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Estimating the Parameters of the Generalized Inverted Kumaraswamy Distribution through the Utilization of a First Failure-Censored Sampling Plan

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Abstract: This paper focuses on the development of approximate Bayes estimators for the shape parameters of the generalized inverted Kumaraswamy (GIKum) distribution. The estimators are based on a progressive first-failure censored plan. The study considers both maximum likelihood and Bayesian estimations using a gamma-informative prior distribution for the parameters, as well as the reliability function, hazard rate, and reversed hazard rate functions. To obtain the estimators, the paper employs Lindley's approximation and utilizes Markov Chain Monte Carlo (MCMC) methods. The Bayes estimators are derived with respect to both symmetric (squared error) and asymmetric (linex and general entropy) loss functions. In order to assess the performance of the proposed estimators, the paper presents numerical results obtained through a simulation study involving different sample sizes.

Keywords: Generalized inverted Kumaraswamy distribution, Lindley's approximation, progressive first-failure censored, loss functions
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1 Introduction

Lifetime testing experiments are often time-consuming and expensive. Therefore, we used different types of censorship protocols to terminate the experiment. The censoring scheme in an experiment can also occur naturally, without the experimenter's control. For instance, in medical studies, a patient may withdraw from the study before its completion. Initially, conventional type I and type II censoring schemes were commonly used. However, in certain life tests, there may be an urgent need to utilize some test items for other purposes before the test concludes. This point was considered by Cohen [6], who introduced the progressive type II censoring scheme, which allows for the removal of items from the experiment before its final termination point. The work on progressive censoring up until 1999 was compiled by Balakrishnan and Aggarwala [2]. Progressive censoring has also been explored by other researchers such as Pradhan and Kundu [16], as well as Krishna and Kumar [10]. Real-life situations often arise where the lifetimes of items are extremely long, while test facilities remain limited. In cases where the test material is relatively inexpensive, it is possible to increase the number of items under test from n to $k \times n$. This approach involves conducting the experiment with n sets or groups, each consisting of k items. Only the first failure is observed within each set, and a progressive censoring strategy is employed across the n groups. The grouping of units and observation of only the first failure was studied by Johnson [9]. Other studies exploring this grouping approach were conducted by Balasooriya [4], Wu et al. [19], and Wu and Yu [21]. The combination of first-failure observation and progressive censoring is referred to as the progressive first-failure censoring scheme, as introduced by Wu and Kus [20]. They developed estimation methods for the Weibull distribution under this novel censoring plan. More recent references on this topic can be found in Lio and Tsai [14], Kumar et al. [11], and Dube et al. [7]. Now, let's delve into the details of the progressive first-failure censoring scheme. Assuming $k \times n$ items are placed on test, distributed among n independent groups, with k items in each group, we adopt a progressive

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censoring scheme denoted as $\underline{R} = (R_1, R_2, ..., R_m)$. When the first failure occurs in a unit, the group in which the first failure happened is removed, along with R_1 additional groups randomly selected from the remaining $(n-1)$ groups. Upon the occurrence of the second failure, the group with the second failure and R_2 additional groups are randomly removed from the remaining $(n - R_1 - 2)$ groups, and so on. This procedure continues until the mth failure occurs, at which point the remaining R_m groups, along with the group in which the last failure took place, are removed.

It is evident that $\sum_{i=1}^{m} R_i + m = n$. Furthermore, if $R_1 = R_2 = ... = R_m = 0$, the progressive first-failure censoring scheme reduces to the first-failure censoring scheme. Similarly, if $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n-m$, it reduces to the first-failure type II censoring scheme, specifically the progressive type II censored scheme with $k = 1$. It is worth noting that the progressive first-failure censoring scheme with a cumulative distribution function (cdf) $F(y)$ can be interpreted as a progressive type II censored sample from a population with a cdf of $1-(1-F(y))^k$. Consequently, results obtained for the progressive type II censored scheme can be readily extended to the progressive first-failure censoring scheme. Therefore, progressive first-failure censoring serves as a generalization of progressive censoring. Despite employing a larger number of items in the progressive first-failure censoring plan compared to other schemes, it offers the advantage of reducing test time.

Let $y_{1:m:n:k},\ y_{2:m:n:k}, ...,\ y_{m:m:n:k}$ denote a progressive first-failure censored sample obtained from a population with a probability density function (pdf) $f(.)$ and cdf $F(.)$, using the progressive censoring scheme $\underline{R}$. For simplicity, we can represent $(y_{1:m:n:k},\ y_{2:m:n:k}, ...,\ y_{m:m:n:k})$ as $\underline{y} = (y_1,\ y_2, ...,\ y_m)$. Based on a progressive first-failure censored sample $\underline{y}$, the likelihood function is given by [refer to Balakrishnan and Aggarwala [2] and Wu and Kus [20]].

$$L(\underline{y}) = \tau k^m \prod_{i=1}^{m} f(y_i)[1-F(y_i)]^{(k(R_i+1)-1)}, \tag{1}$$

where $\tau = n(n-1-R_1)...(n-R_1-...-R_{m-1}-m+1)$.

