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Improving the Performance of a Parallel-Series System Based on Lindley Distribution

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Abstract: In this paper, a parallel-series system is improved. All components assume independent and identically distributed. The Lindley distribution with three parameters is assumed to be a lifetime distribution for the components. Four methods are used to improve the performance of the parallel-series system. The γ-fractiles and equivalence factors are derived. Finally, numerical results are discussed.

Keywords: Lindley distribution, Reliability equivalence, Parallel-series, Improving methods.

1 Introduction

The concept of reliability equivalence factor is introduced by [1]. Several authors studied simple and complex systems and established the reliability equivalence factors, see [2] - [21].
The parallel-series systems are found in many applications of real life. This system is improved and discussed by [9], [22] - [26].

A random variable T has a three-parameters Lindley distribution (TPLD), [27], if it has the pdf given by

$$f(t;\alpha,\beta,\theta)=\frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta t)e^{-\theta t},\ t\geq 0,\ \beta,\theta>0,\ \alpha\theta+\beta>0.$$

The TPLD can be expressed as a mixture of gamma (2,θ) and exponential (θ) distributions as follows.

$$f(t;\alpha,\beta,\theta)=pg_1(t)+(1-p)g_2(t),$$

where $p=\alpha\theta/(\alpha\theta+\beta)$, $g_1(t)=\theta e^{-\theta t}$, and $g_2(t)=\theta^2 te^{-\theta t}$.

The cumulative distribution function (CDF) of TPLD is given by

$$F(t;\alpha,\beta,\theta)=1-\left(1+\frac{\theta\beta t}{\alpha\theta+\beta}\right)e^{-\theta t},\ \beta,\theta>0,\ \alpha\theta+\beta>0.$$

It can be easily verified that TPLD contains the following particular cases:

(i)Two-parameter quasi-Lindley distribution, [28,29],
(ii)Two-parameter Lindley distribution, [30,31],
(iii)Lindley distribution introduced by [32],
(iv)Gamma (2,θ) distribution, when $\alpha=\theta$ and exponential distribution, when $\beta=0$.

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The failure rate of TPLD is

$$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{\theta^2(\alpha+\beta t)}{\beta+\theta(\alpha+\beta t)}.$$

The $\lambda(t)$ is a function of time. Since

$$\frac{d}{dt}\lambda(t) = \left(\frac{\beta\theta}{\alpha\theta+\beta+\theta\beta t}\right)^2 > 0, \quad \text{for all} \quad t \geq 0.$$

Therefore, $\lambda(t)$ is increasing failure rate function. The TPLD has found applications in various fields (Engineering, economics, etc.)

2 Parallel - Series System

The construction of the original system is represented in Figure 1. The original system contains n subsystems, and each subsystem i has m_i items, $i = 1, 2, \cdots, n$, [33],

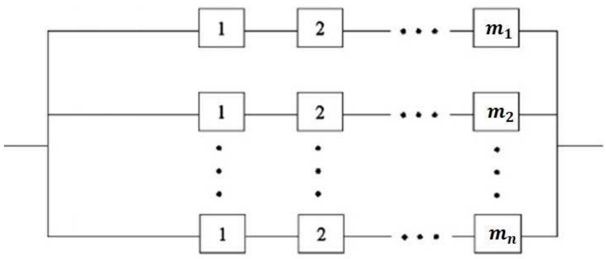

Fig. 1: Parallel-series system

The lifetime of the system components is independent and identically TPLD. The reliability function (RF) for component j is

$$\mathscr{R}_{ij}(t) = \left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}, \; t \geq 0, \tag{1}$$

where $\theta > 0$ and $j = 1, \cdots, m_i, i = 1, 2, \cdots, n$.
The subsystem i, has the following RF,

$$\mathscr{R}_i(t) = \prod_{i=1}^{m_i}\mathscr{R}_{ij}(t) = \left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i} e^{-m_i\theta t}. \tag{2}$$

Therefore, the RF of the original system, can be derived as follows.

$$\mathscr{R}(t) = 1-\prod_{i=1}^{n}[1-\mathscr{R}_i(t)] = 1-\prod_{i=1}^{n}\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i} e^{-m_i\theta t}\right] \tag{3}$$

The expected time to failure (ETTF) to the original system is

$$\mathscr{E}_O = E(T) = \int_0^\infty \mathscr{R}(t)dt \tag{4}$$

Since $\mathscr{E}_O$ does not have closed form, some numerical techniques can be used to calculate $\mathscr{E}_O$.

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3 The Improved Systems

Four different methods are used to improve the parallel-series system as follows.

1. System components are improved by reducing their failure rate by a factor $\rho, 0 < \rho < 1$. This method is called the reduction method (RM).
2. The system is improved by duplicating some components with another standby component connected in parallel. This method is called hot duplication method (HDM).
3. Some of the system components is duplicated by a standby component connected by a perfect switch. This method is called cold duplication method (CDM).
4. The system is improved by assuming that the component is duplicated by a standby component via an imperfect switch, which is called the imperfect duplication method (IDM).

The main objective is obtaining the REFs and comparing the performance of the original and the improved systems.

3.1 The reduction method

Suppose that the failure rates of the set A, are reduced by the factor $\rho, 0 < \rho < 1$, where $|A| = r$, $0 \le r \le N$, and $N = \sum_{i=1}^{n} m_i$. Such that r_i components from the subsystem i, $0 \le r_i \le m_i, i = 1, 2, \cdots, n$. That is $|A| = r = \sum_{i=1}^{n} r_i$. We denote such a set by $A_{|A|}^{(|A_1|,|A_2|,\cdots,|A_n|)}$, where A_i denotes the reducing set from subsystem i.

Since the failure rate of the TPLD is a function of time, the failure rates are reduced from $\lambda(t)$ to $r(t)\lambda(t)$, $0 < r(t) < 1$. Here, the failure rates are reduced by reducing the scale by the factor ρ only.

Let $\mathscr{R}_{ij,\rho}(t)$ denote the RF of component, j, from subsystem i, after reducing its failure rate by the factor ρ, it is given as follows.

$$\mathscr{R}_{ij,\rho}(t) = \left(1 + \frac{\beta\rho\theta t}{\alpha\rho\theta + \beta}\right) e^{-\rho\theta t}. \tag{5}$$

The RF, $\mathscr{R}_{A_i,\rho}(t)$, of the improved subsystem, i, by RM is derived as follows.

$$\mathscr{R}_{A_i,\rho}(t) = \left(1 + \frac{\beta\rho\theta t}{\alpha\rho\theta + \beta}\right)^{r_i} \left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right)^{m_i - r_i} e^{-[m_i - r_i(1-\rho)]\theta t}. \tag{6}$$

The RF, $\mathscr{R}_{A,\rho}(t)$, of the improved system by RM can be obtained as follows.

$$\mathscr{R}_{A,\rho}(t) = 1 - \prod_{i=1}^{n} \left[1 - \left(1 + \frac{\beta\rho\theta t}{\alpha\rho\theta + \beta}\right)^{r_i} \left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right)^{m_i - r_i} e^{-[m_i - r_i(1-\rho)]\theta t}\right]. \tag{7}$$

The ETTF of the improved system, say $\mathscr{E}_{A,\rho}$ can be calculated numerically by the following formula.

$$\mathscr{E}_{A,\rho} = \int_0^\infty \mathscr{R}_{A,\rho}(t)dt. \tag{8}$$

3.2 The hot duplication method

The system will be improved by duplicating the components in the set B by a parallel component, $|B| = h$ and $0 \le h \le N$. Such that, B_i from each subsystem, $|B_i| = h_i$, where $0 \le h_i \le m_i$, $i = 1, 2, \cdots, n$. That is, $|B| = h = \sum_{i=1}^{n} h_i$. We denote such a set by $B_{|B|}^{(|B_1|,|B_2|,\cdots,|B_n|)}$, that is $B = \cup_{i=1}^{n} B_i$.

