1\right\}^{-\left(\frac{s+j-1}{\alpha}\right)},$$

$$\left(y_{1(s)}, y_{2(s)}\right) > 0, (\alpha, \alpha_1, \alpha_2) > 0. \quad (5.8)$$

The joint *Bayesian predictive density* (BPD) of the future-*th* ordered observation and its *s-th* concomitant denoted by

$$h\left(y_{1(s)}, y_{2(s)} | \underline{\omega}\right) = \int_{\underline{\omega}} f\left(y_{1(s)}, y_{2(s)} : \underline{\omega}\right) \pi\left(\underline{\omega} | y_1, y_2\right) d\underline{\omega}, \quad (5.9)$$

where

$$\int_{\underline{\omega}} = \int_{\alpha} \int_{\alpha_1} \int_{\alpha_2} \text{ and } \quad d\underline{\omega} = d\alpha \, d\alpha_1 d\alpha_2. \quad (5.10)$$

Substituting (5.5) and (5.8) in (5.9), yields the joint BPD of $\left(y_{1(s)}, y_{2(s)}\right)$ as

$$h\left(y_{1(s)}, y_{2(s)} | \underline{\omega}\right) = \int_{\underline{\omega}} I_1 I_2 I_3 I_4 I_5 I_6 d\underline{\omega}, \quad (5.11)$$

where

$I_1 = (\alpha+1)^{n+1} \alpha^{a_1-1} \alpha_1^{n+a_2} \alpha_2^{n+a_3}$,

$I_2 = e^{-\left(\frac{\alpha}{b_1}+\frac{\alpha_1}{b_2}+\frac{\alpha_2}{b_3}\right)} \left(1+y_{1(s)}\right)^{-(\alpha_1+1)} \left(1+y_{2(s)}\right)^{-(\alpha_2+1)} e^{-\sum_{i=1}^{n}[(\alpha_1+1)ln(1+y_{1i})+(\alpha_2+1)ln(1+y_{2i})]}$,

$I_3 = \left[1-\left(1+y_{1(s)}\right)^{-\alpha_1}\right]^{-(\alpha+1)} \left[1-\left(1+y_{2(s)}\right)^{-\alpha_2}\right]^{-(\alpha+1)}$,

$I_4 = \left\{\left[1-\left(1+y_{1(s)}\right)^{-\alpha_1}\right]^{-\alpha} + \left[1-\left(1+y_{2(s)}\right)^{-\alpha_2}\right]^{-\alpha} - 1\right\}^{-\left(\frac{1+2\alpha}{\alpha}\right)}$,

$I_5 = e^{-\sum_{i=1}^{n}(\alpha+1)\ln\{[1-(1+y_{1i})^{-\alpha_1}]+(\alpha+1)\ln[1-(1+y_{2i})^{-\alpha_2}]\}+\left(\frac{1+2\alpha}{\alpha}\right)\ln\{[1-(1+y_{1i})^{-\alpha_1}]^{-\alpha}+[1-(1+y_{2i})^{-\alpha_2}]^{-\alpha}-1\}}$,

and

$$I_6 = \sum_{j=0}^{m-s} C_{m,s,j} \left\{\left[1-\left(1+y_{1(s)}\right)^{-\alpha_1}\right]^{-\alpha} + \left[1-\left(1+y_{2(s)}\right)^{-\alpha_2}\right]^{-\alpha} - 1\right\}^{-\left(\frac{s+j-1}{\alpha}\right)} d\underline{\omega}. \quad (5.12)$$

The point predictors of future ordered observation and their concomitants $\left(Y_{1(s)}, Y_{2(s)}\right)$, $s = 1,2,\dots,m$, under SEL function can be obtained as follows:

$$Y_1^* = E\left(y_{1(s)} | \underline{\omega}\right) = \int_{y_{1(s)}=0}^{\infty} y_{1(s)} \int_{y_{2(s)}}^{\infty} h\left(y_{1(s)}, y_{2(s)} | \underline{\omega}\right) dy_{2(s)} dy_{1(s)}, \quad (5.13)$$

and

$$Y_2^* = E\left(y_{2(s)} | \underline{\omega}\right) = \int_{y_{2(s)}=0}^{\infty} y_{2(s)} \int_{y_{1(s)}}^{\infty} h\left(y_{1(s)}, y_{2(s)} | \underline{\omega}\right) dy_{1(s)} dy_{2(s)}. \quad (5.14)$$

The joint Bayes point predictors for the future ordered observation is

$$Y_1^*, Y_2^* = E\left(y_{1(s)}, y_{2(s)} | \underline{\omega}\right) = \int_0^{\infty} \int_0^{\infty} y_{1(s)} y_{2(s)} h\left(y_{1(s)}, y_{2(s)} | \underline{\omega}\right) dy_{1(s)} dy_{2(s)}. \quad (5.15)$$

6. Numerical Illustration for the Bayesian Results

This section aims to examine the precision of the theoretical results of Bayesian estimation and prediction based on simulated data and a data set.