In recent years, there has been a growing interest in the utilization of inverted distributions for data analysis in various fields such as medical, economic, and engineering sciences, lifetime analysis, finance, and insurance.

The Kumaraswamy (Kum) distribution, introduced by Kumaraswamy [12], is defined on the interval (0,1). While it shares similarities with the Beta distribution, the Kum distribution is much simpler to use, particularly in simulation studies, due to its closed-form expressions for the probability density function (pdf) and cumulative distribution function (cdf). For further details on this family of distributions, refer to Barakat et al. [5]. Abd Al-Fattah et al. [1] derived the inverted Kumaraswamy (IKum) distribution through certain transformations applied to the original distribution. Additionally, Iqbal et al. [**?**] derived the GIKum distribution by incorporating a power transformation. The pdf and cdf of the GIKum distribution are given by:

$$f(y;\ \lambda,\ \eta,\ \kappa) = \lambda\ \eta\ \kappa\ y^{\kappa-1}\ (1+y^{\kappa})^{-(\lambda+1)}\ [1-(1+y^{\kappa})^{-\lambda}]^{\eta-1}\ ,\ y>0,\ \lambda,\ \eta,\ \kappa>0 \tag{2}$$

and

$$F(y;\ \lambda,\ \eta,\ \kappa) = \ [1-(1+y^{\kappa})^{-\eta}]^{\lambda}\ ,\ y>0,\ \lambda,\ \eta,\ \kappa>0. \tag{3}$$

The main objective of this paper is to estimate the parameters of the GIKum distribution using Bayes estimators. Both the maximum likelihood estimation (MLE) and Bayesian methods are obtained based on progressive first-failure censoring schemes. The paper is organized as follows: Section 2 covers the MLE of the unknown parameters, along with discussions on reliability functions, hazard rates, and reversed hazard rates. In Section 3, Bayes estimates are calculated using Lindley's approximation, as described in Lindley [13]. Section 4 presents a Monte Carlo simulation to compare the various estimates proposed in this paper. Finally, Section 5 provides concluding remarks.

2 Estimators Based on Maximum Likelihood

In this section, we will derive the Maximum Likelihood Estimators (MLEs) of the unknown parameters, as well as the reliability, hazard rate, and reversed hazard rate functions, based on progressive first-failure censored samples. We assume that the failure time distribution follows the GIKum distribution, with the probability density function (pdf) and cumulative distribution function (cdf) defined in equations (2) and (3) respectively. By combining equations (1), (2), and (3), the likelihood function can be expressed as follows:

$$L(\underline{y};\ \lambda,\ \eta,\ \kappa) = \ \tau(k\lambda\eta\kappa)^m \prod_{i=1}^{m}(y_i^{\kappa-1}(1+y_i^{\kappa})^{-(\lambda+1)}[1-(1+y_i^{\kappa})^{-\lambda}]^{\eta-1}) \times \prod_{i=1}^{m}(1-[1-(1+y_i^{\kappa})^{-\eta}]^{\lambda})^{(k(R_i+1)-1)}. \tag{4}$$

Taking the logarithm of the likelihood function yields the following expression:

$$\log L = \ell = \log\tau + m\log[k\lambda\eta\kappa] + (\kappa-1)\sum_{i=1}^{m}\log y_i - (\lambda+1)\sum_{i=1}^{m}\log[1+y_i^{\kappa}] + (\eta-1)\sum_{i=1}^{m}\log[1-(1+y_i^{\kappa})^{-\lambda}]$$

$$+\sum_{i=1}^{m}((k(R_i+1)-1)\log[1-(1-(1+y_i^{\kappa})^{-\eta})^{\lambda}]). \tag{5}$$

By calculating the first partial derivatives of (5) with respect to λ, η, and κ and setting them equal to zero, we can obtain the likelihood equations:

$$\left.\begin{array}{l}
\frac{m}{\lambda} - \sum_{i=1}^{m}\log[1+y_i^{\kappa}] + (\eta-1)\sum_{i=1}^{m}\frac{(1+y_i^{\kappa})^{-\lambda}\log[1+y_i^{\kappa}]}{1-(1+y_i^{\kappa})^{-\lambda}} \\
= \sum_{i=1}^{m}\frac{(k(1+R_i)-1)\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}\log\left[1-(1+y_i^{\kappa})^{-\eta}\right]}{1-\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}}, \\
\frac{m}{\eta} + \sum_{i=1}^{m}\log\left[1-(1+y_i^{\kappa})^{-\lambda}\right] \\
= \sum_{i=1}^{m}\frac{\lambda(k(1+R_i)-1)(1+y_i^{\kappa})^{-\eta}\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda-1}\log[1+y_i^{\kappa}]}{1-\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}}, \\
\frac{m}{\kappa} + \sum_{i=1}^{m}\log y_i - (1+\lambda)\sum_{i=1}^{m}\frac{y_i^{\kappa}\log y_i}{1+y_i^{\kappa}} + (\eta-1)\sum_{i=1}^{m}\frac{\lambda y_i^{\kappa}(1+y_i^{\kappa})^{-(\lambda+1)}\log y_i}{1-(1+y_i^{\kappa})^{-\lambda}} \\
= \sum_{i=1}^{m}\frac{\eta\lambda(k(1+R_i)-1)y_i^{\kappa}(1+y_i^{\kappa})^{-(\eta+1)}\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda-1}\log y_i}{1-\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}}.
\end{array}\right\} \tag{6}$$