The RF, $\mathscr{R}_{B_i}^{H}(t)$ of the improved subsystem i by improving the components belonging to the set B_i by HDM is given as follows.

$$\mathscr{R}_{B_i}^{H}(t) = \left[2 - \left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right) e^{-\theta t}\right]^{h_i} \left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right)^{m_i} e^{-m_i\theta t}. \tag{9}$$

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Thus, the RF, $\mathscr{R}_B^H(t)$ of the improved system by HDM is

$$\mathscr{R}_B^H(t) = 1 - \prod_{i=1}^{n}\left\{1 - \left[2 - \left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right)e^{-\theta t}\right]^{h_i}\left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right)^{m_i} e^{-m_i\theta t}\right\}. \tag{10}$$

From equation (10), the ETTF to the improved system by HDM, say $\mathscr{E}_B^H$ can be calculated by using

$$\mathscr{E}_B^H = \int_0^\infty \mathscr{R}_B^H(t)dt. \tag{11}$$

3.3 The cold duplication method

In CDM, the components belonging to the set B, are connected via a perfect switch each with an identical component, $|B| = c$. The set $B = B_1 \cup B_2 \cup \cdots \cup B_n$, such that, B_i from the subsystem i, $i = 1, 2, \cdots, n$, $|B_i| = c_i$ and $c = \sum_{i=1}^n c_i$. The set B can be denoted by $B_{|B|}^{(|B_1|,|B_2|,\cdots,|B_n|)}$. The RF, $\mathscr{R}_B^C(t)$, of the improved system by CDM can be derived by

$$\mathscr{R}_B^C(t) = 1 - \prod_{i=1}^{n}[1 - \mathscr{R}_{B_i}^C(t)], \tag{12}$$

where $\mathscr{R}_{B_i}^C(t)$ denotes the RF of the subsystem i after improving by the CDM,

$$\mathscr{R}_{B_i}^C(t) = \left[\mathscr{R}_{ij}^C(t)\right]^{c_i}[\mathscr{R}_{ij}(t)]^{m_i - c_i}. \tag{13}$$

and

$$\mathscr{R}_{ij}^C(t) = \left[1 + \frac{\beta\theta t}{\beta + \alpha\theta} + \frac{\theta^2\left[6\alpha(\beta + \alpha\theta) + 3\beta(\beta + 2\alpha\theta)t + \theta\beta^2 t^2\right]t}{6(\beta + \alpha\theta)^2}\right]e^{-\theta t}. \tag{14}$$

Therefore,

$$\mathscr{R}_B^C(t) = 1 - \prod_{i=1}^{n}\left\{1 - \left[1 + \frac{\beta\theta t}{\beta + \alpha\theta} + \frac{\theta^2\left[6\alpha(\beta + \alpha\theta) + 3\beta(\beta + 2\alpha\theta)t + \theta\beta^2 t^2\right]t}{6(\beta + \alpha\theta)^2}\right]^{c_i}\left(1 + \frac{\beta\theta t}{\alpha\theta + \beta}\right)^{m_i - c_i} e^{-m_i\theta t}.\right\}. \tag{15}$$

The ETTF to the improved system by CDM, can be calculated numerically by using the formula.

$$\mathscr{E}_B^C = \int_0^\infty \mathscr{R}_B^C(t)dt. \tag{16}$$

3.4 Imperfect duplication method

The system will be improved by improving the set of B components by IDM, $|B| = s$, $0 \le s \le N$. By using an imperfect switch, each component belonging to B will relate to an identical component. The lifetime for the imperfect switch is TPLD(α, β, ν). The set B contains B_i from subsystem i, $i = 1, 2, \cdots, n$, such that $|B_i| = s_i$ and $s = \sum_{i=1}^n s_i$. The set B can be denoted by $B_{|B|}^{(|B_1|,|B_2|,\cdots,|B_n|)}$.

Let $\mathscr{R}_B^I(t)$ be the RF of the improved system by IDM, which is given as

$$\mathscr{R}_B^I(t) = 1 - \prod_{i=1}^{n}\left[1 - \mathscr{R}_{B_i}^I(t)\right], \tag{17}$$

where $\mathscr{R}_{B_i}^I(t)$ denotes the RF of the subsystem i, after improving by IDM,

$$\mathscr{R}_{B_i}^I(t) = \left[\mathscr{R}_{ij}^I(t)\right]^{s_i}[\mathscr{R}_{ij}(t)]^{m_i - s_i}. \tag{18}$$

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and

$$\mathscr{R}_{ij}^{I}(t) = \left(1+\frac{\beta\theta t}{\beta+\alpha\theta}\right)e^{-\theta t} + \frac{\theta^2 e^{-(\theta+\nu)t}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3}\Big\{(-1+e^{\nu t})\alpha^3\theta\nu^3 + \alpha\beta^2\nu\Big[-\theta(-1+\nu t)(2+\nu t) -\nu(3+2\nu t-3e^{\nu t})+\theta(-2+3\nu t)e^{\nu t}\Big] + \alpha^2\beta\nu^2\Big[-\nu-2\theta(1+\nu t)+(\nu+\theta(2+\nu t))e^{\nu t}\Big] +\beta^3\Big[-\nu\left(3-3e^{\nu t}+(3+\nu t)\nu t\right)+\theta\left(8+(5+\nu t)\nu t+(-8+3\nu t)e^{\nu t}\right)\Big]\Big\}. \tag{19}$$

Substituting from (19) and (18) into (17), the $\mathscr{R}_B^C(t)$ is given in the following form

$$\mathscr{R}_{B}^{I}(t) = 1-\prod_{i=1}^{n}\Bigg\{1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i-s_i} e^{-m_i\theta t}\Bigg[1+\frac{\beta\theta t}{\beta+\alpha\theta}+\frac{\theta^2 e^{-\nu t}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3}\Big\{(-1+e^{\nu t})\alpha^3\theta\nu^3 +\alpha\beta^2\nu\Big[-\theta(-1+\nu t)(2+\nu t)-\nu(3+2\nu t-3e^{\nu t})+\theta(-2+3\nu t)e^{\nu t}\Big]+\alpha^2\beta\nu^2\Big[-\nu-2\theta(1+\nu t) +(\nu+\theta(2+\nu t))e^{\nu t}\Big]+\beta^3\Big[-\nu\left(3-3e^{\nu t}+(3+\nu t)\nu t\right)+\theta\left(8+(5+\nu t)\nu t+(-8+3\nu t)e^{\nu t}\right)\Big]\Big\}\Bigg]^{s_i}\Bigg\}. \tag{20}$$

The ETTF to the improved system by IDM, is calculated numerically by using the following integration.

$$\mathscr{E}_B^I = \int_0^\infty \mathscr{R}_B^I(t)dt. \tag{21}$$

4 The γ-Fractiles

In this section, γ-fractiles (GF) are presented to measure the performance of reliability of the original and improved systems.