6.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates based on generated data from the BLO distribution. Bayes averages of the parameters, rf and hrf are computed. Moreover, credible intervals of the parameters, rf and hrf are evaluated. Bayes point predictors for a future observation from the BLO distribution are computed, for the two-sample case. All simulation studies are performed using R programming language.

Simulation algorithm

1. Several data sets are generated from the BLO distribution for a combination of the population parameter values of $\underline{\omega} = (\alpha, \alpha_1, \alpha_2)$.
2. Also, for samples of size (30, 50 and 100) using N=10000 replications for each sample size.
3. The population parameter values of α, α_1 and α_2 used in this simulation study are

$(\alpha = 0.3, \alpha_1 = 0.8, \alpha_2 = 0.9)$ and $(\alpha = 7.4, \alpha_1 = 5.1, \alpha_2 = 4.5)$.

4. If ω_j^* is an estimate of ω; based on sample $j,\ j = 1,2,\dots,N$, then the average estimates over the samples are calculated by $\overline{\omega_j^*} = \frac{1}{N}\sum_{j=1}^{N}\omega_j^*$.
5. The ERs of ω^*, over the N samples are computed.

- Tables 11 and 12 display the Bayes averages, RABs, ERs and credible intervals based on samples of size n and N=10000 repetitions with informative prior.
- Tables 13 and 14 present the Bayes averages, ER and credible intervals of rf and hrf for different values of the time t_{01}, t_{02} based on informative priors.
- The averages of the two-sample Bayes predictors using informative priors are presented in Table 17.

6.2. Real data set

In this subsection, the real data set considered was given in Subsection 4.2 and it was analyzed to illustrate the theoretical results of applying the Bayesian approach.

- Tables 15 and 16 display the Bayes estimates of the parameters, rf and hrf, for the real data under informative prior, also the standard errors are calculated.
- Table 18 presents two-sample Bayes predictors.

6.3. Concluding remarks

1. The ERs of the estimates are inversely proportional to the sample size.
2. The lengths of the CIs for the parameters become narrower as the sample size increases.
3. The Bayes averages are very close to the initial values of the rf and hrf as the sample size increases. Also, ERs decrease when the sample size increases.
4. The results become better as the informative sample size gets larger.

Table 11: Bayes averages, estimated risks and 95% credible intervals for the parameters of BLO $(\mathrm{N} = 10000, \alpha = 0.3, \alpha_1 = 0.8, \alpha_2 = 0.9)$

n	parameters	Averages	ER	UL	LL	Length
30	α	0.3991	1.1087e-06	0.3998	0.3978	0.0019
	α_1	0.8001	1.8623e-06	0.8024	0.7985	0.0039
	α_2	0.9026	8.0086e-06	0.9041	0.8998	0.0044
50	α	0.3995	4.6042e-07	0.4005	0.3983	0.0018
	α_1	0.7992	1.3514e-06	0.8009	0.7974	0.0035
	α_2	0.8999	4.1675e-07	0.9009	0.8986	0.0023
100	α	0.3998	1.2963e-07	0.4003	0.3990	0.0013
	α_1	0.7994	4.4998e-07	0.80002	0.7986	0.0014
	α_2	0.8996	3.4311e-07	0.9006	0.8987	0.0019

Table 12: Bayes averages, estimated risks and95% credible intervals for the parameters $(\mathrm{N} = 10000, \alpha = 7.4, \alpha_1 = 5.1, \alpha_2 = 4.5)$

n	parameters	Averages	ER	UL	LL	Length
30	α	7.3992	8.6361e-07	7.4002	7.3981	0.0021
	α_1	5.0976	8.8596e-06	5.1002	5.0947	0.0055
	α_2	4.5020	6.3005e-06	4.4995	4.4995	0.0049
50	α	7.3992	8.2490e-07	7.3999	7.3981	0.0019
	α_1	5.1014	2.5689e-06	5.1023	5.0996	0.0027
	α_2	4.4981	5.6307e-06	4.4999	4.4953	0.0046
100	α	7.4001	5.3946e-07	7.4016	7.3986	0.0029
	α_1	5.0989	1.3778e-06	5.0996	5.0979	0.0017
	α_2	4.0499	1.3472e-06	4.4997	4.4973	0.0024