The solutions to the non-linear equations (6) correspond to the Maximum Likelihood Estimators (MLEs) of the parameters, denoted as $\hat{\lambda}$, $\hat{\eta}$, and $\hat{\kappa}$. Additionally, the MLEs of the reliability, hazard rate, and reversed hazard rate functions can be expressed as follows:

$$\hat{R}(t) = 1-\left[1-\left(1+t^{\hat{\kappa}}\right)^{-\hat{\lambda}}\right]^{\hat{\eta}}, t>0,$$

$$\hat{H}(t) = \frac{\hat{\lambda}\,\hat{\eta}\,\hat{\kappa}\,t^{\hat{\kappa}-1}\left[1-\left(1+t^{\hat{\kappa}}\right)^{-\hat{\lambda}}\right]^{\hat{\eta}-1}}{\left(1+t^{\hat{\kappa}}\right)^{\hat{\lambda}+1}\left(1-\left(1-\left(1+t^{\hat{\kappa}}\right)^{-\hat{\lambda}}\right)^{\hat{\eta}}\right)}, t>0,$$

and

$$\hat{H}^{*}(t) = \frac{\hat{\lambda}\,\hat{\eta}\hat{\kappa}t^{\hat{\kappa}-1}}{\left(1+t^{\hat{\kappa}}\right)^{\hat{\lambda}+1}\left[1-(1+t^{\hat{\kappa}})^{-\hat{\lambda}}\right]}, t>0.$$

3 Estimation Using Bayesian Methods

In this section, we will derive the Bayesian estimators for the unknown parameters λ, η, and κ of the GIKum distribution. Additionally, we will study the reliability, hazard rate, and reversed hazard rate functions based on progressive first-failure censoring samples. We will consider both symmetric (squared error) and asymmetric (linex and general entropy) loss functions.

Furthermore, we will utilize Lindley's approximation and Markov Chain Monte Carlo (MCMC) methods to obtain the Bayesian estimators.

We assume that $\lambda,\ \eta$, and κ are independent random variables with informative prior distributions, specifically gamma distributions, which can be defined as follows:

$$\pi_1(\lambda;\ \zeta_1,\ \nu_1) = \frac{e^{-\zeta_1 \eta} {\zeta_1}^{\nu_1}}{\Gamma(\nu_1)}\ \lambda^{\nu_1-1};\ \ \lambda > 0,\ (\zeta_1,\ \nu_1 > 0),$$

$$\pi_2(\eta;\ \zeta_2,\ \nu_2) = \frac{e^{-\zeta_2 \eta} {\zeta_2}^{\nu_2}}{\Gamma(\nu_2)}\ \eta^{\nu_2-1};\ \ \eta > 0,\ (\zeta_2,\ \nu_2 > 0),$$

and

$$\pi_3(\kappa;\ \zeta_3,\ \nu_3) = \frac{e^{-\zeta_3 \kappa} {\zeta_3}^{\nu_3}}{\Gamma(\nu_3)}\ \kappa^{\nu_3-1};\ \ \kappa > 0,\ (\zeta_3,\ \nu_3 > 0).$$

Then the joint prior distribution for $\lambda,\ \eta$, and κ is defined by

$$\pi(\lambda,\ \eta,\ \kappa) = \frac{e^{-(\zeta_1\lambda+\zeta_2\eta+\zeta_3\kappa)}\ {\zeta_1}^{\nu_1}\ {\zeta_2}^{\nu_2}\ {\zeta_3}^{\nu_3}}{\Gamma(\nu_1)\ \Gamma(\nu_2)\ \Gamma(\nu_3)}\ \lambda^{\nu_1-1}\ \eta^{\nu_2-1}\ \kappa^{\nu_3-1}; \tag{7}$$

$$\lambda > 0,\ \eta > 0,\ \kappa > 0,\ (\zeta_1,\ \nu_1,\ \zeta_2,\ \nu_2,\ \zeta_3,\ \nu_3 > 0).$$