The GF of the original system, $\mathscr{L}(\gamma)$, can be obtained by using the following equation.

$$\mathscr{R}\left(\frac{\mathscr{L}(\gamma)}{\Theta}\right)=\gamma, \tag{22}$$

where $\Theta = N\theta$, $N=\sum_{i=1}^{n} m_i$.

From equations (3) and (22), $\mathscr{L}=\mathscr{L}(\gamma)$ satisfies the following equation.

$$\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{\beta\theta\mathscr{L}}{(\alpha\theta+\beta)\Theta}\right)^{m_i} e^{-m_i\frac{\theta\mathscr{L}}{\Theta}}\right]-\ln(1-\gamma)=0. \tag{23}$$

The GF of the improved system according to duplication methods, $\mathscr{L}_B^D(\gamma)$, is defined as

$$\mathscr{R}_B^D\left(\frac{\mathscr{L}(\gamma)}{\Theta}\right)=\gamma, \quad D=H,I, \text{ and } C. \tag{24}$$

For $D=H$, from equations (10) and (24), $\mathscr{L}=\mathscr{L}_B^H(\gamma)$ is a solution of the following equation.

$$\sum_{i=1}^{n}\ln\left\{1-\left[2-\left(1+\frac{\beta\theta\mathscr{L}}{(\alpha\theta+\beta)\Theta}\right)e^{-\frac{\theta\mathscr{L}}{\Theta}}\right]^{h_i}\left(1+\frac{\beta\theta\mathscr{L}}{(\alpha\theta+\beta)\Theta}\right)^{m_i}e^{-\frac{m_i\theta}{\Theta}\mathscr{L}}\right\}-\ln(1-\gamma)=0. \tag{25}$$

For $D=C$, and from equations (15) and (24), $\mathscr{L}=\mathscr{L}_B^C(\gamma)$ can be derived by solve the following equation.

$$\sum_{i=1}^{n}\ln\left\{1-\left[1+\frac{\beta\theta\mathscr{L}}{(\beta+\alpha\theta)\Theta}+\frac{\theta^2\left[6\alpha(\beta+\alpha\theta)\Theta^2+3\beta(\beta+2\alpha\theta)\Theta\mathscr{L}+\theta\beta^2\mathscr{L}^2\right]\mathscr{L}}{6(\beta+\alpha\theta)^2\Theta^3}\right]^{c_i}\times \left(1+\frac{\beta\theta\mathscr{L}}{(\alpha\theta+\beta)\Theta}\right)^{m_i-c_i}e^{-\frac{m_i\theta}{\Theta}\mathscr{L}}\right\}-\ln(1-\gamma)=0. \tag{26}$$

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Setting $D = I$, in (24), substituting from (20) into (24), $\mathscr{L} = \mathscr{L}_B^I(\gamma)$, satisfies the following equation.

$$\sum_{i=1}^{n} \ln \Bigg\{ 1 - \left(1 + \frac{\beta\theta\mathscr{L}}{(\alpha\theta+\beta)\Theta}\right)^{m_i - s_i} e^{-\frac{m_i\theta}{\Theta}\mathscr{L}} \Bigg[1 + \frac{\beta\theta\mathscr{L}}{(\beta+\alpha\theta)\Theta} + \frac{\theta^2 e^{-\frac{\nu}{\Theta}\mathscr{L}}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3} \Big\{ (-1 + e^{\frac{\nu}{\Theta}\mathscr{L}}) \times$$
$$\alpha^3\theta\nu^3 + \alpha\beta^2\nu \left[-\theta\left(-1+\frac{\nu}{\Theta}\mathscr{L}\right)\left(2+\frac{\nu}{\Theta}\mathscr{L}\right) - \nu\left(3 + \frac{2\nu}{\Theta}\mathscr{L} - 3e^{\frac{\nu}{\Theta}\mathscr{L}}\right) + \theta\left(-2+\frac{3\nu}{\Theta}\mathscr{L}\right) e^{\frac{\nu}{\Theta}\mathscr{L}} \right] +$$
$$\alpha^2\beta\nu^2\left[-\nu - 2\theta\left(1+\frac{\nu}{\Theta}\mathscr{L}\right) + \left(\nu + \theta\left(2+\frac{\nu}{\Theta}\mathscr{L}\right)\right) e^{\frac{\nu}{\Theta}\mathscr{L}} \right] + \beta^3\left[-\nu\left(3 - 3e^{\frac{\nu}{\Theta}\mathscr{L}} + \left(3+\frac{\nu}{\Theta}\mathscr{L}\right)\frac{\nu}{\Theta}\mathscr{L}\right) + \right.$$
$$\left. \theta\left(8 + \left(5+\frac{\nu}{\Theta}\mathscr{L}\right)\frac{\nu}{\Theta}\mathscr{L} + \left(-8+3\frac{\nu}{\Theta}\mathscr{L}\right) e^{\frac{\nu}{\Theta}\mathscr{L}}\right) \right] \Big\} \Bigg]^{s_i} \Bigg\} - \ln(1-\gamma) = 0. \tag{27}$$

A numerical Program is used to solve the equations (23), (25) - (27).

5 The Reliability Equivalence Factors

The reliability equivalence factor (REF) is defined as that factor by which the failure rates of the set A, of system's components should be reduced to reach the reliability of that system which improved by improving the set B of system's, according to duplication methods.
The failure rate of TPLD, is reduced by the factor $r(t)$, we consider the scale parameter of TPLD is reduced from θ to $\rho\theta$ only.

$$r(t)\lambda(t) = \frac{\rho^2\theta^2(\alpha+\beta t)}{\beta+\rho\theta(\alpha+\beta t)}. \tag{28}$$

In this section, we will deduce two types of REFs of the parallel-series system: (i) the survival reliability equivalence factor (SREF), (ii) mean reliability equivalence factor (MREF) as follows.