Table 13: Bayes averages, estimated risks and 95% credible intervals for the reliability and hazard rate functions
$(N = 10000, \alpha = 0.3, \alpha_1 = 0.8, \alpha_2 = 0.9, t_{01} = 3, t_{02} = 2)$

n	rf and hrf	Averages	ER	UL	LL	Length
30	$R(t_{01}, t_{02})$	0.1426	3.0099e-06	0.1442	0.1406	0.0036
	$h(t_{01}, t_{02})$	0.0546	8.9195e-07	0.0552	0.0538	0.0014
50	$R(t_{01}, t_{02})$	0.1433	1.9819e-06	0.1449	0.1406	0.0043
	$h(t_{01}, t_{02})$	0.0549	5.3266e-07	0.0557	0.0535	0.0022
100	$R(t_{01}, t_{02})$	0.1444	1.8552e-07	0.1449	0.1436	0.0013
	$h(t_{01}, t_{02})$	0.0559	4.8041e-07	0.0569	0.0548	0.0021

Table 14: Bayes averages, estimated risks and 95% credible intervals for the reliability and hazard rate functions of
$(N = 10000, \alpha = 1.6, \alpha_1 = 1.0, \alpha_2 = 1.8, t_{01} = 2, t_{02} = 4)$

n	rf and hrf	Averages	ER	UL	LL	Length
30	$R(t_{01}, t_{02})$	0.0328	7.5963e-06	0.0351	0.0310	0.0040
	$h(t_{01}, t_{02})$	0.0852	3.9061e-06	0.0871	0.0831	0.0041
50	$R(t_{01}, t_{02})$	0.0357	3.6245e-07	0.0366	0.0344	0.0021
	$h(t_{01}, t_{02})$	0.0871	1.1231e-06	0.0889	0.0856	0.0034
100	$R(t_{01}, t_{02})$	0.0354	3.1949e-07	0.0366	0.0343	0.0022
	$h(t_{01}, t_{02})$	0.0865	5.5949e-07	0.0877	0.0849	0.0027

Table 15: Bayes estimates and standard errors for the parameters for the real data set
$(\alpha = 0.3, \alpha_1 = 0.8, \alpha_2 = 0.9)$

n	Parameters	Estimates	SE
42	α	0.2980	0.0008
	α_1	0.7992	0.0006
	α_2	0.9008	0.0007

Table 16: Bayes estimates, standard errors for the reliability and hazard rate functions of BLO for the real data set
$(\alpha = 0.3, \alpha_1 = 0.8, \alpha_2 = 0.9, t_{01} = 3, t_{02} = 2)$

n	rf and hrf	Estimates	SE
42	$R(t_{01}, t_{02})$	0.1434	0.0005
	$h(t_{01}, t_{02})$	0.0559	0.0006

Table 17: Bayes predictive, estimated risks and 95% credible intervals of the future observation under two-sample prediction
$(N = 10000, \alpha = 0.3, \alpha_1 = 0.8, \alpha_2 = 0.9)$

n	s	$\hat{y}_{(s)}$	Predictor	UL	LL	Length
30	1	$\hat{y}_{1(s)}$	3.9995	4.0009	3.9966	0.0043
		$\hat{y}_{2(s)}$	6.9994	7.0011	6.9979	0.0031
	12	$\hat{y}_{1(s)}$	4.001	4.0027	3.9996	0.0003
		$\hat{y}_{2(s)}$	6.9989	7.0003	6.9976	0.0028

	18	$\hat{y}_{1(s)}$	4.0029	4.0034	3.9999	0.0039
		$\hat{y}_{2(s)}$	7.0013	7.0035	6.9999	0.0035
50	1	$\hat{y}_{1(s)}$	3.9997	4.0009	3.9975	0.0034
		$\hat{y}_{2(s)}$	6.9995	0.0005	6.9979	0.0025
	12	$\hat{y}_{1(s)}$	4.0002	4.0018	3.9983	0.0034
		$\hat{y}_{2(s)}$	6.9991	6.9998	6.9984	0.0014
	18	$\hat{y}_{1(s)}$	3.9989	4.0003	3.9967	0.0036
		$\hat{y}_{2(s)}$	7.0015	7.0023	6.9997	0.0026
100	1	$\hat{y}_{1(s)}$	4.0006	4.0013	3.9994	0.0019
		$\hat{y}_{2(s)}$	7.0005	7.0016	6.9994	0.0021
	12	$\hat{y}_{1(s)}$	3.9994	4.0002	3.9976	0.0026
		$\hat{y}_{2(s)}$	6.9994	7.0004	6.9976	00028
	18	$\hat{y}_{1(s)}$	3.9989	4.0002	3.9976	0.0025
		$\hat{y}_{2(s)}$	69987	7.0000	6.9974	0.0026

Table 18: Bayes predictive and standard errors for a future observation under two-sample prediction for the real data set

s	$\hat{y}_{(s)}$	Predictor
1	$\hat{y}_{(s)}$	4.0001
	$\hat{y}_{1(s)}$	7.0010
12	$\hat{y}_{2(s)}$	4.0009
	$\hat{y}_{1(s)}$	6.9984
18	$\hat{y}_{2(s)}$	3.9977
	$\hat{y}_{1(s)}$	6.9982

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