By utilizing equations (4) and (7), we can derive the posterior distribution of $\lambda,\ \eta$ and κ as follows:

$$\pi\left(\lambda,\ \eta,\ \kappa \,|\, \underline{y}\right) = \frac{\alpha\ \beta}{\int_0^\infty \int_0^\infty \int_0^\infty \alpha\ \beta\ d\lambda\ d\eta\ d\kappa}, \tag{8}$$

where

$$\alpha = e^{-(\zeta_1\lambda+\zeta_2\eta+\zeta_3\kappa)}\ \lambda^{\nu_1+m-1}\ \eta^{\nu_2+m-1}\ \kappa^{\nu_3+m-1} \prod_{i=1}^{m} y_i^{\kappa-1}(1+{y_i}^{\kappa})^{-(\lambda+1)},$$

and

$$\beta = \prod_{i=1}^{m}\left[1-(1+{y_i}^{\kappa})^{-\lambda}\right]^{\eta-1}\left(1-\left[1-(1+{y_i}^{\kappa})^{-\eta}\right]^{\lambda}\right)^{(k(R_i+1)-1)}.$$

Since the integration in equation (8) cannot be obtained in closed form, we need to solve it numerically. In the subsequent subsections, we will derive Bayesian estimators for the parameters $\lambda,\ \eta,\ \kappa$, as well as the reliability, hazard rate, and reversed hazard rate functions under various loss functions.

3.1 Bayesian estimator under the squared error loss function

1.The Bayesian estimator for the parameter λ is given by

$$\hat{\lambda}_{sq} = E(\lambda) = \int_0^\infty \int_0^\infty \int_0^\infty \left(\lambda\ \pi\left(\lambda,\ \eta,\ \kappa \,|\, \underline{y}\right)\right)\ d\lambda\ d\eta\ d\kappa,$$

provided that $E(\lambda)$ exists. Since this integral cannot be solved analytically,we use Lindley's Bayesian approximation for any function ψ of parameter $\omega,\ \omega = (\theta_1,\ \theta_2,\ \theta_3)$ and $Q(\theta_1,\ \theta_2,\ \theta_3) = \log \pi(\theta_1,\ \theta_2,\ \theta_3)$, which is defined by

$$E\left(\psi(\omega) \,|\, \underline{y}\right) \approx \left(\psi(\theta_1,\ \theta_2,\ \theta_3) + \frac{1}{2}\left[\sum_r \sum_s (\psi_{rs} + 2\psi_r Q_s)\,\sigma_{rs}\ + \sum_r \sum_s \sum_z \sum_w L_{rsz}\psi_w \sigma_{rs}\sigma_{zw}\right]\right)_{(\theta_1,\ \theta_2,\ \theta_3)_{ML}}, \tag{9}$$

$$\forall\ r,\ s,\ z,\ w = 1,\ 2,\ 3,$$

where $Q_i = \frac{\partial Q(\theta_1,\ \theta_2,\ \theta_3)}{\partial \theta_i}$, $\psi_i = \frac{\partial \psi(\theta_1,\ \theta_2,\ \theta_3)}{\partial \theta_i}$, $\psi_{ij} = \frac{\partial^2 \psi(\theta_1,\ \theta_2,\ \theta_3)}{\partial \theta_i\ \partial \theta_j}$, $L_{ij} = \frac{\partial^2 \ell}{\partial \theta_i\ \partial \theta_j}$, $L_{ijk} = \frac{\partial^3 \ell}{\partial \theta_i\ \partial \theta_j\ \partial \theta_k}$, $\forall\ i,\ j,\ k = 1,\ 2,\ 3$, and $\sigma_{ij} = (i,j)^{th}$ element in the matrix

$$\begin{pmatrix} -L_{11} & -L_{12} & -L_{13} \\ -L_{21} & -L_{22} & -L_{23} \\ -L_{31} & -L_{32} & -L_{33} \end{pmatrix}^{-1},\forall\ i,\ j = 1,\ 2,\ 3.$$

Substitute in the equation (9), $\psi=\lambda$, the Bayesian estimator of the shape parameter λ is given as

$$\hat{\lambda}_{sq}\approx\lambda+Q_1\sigma_{11}+Q_2\sigma_{12}+Q_3\sigma_{13}+\frac{1}{2}\left(A\sigma_{11}+B\sigma_{21}+C\sigma_{31}\right),$$

where

$$A=\sigma_{11}L_{111}+\sigma_{22}L_{221}+\sigma_{33}L_{331}+2\left(\sigma_{12}L_{121}+\sigma_{13}L_{131}+\sigma_{23}L_{231}\right),$$
$$B=\sigma_{11}L_{112}+\sigma_{22}L_{222}+\sigma_{33}L_{332}+2\left(\sigma_{12}L_{122}+\sigma_{13}L_{132}+\sigma_{23}L_{232}\right),$$
$$C=\sigma_{11}L_{113}+\sigma_{22}L_{223}+\sigma_{33}L_{333}+2\left(\sigma_{12}L_{123}+\sigma_{13}L_{133}+\sigma_{23}L_{233}\right).$$