5.1 The SREF

The SREF, $\rho_{A,B}^D(\gamma)$, is obtained by equating the reliability function of the improved system that is obtained by reduction method with duplication method at the level γ. $\rho_{A,B}^D(\gamma)$, can be obtained by solving the following system:

$$\mathscr{R}_{A,\rho}(t) = \gamma, \qquad \mathscr{R}_B^D(t) = \gamma, \quad D = H, C, I. \tag{29}$$

1. Using equation (29) together with equations (7) and (10), the $\rho = \rho_{A,B}^H(\gamma)$, is obtained by solve the following system.

$$\left.\begin{aligned} &\sum_{i=1}^{n} \ln\left[1 - \left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)^{r_i}\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i - r_i} e^{-[m_i - r_i(1-\rho)]\theta t}\right] - \ln(1-\gamma) = 0 \\ &\sum_{i=1}^{n} \ln\left\{1 - \left[2 - \left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{h_i}\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i} e^{-m_i\theta t}\right\} - \ln(1-\gamma) = 0 \end{aligned}\right\}. \tag{30}$$

2. Substituting from equations (7) and (15) into equation (29), the $\rho = \rho_{A,B}^C(\gamma)$, satisfies the following system.

$$\left.\begin{aligned} &\sum_{i=1}^{n} \ln\left[1 - \left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)^{r_i}\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i - r_i} e^{-[m_i - r_i(1-\rho)]\theta t}\right] - \ln(1-\gamma) = 0 \\ &\sum_{i=1}^{n} \ln\left\{1 - \left(1 + \frac{\beta\theta t}{\beta+\alpha\theta} + \frac{\theta^2\left[6\alpha(\beta+\alpha\theta) + 3\beta(\beta+2\alpha\theta)t + \theta\beta^2 t^2\right]t}{6(\beta+\alpha\theta)^2}\right)^{c_i} \times \right. \\ &\left. \qquad \left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i - c_i} e^{-m_i\theta t}\right\} - \ln(1-\gamma) = 0 \end{aligned}\right\}. \tag{31}$$

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3.Using equations (7) and (20) together with equation (29), the $\rho = \rho^{I}_{A,B}(\gamma)$, satisfies the following system.

$$\left.\begin{aligned} \sum_{i=1}^{n}\ln\left[1-\left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)^{r_i}\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i-r_i}e^{-[m_i-r_i(1-\rho)]\theta t}\right]-\ln(1-\gamma)=0 \\ \sum_{i=1}^{n}\ln\left\{1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)^{m_i-s_i}e^{-m_i\theta t}\left[1+\frac{\beta\theta t}{\beta+\alpha\theta}-\frac{\theta^2e^{-\nu t}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3}\left\{(-1+e^{\nu t})\alpha^3\theta\nu^3\right.\right.\right. \\ +\alpha\beta^2\nu\left[-\theta(-1+\nu t)(2+\nu t)-\nu(3+2\nu t-3e^{\nu t})+\theta(-2+3\nu t)e^{\nu t}\right] \\ +\beta^3\left[-\nu(3-3e^{\nu t}+(3+\nu t)\nu t)+\theta(8+(5+\nu t)\nu t+(-8+3\nu t)e^{\nu t})\right]+ \\ \left.\left.\left.\alpha^2\beta\nu^2\left[-\nu-2\theta(1+\nu t)+(\nu+\theta(2+\nu t))e^{\nu t}\right]\right\}\right]^{s_i}\right\}-\ln(1-\gamma)=0 \end{aligned}\right\}. \quad (32)$$

The solutions for the systems (30)- (32) can be obtained numerically.

5.2 The MREF

The MREF, $\xi^{D}_{A,B}$, can be derived by equating the ETTF of the improved system that obtained by improving the system according to RM with the duplication method. The $\xi = \xi^{D}_{A,B}$ is the solution of the following equation.

$$\mathscr{E}_{A,\rho} = \mathscr{E}^{D}_{B}, \quad D = H,C,I. \quad (33)$$

By substituting from (8), (11), (16) and (21) into (33), the $\xi = \xi^{D}_{A,B}$ can be obtained for $D = H,C$ and I, respectively.

6 Numerical Results

To explain the previous theoretical results a numerical example is introduced, under the following assumptions:

1.There are two subsystems in the parallel-series system.
2.The system contains three components, such that, $m_1 = 1$, and $m_2 = 2$, see Figure 2.
3.The parameters $\alpha = 0.1, \beta = 0.2, \theta = 0.7$ and $\nu = 0.3$.

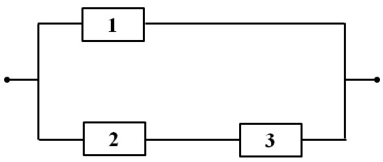

Fig. 2: The parallel series system

For this example, the ETTF of the original system is 2.879998 and Table 1 shows the values of $\mathscr{E}^{D}_{B}$ for the improved systems.

Table 1: The values of $\mathscr{E}^{D}_{B}$, for different values of $B^{(|B_1|,|B_2|)}_{|B|}$, $D = H,I$ and C.

	$B^{(1,0)}_{1}$	$B^{(0,1)}_{1}$	$B^{(1,1)}_{2}$	$B^{(0,2)}_{2}$	$B^{(1,2)}_{3}$
$\mathscr{E}^{H}_{B}$	3.71448	3.06028	3.81752	3.34361	3.98714
$\mathscr{E}^{I}_{B}$	4.54785	3.16994	4.67554	3.79786	4.99114
$\mathscr{E}^{C}_{B}$	5.07075	3.21837	5.20184	4.08258	5.60702

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Figures 3-5 show the RF of the parallel-series system and improved systems for $D = H, I$ and C.

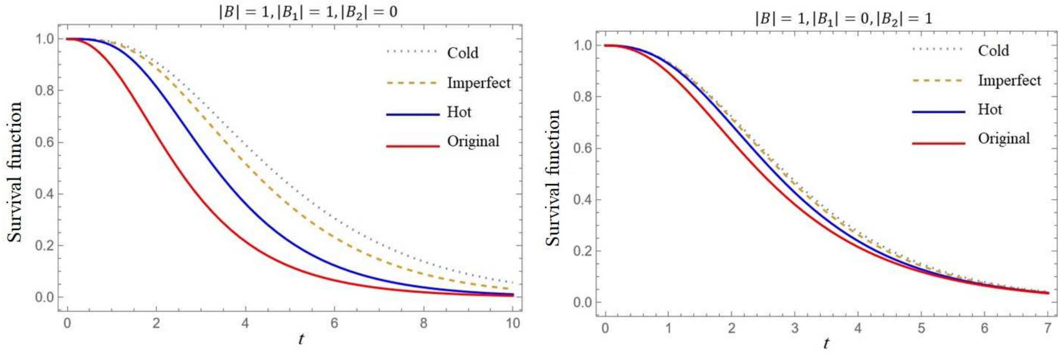

Fig. 3: The $\mathscr{R}(t), \mathscr{R}_B^D(t)$, for $|B| = 1$.