2.Substitute into equation (9), $\psi=\eta$, the Bayesian estimator of the parameter η is given as

$$\hat{\eta}_{sq}\approx\left(\eta+Q_1\sigma_{21}+Q_2\sigma_{22}+Q_3\sigma_{23}+\frac{1}{2}\left(A\sigma_{12}+B\sigma_{22}+C\sigma_{32}\right)\right).$$

3.Substitute in the equation (9), $\psi=\kappa$, the Bayesian estimator of the parameter κ is given as

$$\hat{\kappa}_{sq}\approx\left(\kappa+Q_1\sigma_{31}+Q_2\sigma_{32}+Q_3\sigma_{33}+\frac{1}{2}\left(A\sigma_{13}+B\sigma_{23}+C\sigma_{33}\right)\right).$$

4.Substitute into equation (9), $\psi=R\left(t\right)$, the Bayesian estimator of the reliability function $R(t)$ is given as

$$\hat{R}_{sq}\left(t\right)\approx R\left(t\right)+\left(\psi_1a_1+\psi_2a_2+\psi_3a_3+a_4+a_5\right)+\frac{1}{2}[A\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}\right)$$
$$+B\left(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}\right)+C\left(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33}\right)],$$

where

$$a_i=Q_1\sigma_{i1}+Q_2\sigma_{i2}+Q_3\sigma_{i3};i=1,2,3,$$
$$a_4=\psi_{12}\sigma_{12}+\psi_{13}\sigma_{13}+\psi_{23}\sigma_{23},$$
$$a_5=\frac{1}{2}\left(\psi_{11}\sigma_{11}+\psi_{22}\sigma_{22}+\psi_{33}\sigma_{33}\right).$$

5.Substitute in the equation (9), $\psi=H\left(t\right)$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{sq}\left(t\right)\approx H\left(t\right)+\left(\psi_1a_1+\psi_2a_2+\psi_3a_3+a_4+a_5\right)+\frac{1}{2}[A\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}\right)]$$
$$+B\left(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}\right)+C\left(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33}\right).$$

6.Substitute into equation (9), $\psi=H^{\star}\left(t\right)$, the Bayesian estimator of the reversed hazard rate function $H^{\star}(t)$ is given by

$$\hat{H}^{\star}_{sq}\left(t\right)\approx H^{\star}\left(t\right)+\left(\psi_1a_1+\psi_2a_2+\psi_3a_3+a_4+a_5\right)+\frac{1}{2}[A\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}\right)$$
$$+B\left(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}\right)+C\left(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33}\right)].$$

3.2 Bayesian Estimators Under Linear-Exponential Loss Function (LINEX)

1.Substitute into equation (9), $\psi=e^{-\rho\lambda}$, the Bayesian estimator of the shape parameter λ is given as

$$\hat{\lambda}_{LINEX}\approx$$
$$-\tfrac{1}{\rho}\log\left[e^{-\rho\lambda}-\tfrac{\rho}{e^{\lambda\rho}}\left(Q_1\sigma_{11}+Q_2\sigma_{12}+Q_3\sigma_{13}\right)+\tfrac{\rho^2}{2e^{\lambda\rho}}\sigma_{11}-\tfrac{\rho}{2e^{\lambda\rho}}\left(A\sigma_{11}+B\sigma_{21}+C\sigma_{31}\right)\right].$$

2.The Bayesian estimator for the parameter η is given by

$$\hat{\eta}_{LINEX}=-\frac{1}{\rho}\log\left[E\left(e^{-\rho\eta}\right)\right],$$

provided that $E\left(e^{-\rho\eta}\right)$ exists. Substitute into equation (9), $\psi=e^{-\rho\eta}$, the Bayesian estimator of the parameter η is given by

$$\hat{\eta}_{LINEX}\approx$$
$$-\tfrac{1}{\rho}\log\left[e^{-\rho\eta}-\tfrac{\rho}{e^{\eta\rho}}\left(Q_1\sigma_{21}+Q_2\sigma_{22}+Q_3\sigma_{23}\right)+\tfrac{\rho^2}{2e^{\eta\rho}}\sigma_{22}-\tfrac{\rho}{2e^{\eta\rho}}\left(A\sigma_{12}+B\sigma_{22}+C\sigma_{32}\right)\right].$$

3.Substitute into equation (9), $\psi = e^{-\rho\kappa}$, the Bayesian estimator of the shape parameter κ is given by

$$\hat{\kappa}_{LINEX} \approx -\frac{1}{\rho}\log\left[e^{-\rho\kappa} - \frac{\rho}{e^{\kappa\rho}}\left(Q_1\sigma_{31} + Q_2\sigma_{32} + Q_3\sigma_{33}\right) + \frac{\rho^2}{2e^{\kappa\rho}}\sigma_{33} - \frac{\rho}{2e^{\kappa\rho}}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)\right].$$

4.Substitute into equation (9), $\psi = e^{-\rho R(t)}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{LINEX}(t) \approx -\frac{1}{\rho}\log\left[e^{-\rho R(t)} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right)\right.$$
$$\left.+\frac{1}{2}[A(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}) + B(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}) + C(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33})]\right].$$

5.Substitute into equation (9), $\psi = e^{-\rho H(t)}$, the Bayesian estimator of the hazard rate function $H(t)$ is given as

$$\hat{H}_{LINEX}(t) \approx -\frac{1}{\rho}\log\left[e^{-\rho H(t)} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right)\right.$$
$$\left.+\frac{1}{2}[A(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}) + B(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}) + C(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33})]\right].$$

6.Substitute into equation (9), $\psi = e^{-\rho H^\star(t)}$, the Bayesian estimator of the reversed hazard rate function $H^\star(t)$ is given as

$$\hat{H}^\star_{LINEX}(t) \approx -\frac{1}{\rho}\log\left[e^{-\rho H^\star(t)} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right)\right.$$
$$\left.+\frac{1}{2}[A(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}) + B(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}) + C(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33})]\right].$$

3.3 Bayesian estimator under general entropy loss function

1.The Bayesian estimator for the shape parameter λ is given by

$$\hat{\lambda}_{Gentropy} = \left[E\left(\lambda^{-q}\right)\right]^{\frac{-1}{q}},$$

provided that $E\left(\lambda^{-q}\right)$ exists. Substitute into equation (9), $\psi = \lambda^{-q}$, the Bayesian estimator of the parameter λ is given by

$$\hat{\lambda}_{\text{Gentropy}} \approx \begin{bmatrix} \lambda^{-q} - q\lambda^{-(q+1)}\left(Q_1\sigma_{11} + Q_2\sigma_{12} + Q_3\sigma_{13}\right) + \frac{\left((q+1)\,q\lambda^{-(q+2)}\right)}{2}\sigma_{11} \\ -\frac{q\lambda^{-(q+1)}}{2}\left(A\sigma_{11} + B\sigma_{21} + C\sigma_{31}\right) \end{bmatrix}^{\frac{-1}{q}}.$$

2.Substitute into equation (9), $\psi = \eta^{-q}$, the Bayesian estimator of the shape parameter η is given by

$$\hat{\eta}_{\text{Gentropy}} \approx \begin{bmatrix} \eta^{-q} - q\eta^{-(q+1)}\left(Q_1\sigma_{21} + Q_2\sigma_{22} + Q_3\sigma_{23}\right) + \frac{\left((q+1)\,q\eta^{-(q+2)}\right)}{2}\sigma_{22} \\ -\frac{q\eta^{-(q+1)}}{2}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right) \end{bmatrix}^{\frac{-1}{q}}.$$

3.Substitute into equation (9), $\psi = \kappa^{-q}$, the Bayesian estimator of the shape parameter κ is given by

$$\hat{\kappa}_{\text{Gentropy}} \approx \begin{bmatrix} \kappa^{-q} - q\kappa^{-(q+1)}\left(Q_1\sigma_{31} + Q_2\sigma_{32} + Q_3\sigma_{33}\right) + \frac{\left((q+1)\,q\kappa^{-(q+2)}\right)}{2}\sigma_{33} \\ -\frac{q\kappa^{-(q+1)}}{2}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right) \end{bmatrix}^{\frac{-1}{q}}.$$

4.Substitute into equation (9), $\psi = R(t)^{-q}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{Gentropy}(t) \approx \left[(R(t))^{-q} + (\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5) + \frac{1}{2} \begin{bmatrix} A(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}) \\ +B(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}) \\ +C(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}) \end{bmatrix} \right]^{\frac{-1}{q}}.$$

5.Substitute into equation (9), $\psi = H(t)^{-q}$, the Bayesian estimator of the hazard rate function $H(t)$ is given as

$$\hat{H}_{Gentropy}(t) \approx \left[(H(t))^{-q} + (\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5) + \frac{1}{2} \begin{bmatrix} A(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}) \\ +B(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}) \\ +C(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}) \end{bmatrix} \right]^{\frac{-1}{q}}.$$

6.Substitute in the equation (9), $\psi = H^{\star}(t)^{-q}$, the Bayesian estimator of the reversed hazard rate function $H^{\star}(t)$ is given as

$$\hat{H}^{\star}_{Gentropy}(t) \approx \left[(H^{\star}(t))^{-q} + (\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5) + \frac{1}{2} \begin{bmatrix} A(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}) \\ +B(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}) \\ +C(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}) \end{bmatrix} \right]^{\frac{-1}{q}}.$$

4 Simulation studies

In this section, we conduct a Monte Carlo simulation study to compare the performance of the various estimates developed in the previous sections. A large number of (10000) progressively first-time failure censored samples are generated from model (4). These generated samples have different combinations of $k = 3;\ 6$; number of groups in the sample $n = 50;\ 80$, effective sample size $m = 25;\ 40$ out of n, and progressive censoring scheme $\underline{R}$.