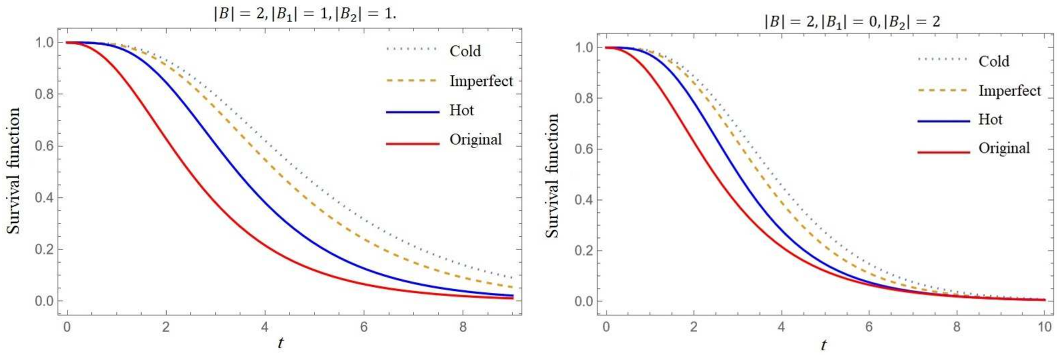

Fig. 4: The $\mathscr{R}(t), \mathscr{R}_B^D(t)$, for $|B| = 2$.

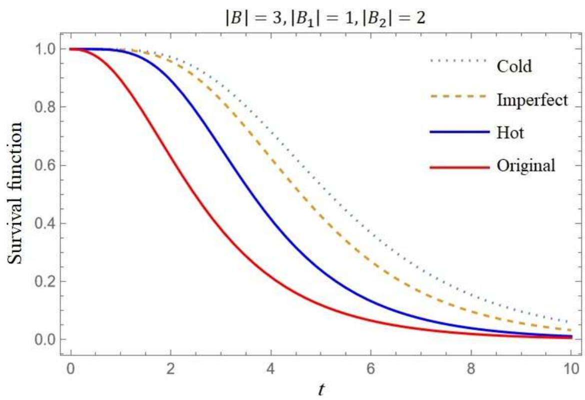

Fig. 5: The $\mathscr{R}(t), \mathscr{R}_B^D(t)$, for $|B| = 3$.

Figures 6-8 show $\mathscr{R}(t), \mathscr{R}_B^D(t)$ for different $|B|$ and improving method.

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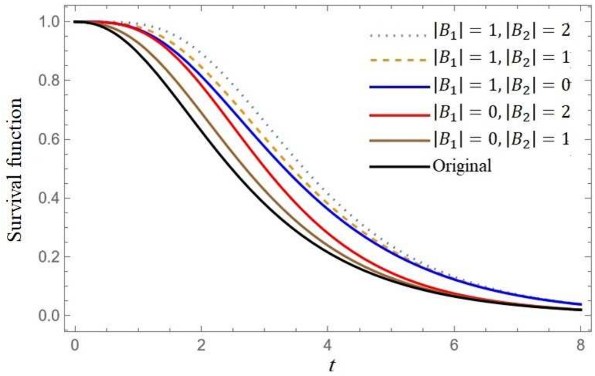

Fig. 6: The $\mathscr{R}(t), \mathscr{R}_B^H(t)$, for $|B| = 1,2$ and 3.

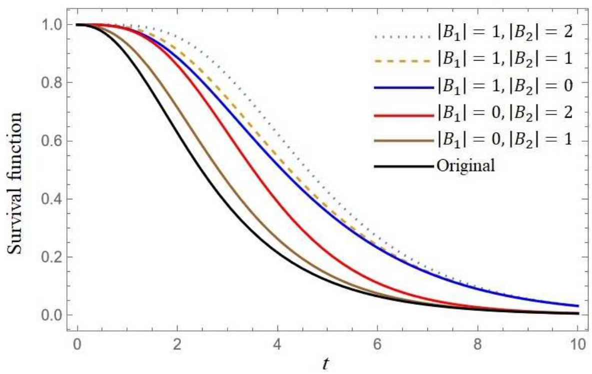

Fig. 7: The $\mathscr{R}(t), \mathscr{R}_B^I(t)$, for $|B| = 1,2$ and 3.

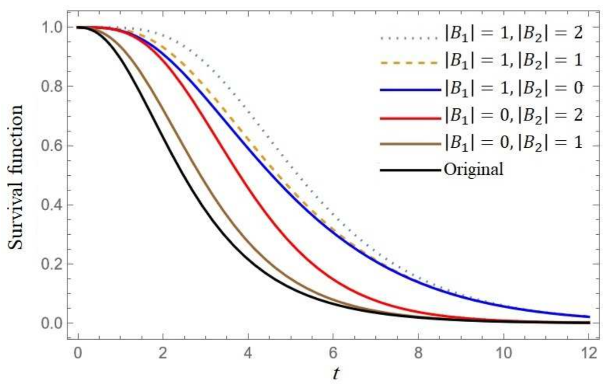

Fig. 8: The $\mathscr{R}(t), \mathscr{R}_B^C(t)$, for $|B| = 1,2$ and 3.

According to the previous theoretical formulae, the Mathematica Program System is used to calculate $\mathscr{L}(\gamma)$, $\mathscr{L}_B^D(\gamma)$ and $\rho_{A,B}^D(\gamma)$, $D = H, I, C$, such that $\gamma = 0.1, 0.2, \cdots, 0.9$. The values of $\mathscr{L}(\gamma)$ and $\mathscr{L}_B^D(\gamma)$ are presented in Tables 2 and 3.

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Table 2: The $\mathscr{L}(\gamma), \mathscr{L}_B^H(\gamma)$, $D = H, I$ and C and $|B| = 1$ and 2.

		$B_1^{(1,0)}$			$B_1^{(0,1)}$			$B_2^{(1,1)}$		
γ	$\mathscr{L}$	$\mathscr{L}^H$	$\mathscr{L}^I$	$\mathscr{L}^C$	$\mathscr{L}^H$	$\mathscr{L}^I$	$\mathscr{L}^C$	$\mathscr{L}^H$	$\mathscr{L}^I$	$\mathscr{L}^C$
0.1	11.1086	13.3537	16.3471	18.3731	11.3356	11.6556	11.8377	13.4299	16.4100	18.4180
0.2	8.6646	10.7704	13.2900	14.9161	9.0030	9.3489	9.5166	10.9063	13.4230	15.0257
0.3	7.1937	9.1759	11.3470	12.7054	7.5962	7.9276	8.0741	9.3591	11.5469	12.8856
0.4	6.1036	7.9743	9.8572	11.0041	6.5445	6.8479	6.9730	8.1954	10.1177	11.2542
0.5	5.2058	6.9720	8.6024	9.5688	5.6682	5.9373	6.0419	7.2236	8.9153	9.8841
0.6	4.4113	6.0747	7.4741	8.2795	4.8816	5.1123	5.1972	6.3503	7.8302	8.6516
0.7	3.6628	5.2188	6.3975	7.0535	4.1276	4.3162	4.3819	5.5118	6.7862	7.4709
0.8	2.9051	4.3384	5.2939	5.8046	3.3476	3.4892	3.5355	4.6409	5.7009	6.2503
0.9	2.0404	3.3056	4.0085	4.3643	2.4287	2.5146	2.5406	3.6031	4.4085	4.8072

Table 3: The $\mathscr{L}(\gamma), \mathscr{L}_B^H(\gamma)$, $D = H, I$ and C and $|B| = 2$ and 3.