This study will include the following steps:

1.To obtain a progressive first-failure censored sample based on the model (4) and the specified values of $(k,n,m,\underline{R})$, we can employ the algorithm proposed by Balakrishnan and Sandhu [3].
2.Section 2 provides the necessary guidelines for calculating the maximum likelihood estimates of λ, η, κ, $R(t)$, $H(t)$, and $H^{\star}(t)$. Apply these guidelines to obtain the respective maximum likelihood estimates.
3.Referencing Section 3, compute the Bayes estimates of λ, η, κ, $R(t)$, $H(t)$, and $H^{\star}(t)$.
4.Repeat steps (1)-(3) a total of 10,000 times, using varying values of $(k,n,m,\underline{R})$.

In this section, we examine two key performance measures: the estimation average and the mean square error. The estimation average $= \frac{\sum_{i=1}^{10000} \hat{\theta}_i}{10000}$, the mean square error $= \frac{\sum_{i=1}^{10000} (\hat{\theta}_i - \theta)^2}{10000}$, where θ is the true parameter and $\hat{\theta}$ is its estimator. Due to the non-analytical solvability of the nonlinear equations (6), extensive computations are carried out using Mathematica 11. Numerical methods, such as the Newton-Raphson method, are employed with initial values close to the actual parameter values.
Throughout this section, we will use the following abbreviations for brevity and clarity:

1.$MSEs$: Mean square errors.
2.ML: Maximum likelihood estimate.
3.B_{Sq}: Bayes estimate under squared error loss function.
4.$B_{Lx,c=3}$: Bayes estimate under linex loss function with $c = 3$.
5.$B_{Lx,c=6}$: Bayes estimate under linex loss function with $c = 6$.
6.$B_{Ge,q=4}$: Bayes estimate under general entropy loss function with $q = 4$.
7.$B_{Ge,q=8}$: Bayes estimate under general entropy loss function with $q = 8$.

Table 1. The average values of the estimates for the parameters $\lambda,\ \eta,\ \kappa$, along with their corresponding mean square errors (MSEs) in parentheses. These estimates are computed under the conditions $\lambda = 1.2, \eta = 0.7, \kappa = 0.9, \zeta_1 = 2, \nu_1 = 3, \zeta_2 = 2, \nu_2 = 3$, and $\zeta_3 = 2,\ \nu_3 = 3$.

$B_{Lx,c=3}$	$B_{Lx,c=6}$	$B_{Ge,q=4}$	$B_{Ge,q=8}$	B_{Sq}	ML	Scheme	(k,n,m)
The average estimates of λ (provided with the MSEs)							
1.35944	1.68121	1.27857	1.19887	1.31237	1.25973	(10,18*0,10)	(3,40,20)
(0.02184)	(0.18051)	(0.00449)	(0.00026)	(0.00002)	(0.05445)		
1.30942	1.4555	1.2572	1.2244	1.2731	1.2469	(20,19*0)	
(0.02142)	(0.17671)	(0.02441)	(0.00261)	(0.00003)	(0.04371)		
1.40032	1.27784	1.25572	1.23663	1.26893	1.22817	(19*0,20)	
(0.03651)	(0.021420)	(0.00191)	(0.00062)	(0.00004)	(0.04016)		
1.32197	1.37235	1.25391	1.2262	1.26054	1.23437	(20,38*0,20)	(6,80,40)
(0.01206)	(0.03731)	(0.00172)	(0.00021)	(0.00006)	(0.02232)		
1.29565	1.37632	1.25821	1.22872	1.27255	1.24205	(40,39*0)	
(0.00694)	(0.02735)	(0.00211)	(0.00272)	(0.00007)	(0.02094)		
1.3478	1.39252	1.29421	1.22993	1.2842	1.25731	(39*0,40)	
(0.02521)	(0.04373)	(0.00578)	(0.00488)	(0.00894)	(0.03191)		
The average estimates of η (provided with the MSEs)							
0.76092	0.75497	0.73011	0.71703	0.77173	0.73354	(10,18*0,10)	(3,40,20)
(0.00292)	(0.002282)	(0.00052)	(0.00011)	(0.00418)	(0.01620)		
0.75342	0.74311	0.72110	0.71320	0.75183	0.72161	(20,19*0)	
(0.00186)	(0.002271)	(0.00412)	(0.00012)	(0.00324)	(0.01420)		
0.76736	0.76834	0.73023	0.72221	0.75141	0.72486	(19*0,20)	
(0.000362)	(0.00374)	(0.00053)	(0.00022)	(0.00196)	(0.02153)		
0.74137	0.73562	0.72330	0.70886	0.73372	0.71712	(20,38*0,20)	(6,80,40)
(0.00116)	(0.00081)	(0.00026)	(0.00043)	(0.00071)	(0.00978)		
0.75276	0.72147	0.73750	0.69854	0.75754	0.73563	(40,39*0)	
(0.00206)	(0.00020)	(0.00093)	(0.00007)	(0.00252)	(0.00988)		
0.75482	0.73024	0.73121	0.70817	0.75043	0.72993	(39*0,40)	
(0.00316)	(0.00021)	(0.00081)	(0.00067)	(0.00194)	(0.00873)		
The average estimates of κ (provided with the MSEs)							
1.00306	0.92721	0.97194	0.89376	1.04754	0.98601	(10,18*0,10)	(3,40,20)
(0.00877)	(0.00031)	(0.00391)	(0.00078)	(0.01912)	(0.03760)		
0.96401	0.92512	0.95410	0.90124	0.99841	0.94261	(20,19*0)	
(0.00672)	(0.00032)	(0.00291)	(0.00054)	(0.00943)	(0.02541)		
0.99701	0.92188	0.95800	0.88004	1.00548	0.95994	(19*0,20)	
(0.00771)	(0.00015)	(0.00237)	(0.00083)	(0.00925)	(0.03973)		
0.96373	0.93342	0.94442	0.90891	0.96632	0.94035	(20,38*0,20)	(6,80,40)
(0.00296)	(0.00058)	(0.00123)	(0.00001)	(0.00324)	(0.01567)		
0.95894	0.91474	0.94266	0.89345	0.97461	0.94318	(40,39*0)	
(0.00246)	(0.00003)	(0.00113)	(0.00024)	(0.00426)	(0.01732)		
0.96941	0.93681	0.95321	0.91861	0.97643	0.95321	(39*0,40)	
(0.00345)	(0.00043)	(0.00293)	(0.00942)	(0.00541)	(0.02413)		