		$B_2^{(0,2)}$			$B_3^{(1,2)}$		
γ	$\mathscr{L}$	$\mathscr{L}^H$	$\mathscr{L}^I$	$\mathscr{L}^C$	$\mathscr{L}^H$	$\mathscr{L}^I$	$\mathscr{L}^C$
0.1	11.1086	11.6934	12.9112	13.8514	13.5673	16.6702	18.7497
0.2	8.6646	9.5003	10.7461	11.5818	11.1364	13.8660	15.5976
0.3	7.1937	8.1731	9.3712	10.1178	9.6573	12.1238	13.6352
0.4	6.1036	7.1741	8.3024	8.9706	8.5468	10.7961	12.1378
0.5	5.2058	6.3351	7.3832	7.9790	7.6180	9.6721	10.8692
0.6	4.4113	5.5755	6.5350	7.0610	6.7807	8.6476	9.7123
0.7	3.6628	4.8406	5.7008	6.1564	5.9729	7.6491	8.5845
0.8	2.9051	4.0712	4.8146	5.1943	5.1281	6.5940	7.3929
0.9	2.0404	3.1476	3.7360	4.0235	4.1108	5.3090	5.9421

Based on the results shown in Tables 2, 3 and Figures 3-6, we can conclude that:

1. $\mathscr{R}(t) < \mathscr{R}_B^H(t) < \mathscr{R}_B^I(t) < \mathscr{R}_B^C(t)$, in all studied cases.
2. $\mathscr{E} < \mathscr{E}_B^H < \mathscr{E}_B^I < \mathscr{E}_B^C$, in all studied cases.
3. $\mathscr{L}(\gamma) < \mathscr{L}_B^H(\gamma) < \mathscr{L}_B^I(\gamma) < \mathscr{L}_B^C(\gamma)$, in all studied cases.
4. According to the duplication methods: (i) Improving one component from the first subsystem gives a better design than improving one component from the second subsystem. (ii) Improving two components selected from both subsystems produces a better design than improving two components selected from the second subsystem. (iii) Improving all components of the system gives the best design.
5. Cold duplication method gives the best improvement than other methods.

The SREF is show in Tables 4 and 5, for each duplication method and the sets A and B.

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Table 4: The values of $\rho^D_{A,B}(\gamma)$, $D = H, I$ and C.

γ	A	$B_1^{(1,0)}$			$B_1^{(0,1)}$			$B_2^{(1,1)}$		
		ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C
0.1	$A_1^{(1,0)}$	0.82531	0.67693	0.60486	0.97818	0.94937	0.93389	0.82063	0.67443	0.60344
	$A_1^{(0,1)}$	0.00000	0.00000	0.00000	0.79503	0.60850	0.52680	–	–	0.00084
	$A_2^{(1,1)}$	0.83101	0.68025	0.60673	0.97973	0.95256	0.93777	0.82630	0.67769	0.60529
	$A_2^{(0,2)}$	0.59086	0.43732	0.37978	0.89868	0.80943	0.77174	0.58490	0.43518	0.37872
	$A_3^{(1,2)}$	0.84060	0.69362	0.62035	0.98116	0.95579	0.94193	0.83608	0.69109	0.61890
0.2	$A_1^{(1,0)}$	0.78493	0.63536	0.56806	0.95571	0.91520	0.89706	0.77475	0.62921	0.56407
	$A_1^{(0,1)}$	–	–	–	0.74733	0.56302	0.48795	–	–	–
	$A_2^{(1,1)}$	0.79766	0.64381	0.57353	0.96077	0.92371	0.90677	0.78743	0.63738	0.56936
	$A_2^{(0,2)}$	0.58942	0.42924	0.37103	0.87585	0.78944	0.75586	0.57678	0.42362	0.36778
	$A_3^{(1,2)}$	0.81671	0.67031	0.60095	0.96513	0.93192	0.91665	0.80719	0.66404	0.59680
0.3	$A_1^{(1,0)}$	0.75147	0.60298	0.53944	0.93401	0.88705	0.86807	0.73559	0.59258	0.53210
	$A_1^{(0,1)}$	–	–	–	0.70906	0.52915	0.45909	–	–	–
	$A_2^{(1,1)}$	0.77171	0.61764	0.54980	0.94356	0.90154	0.88411	0.75573	0.60657	0.54196
	$A_2^{(0,2)}$	0.58392	0.42293	0.36494	0.85789	0.77520	0.74476	0.56473	0.41304	0.35861
	$A_3^{(1,2)}$	0.79943	0.65606	0.59003	0.95140	0.91485	0.89958	0.78493	0.64542	0.58227
0.4	$A_1^{(1,0)}$	0.72022	0.57395	0.51376	0.91254	0.86198	0.84298	0.69851	0.55889	0.50246
	$A_1^{(0,1)}$	–	–	–	0.67394	0.49947	0.43382	–	–	–
	$A_2^{(1,1)}$	0.74828	0.59563	0.53013	0.92727	0.88279	0.86560	0.72643	0.57933	0.51777
	$A_2^{(0,2)}$	0.57559	0.41673	0.35965	0.84175	0.76328	0.73563	0.55007	0.40201	0.34945
	$A_3^{(1,2)}$	0.78422	0.64502	0.58227	0.93890	0.90110	0.88639	0.76479	0.62960	0.57019
0.5	$A_1^{(1,0)}$	0.68899	0.54589	0.48892	0.89071	0.83852	0.82003	0.66132	0.52582	0.47314
	$A_1^{(0,1)}$	–	–	–	0.63917	0.47108	0.40969	–	–	–
	$A_2^{(1,1)}$	0.72522	0.57527	0.51229	0.91123	0.86597	0.84947	0.69734	0.55324	0.49466
	$A_2^{(0,2)}$	0.56452	0.40980	0.35429	0.82617	0.75244	0.72746	0.53287	0.38985	0.33955
	$A_3^{(1,2)}$	0.76926	0.63504	0.57572	0.92691	0.88919	0.87535	0.74490	0.61452	0.55871
0.6	$A_1^{(1,0)}$	0.65600	0.51715	0.46347	0.86777	0.81566	0.79815	0.62219	0.49173	0.44271
	$A_1^{(0,1)}$	–	–	–	0.60259	0.44208	0.38505	–	–	–
	$A_2^{(1,1)}$	0.70087	0.55487	0.49478	0.89478	0.85014	0.83471	0.66670	0.52663	0.47116
	$A_2^{(0,2)}$	0.55023	0.40139	0.34810	0.81027	0.74200	0.71974	0.51254	0.37585	0.32820
	$A_3^{(1,2)}$	0.75329	0.62482	0.56919	0.91486	0.87830	0.86557	0.72390	0.59892	0.54673
0.7	$A_1^{(1,0)}$	0.61897	0.48585	0.43578	0.84271	0.79248	0.77642	0.57878	0.45472	0.40953
	$A_1^{(0,1)}$	–	–	–	0.56157	0.41041	0.35816	–	–	–
	$A_2^{(1,1)}$	0.67319	0.53263	0.47601	0.87711	0.83456	0.82058	0.63234	0.49765	0.44564
	$A_2^{(0,2)}$	0.53141	0.39036	0.34010	0.79313	0.73141	0.71207	0.48772	0.35892	0.31444
	$A_3^{(1,2)}$	0.73480	0.61309	0.56157	0.90210	0.86779	0.85646	0.70015	0.58147	0.53313
0.8	$A_1^{(1,0)}$	0.57371	0.44877	0.40306	0.81364	0.76767	0.75374	0.52680	0.41150	0.37072
	$A_1^{(0,1)}$	–	–	–	0.51130	0.37261	0.32606	–	–	–
	$A_2^{(1,1)}$	0.63841	0.50553	0.45334	0.85686	0.81829	0.80632	0.59020	0.46312	0.41534
	$A_2^{(0,2)}$	0.50499	0.37448	0.32837	0.77329	0.71998	0.70401	0.45526	0.33681	0.29636
	$A_3^{(1,2)}$	0.71108	0.59774	0.55106	0.88764	0.85703	0.84748	0.67069	0.55992	0.51599
0.9	$A_1^{(1,0)}$	0.50832	0.39701	0.35754	0.77575	0.73841	0.72779	0.45440	0.35320	0.31847
	$A_1^{(0,1)}$	–	–	–	0.43835	0.31929	0.28069	–	–	–
	$A_2^{(1,1)}$	0.58555	0.46514	0.41944	0.83059	0.79955	0.79056	0.52877	0.41430	0.37262
	$A_2^{(0,2)}$	0.46129	0.34702	0.30725	0.74719	0.70629	0.69472	0.40558	0.30289	0.26830
	$A_3^{(1,2)}$	0.67407	0.57277	0.53255	0.86907	0.84485	0.83781	0.62699	0.52795	0.48991

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Table 5: The values of $\rho^{D}_{A,B}(\gamma)$, $D = H, I$ and C.

γ	A	$B_2^{(0,2)}$			$B_3^{(1,2)}$		
		ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C
0.1	$A_1^{(1,0)}$	0.94610	0.85382	0.79577	0.81233	0.66426	0.59317
	$A_1^{(0,1)}$	0.59046	0.11066	0.00106	–	0.00001	0.00000
	$A_2^{(1,1)}$	0.94945	0.85948	0.80126	0.81797	0.66730	0.59484
	$A_2^{(0,2)}$	0.80102	0.62946	0.55486	0.57460	0.42662	0.37111
	$A_3^{(1,2)}$	0.95288	0.86784	0.81196	0.82807	0.68079	0.60841
0.2	$A_1^{(1,0)}$	0.89879	0.78678	0.72845	0.75821	0.60962	0.54415
	$A_1^{(0,1)}$	0.49496	–	0.00034	–	–	–
	$A_2^{(1,1)}$	0.90839	0.79952	0.74034	0.77071	0.61692	0.54863
	$A_2^{(0,2)}$	0.75894	0.59176	0.52289	0.55687	0.40616	0.35191
	$A_3^{(1,2)}$	0.91811	0.81844	0.76298	0.79156	0.64397	0.57600
0.3	$A_1^{(1,0)}$	0.85578	0.73457	0.67749	0.71138	0.56473	0.50376
	$A_1^{(0,1)}$	0.41383	–	–	–	–	–
	$A_2^{(1,1)}$	0.87270	0.75470	0.69614	0.73111	0.57684	0.51171
	$A_2^{(0,2)}$	0.72582	0.56351	0.49881	0.53647	0.38733	0.33482
	$A_3^{(1,2)}$	0.88955	0.78400	0.73013	0.76243	0.61659	0.55206
0.4	$A_1^{(1,0)}$	0.81444	0.68855	0.63322	0.66703	0.52359	0.46663
	$A_1^{(0,1)}$	0.33216	–	–	–	–	–
	$A_2^{(1,1)}$	0.83929	0.71631	0.65891	0.69420	0.54088	0.47856
	$A_2^{(0,2)}$	0.69572	0.53861	0.47752	0.51433	0.36866	0.31817
	$A_3^{(1,2)}$	0.86376	0.75574	0.70377	0.73586	0.59272	0.53127
0.5	$A_1^{(1,0)}$	0.77286	0.64506	0.59178	0.62277	0.48362	0.43049
	$A_1^{(0,1)}$	0.23748	–	–	–	–	–
	$A_2^{(1,1)}$	0.80623	0.68069	0.62483	0.65755	0.50638	0.44683
	$A_2^{(0,2)}$	0.66613	0.51467	0.45703	0.49018	0.34931	0.30110
	$A_3^{(1,2)}$	0.83883	0.73025	0.68035	0.70972	0.57003	0.51156
0.6	$A_1^{(1,0)}$	0.72917	0.60164	0.55075	0.57659	0.44293	0.39366
	$A_1^{(0,1)}$	0.08807	–	–	–	–	–
	$A_2^{(1,1)}$	0.77172	0.64549	0.59154	0.61918	0.47140	0.41481
	$A_2^{(0,2)}$	0.63515	0.49009	0.43596	0.46318	0.32840	0.28281
	$A_3^{(1,2)}$	0.81322	0.70547	0.65786	0.68239	0.54697	0.49155
0.7	$A_1^{(1,0)}$	0.68083	0.55575	0.50773	0.52592	0.39939	0.35426
	$A_1^{(0,1)}$	–	–	–	–	–	–
	$A_2^{(1,1)}$	0.73343	0.60831	0.55676	0.57651	0.43380	0.38058
	$A_2^{(0,2)}$	0.60053	0.46312	0.41281	0.43171	0.30467	0.26218
	$A_3^{(1,2)}$	0.78508	0.67952	0.63453	0.65190	0.52186	0.46979
0.8	$A_1^{(1,0)}$	0.62305	0.50334	0.45903	0.46615	0.34944	0.30913
	$A_1^{(0,1)}$	–	–	–	–	–	–
	$A_2^{(1,1)}$	0.68702	0.56533	0.51700	0.52488	0.38990	0.34095
	$A_2^{(0,2)}$	0.55823	0.43071	0.38496	0.39226	0.27560	0.23706
	$A_3^{(1,2)}$	0.75116	0.64956	0.60779	0.61468	0.49189	0.44384
0.9	$A_1^{(1,0)}$	0.54242	0.43383	0.39517	0.38477	0.28381	0.25011
	$A_1^{(0,1)}$	–	–	–	–	–	–
	$A_2^{(1,1)}$	0.62004	0.50635	0.46305	0.45115	0.32986	0.28737
	$A_2^{(0,2)}$	0.49685	0.38450	0.34516	0.33465	0.23416	0.20145
	$A_3^{(1,2)}$	0.70220	0.60808	0.57095	0.56058	0.44932	0.40693

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According to the results presented in Tables 2-5:

1. Improving one component from the first subsystem, $|B_1| = 1$, according to HDM, will increase $\mathscr{L}(0.1)$ from $\frac{11.1086}{\Theta}$ to $\frac{13.3537}{\Theta}$, see Table 2. The same effect can occur by reducing the failure rates of (i) one component, $|A_1| = 1$ by the factor $\rho^H = 0.82531$, (ii) one component, $|A_2| = 1$, by $\rho^H = 0.000004$, (iii) two components, $|A_1| = |A_2| = 1$, by $\rho^H = 0.83101$, (iv) two components, $|A_2| = 2$, by $\rho^H = 0.59086$, (v) three components, $|A_1| = 1$, $|A_2| = 2$, by the factor, $\rho^H = 0.84060$, see Table 4.
2. Imperfect duplication of $|B_1| = 1$, will increase $\mathscr{L}(0.1)$ from $\frac{11.1086}{\Theta}$ to $\frac{16.3471}{\Theta}$, see Table 2. The same effect can be obtained by reducing the failure rates of (i) one component, $|A_1| = 1$ by the factor $\rho^I = 0.67693$, (ii) one component, $|A_2| = 1$, by $\rho^I = 0.00002$, (iii) two components, $|A_1| = |A_2| = 1$, by $\rho^I = 0.68025$, (iv) two components, $|A_2| = 2$, by $\rho^I = 0.43732$, (v) three components, $|A_1| = 1$, $|A_2| = 2$, by the factor $\rho^I = 0.69362$, see Table 4.
3. Improving one component, $|B_1| = 1$, by using CDM will increase $\mathscr{L}(0.1)$ from $\frac{11.1086}{\Theta}$ to $\frac{18.3731}{\Theta}$, see Table 2. The same effect can occur by reducing the failure rates of (i) one component, $|A_1| = 1$ by the factor $\rho^C = 0.60486$, (ii) one component, $|A_2| = 1$, by $\rho^C = 0.000001$, (iii) two components, $|A_1| = |A_2| = 1$, by the same factor $\rho^C = 0.60673$, (iv) two components, $|A_2| = 2$, by the same factor $\rho^C = 0.37978$, (v) three components, $|A_1| = 1$, $|A_2| = 2$, by the same factor $\rho^C = 0.62035$, see Table 4.
4. The rest results in Tables 4 and 5, can be explained by the same manner.
5. The notation "–" means that there is no equivalence between the reduction and duplication methods.

Table 6 shows the values of the MREF.

Table 6: The values of $\xi^D_{A,B}(\gamma)$, for $D = H, I$ and C.

A	$B_1^{(1,0)}$ H	$B_1^{(1,0)}$ I	$B_1^{(1,0)}$ C	$B_1^{(0,1)}$ H	$B_1^{(0,1)}$ I	$B_1^{(0,1)}$ C	$B_2^{(1,1)}$ H	$B_2^{(1,1)}$ I	$B_2^{(1,1)}$ C
$A_1^{(1,0)}$	0.74596	0.60181	0.53824	0.92929	0.89162	0.87607	0.72410	0.58484	0.52448
$A_1^{(0,1)}$	–	–	–	0.65761	0.50309	0.44202	–	–	–
$A_2^{(1,1)}$	0.76636	0.62050	0.55430	0.93854	0.90462	0.89038	0.74474	0.60291	0.53987
$A_2^{(0,2)}$	0.56262	0.41831	0.36435	0.83432	0.76489	0.73891	0.53818	0.40345	0.35323
$A_3^{(1,2)}$	0.79281	0.65764	0.59429	0.94638	0.91654	0.90396	0.77312	0.64095	0.58030

A	$B_2^{(0,2)}$ H	$B_2^{(0,2)}$ I	$B_2^{(0,2)}$ C	$B_3^{(1,2)}$ H	$B_3^{(1,2)}$ I	$B_3^{(1,2)}$ C
$A_1^{(1,0)}$	0.83857	0.72816	0.67386	0.69105	0.54698	0.48628
$A_1^{(0,1)}$	0.29521	–	–	–	–	–
$A_2^{(1,1)}$	0.85552	0.74877	0.69432	0.71168	0.56345	0.49972
$A_2^{(0,2)}$	0.68142	0.54263	0.48584	0.50319	0.37151	0.32325
$A_3^{(1,2)}$	0.87301	0.77680	0.72674	0.74277	0.60313	0.54096

One can conclude that, the improved system that can be obtained by improving the system according to:

1. Improving one component, $|B_1| = 1$, by the HDM, has the same expected time to failure of that system which can be obtained by reducing the failure rate of (i) one component, $|A_1| = 1$ by $\xi^H = 0.74596$, (ii) two components, $|A_1| = |A_2| = 1$, by $\xi^H = 0.76636$, (iii) two components, $|A_2| = 2$, by $\xi^H = 0.56262$, (iv) three components, $|A_1| = 1, |A_2| = 2$, by $\xi^H = 0.79281$, see Table 6.
2. Improving one component, $|B_1| = 1$,by the IDM has the same expected time to failure of that system which can be obtained by reducing the failure rate of (i) one component, $|A_1| = 1$ by $\xi^I = 0.60181$, (ii) two components, $|A_1| = |A_2| = 1$, by $\xi^I = 0.62050$, (iii) two components, $|A_2| = 2$, by $\xi^I = 0.41831$ (iv) three components, $|A_1| = 1, |A_2| = 2$, by $\xi^I = 0.65764$, see Table 6.
3. Improving one component, $|B_1| = 1$, by CDM has the same expected time to failure of that system which can be obtained by reducing the failure rate of (i) one component, $|A_1| = 1$ by the factor $\xi^C = 0.53824$, (ii) two components, $|A_1| = |A_2| = 1$, by $\xi^C = 0.55430$, (iii) two components, $|A_2| = 2$, by $\xi^C = 0.36435$ (iv) three components, $|A_1| = 1, |A_2| = 2$, by $\xi^C = 0.59429$, see Table 6.
4. In the same manner, the rest of results presented in Table 6 can be explained.

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7 Conclusion

The performance of parallel-series system based on TPLD was improved. The lifetimes of the components are assumed to be independently and identically Lindley distributed with three parameters. Four different methods were used to improve the system reliability. The reliability function and expected time to failure for each method was derived. The reliability equivalence factors and γ-fractiles were established. Numerical example was discussed to apply the theoretical results. Cold duplication method gives the best improvement than other methods.

Author's contributions

All authors equally contributed to the research article and agreed to be published.

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Competing interests

We declare that we have no competing interests.

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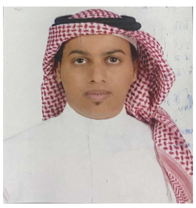

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