Table 2. The average values of the estimates for the reliability, hazard rate, and reversed hazard rate functions, accompanied by their respective mean square errors (MSEs) in parentheses. These estimates are computed under the conditions $\lambda = 1.2, \eta = 0.7, \kappa = 0.9, \zeta_1 = 2, \nu_1 = 3, \zeta_2 = 2, \nu_2 = 3$, and $\zeta_3 = 2,\ \nu_3 = 3$.

$B_{Lx,c=3}$	$B_{Lx,c=6}$	$B_{Ge,q=4}$	$B_{Ge,q=8}$	B_{Sq}	ML	Scheme	(k,n,m)
The average estimates of relaibility function R(t=2)=0.166423 (*with the MSEs*)							
0.17911	0.16931	0.16722	0.16687	0.17942	0.16747	(10,18*0,10)	(3,40,20)
(0.00213)	(0.00212)	(0.00202)	(0.00043)	(0.00432)	(0.00362)		
0.16544	0.16697	0.16843	0.16649	0.16941	0.16720	(20,19*0)	
(0.00146)	(0.00742)	(0.00426)	(0.00863)	(0.00123)	(0.02611)		
0.16844	0.16978	0.16852	0.16792	0.17242	0.16252	(19*0,20)	
(0.01541)	(0.00856)	(0.07023)	(0.00134)	(0.00751)	(0.04214)		
0.16744	0.16671	0.16653	0.16633	0.16943	0.16694	(20,38*0,20)	(6,80,40)
(0.00323)	(0.00332)	(0.00422)	(0.00363)	(0.00445)	(0.00653)		
0.16842	0.16644	0.16642	0.166384	0.166872	0.16743	(40,39*0)	
(0.00125)	(0.00011)	(0.00032)	(0.00054)	(0.00156)	(0.00871)		
0.16793	0.16711	0.16832	0.16856	0.16997	0.16774	(39*0,40)	
(0.09472)	(0.08120)	(0.00574)	(0.00445)	(0.00842)	(0.00493)		
The average estimates of hazard rate function H(t=2)=0.427115 (*with the MSEs*)							
0.49872	0.49432	0.54467	0.49935	0.50242	0.46943	(10,18*0,10)	(3,40,20)
(0.00761)	(0.00833)	(0.00092)	(0.00433)	(0.00231)	(0.01343)		
0.46423	0.45872	0.45992	0.48232	0.49702	0.45871	(20,19*0)	
(0.00695)	(0.02724)	(0.00026)	(0.00294)	(0.00122)	(0.00297)		
0.54212	0.48682	0.46831	0.47331	0.49942	0.47813	(19*0,20)	
(0.00623)	(0.00512)	(0.00132)	(0.00297)	(0.02565)	(0.00246)		
0.46833	0.48782	0.46283	0.47682	0.49202	0.44681	(20,38*0,20)	(6,80,40)
(0.03802)	(0.00077)	(0.00322)	(0.00876)	(0.02462)	(0.01324)		
0.45255	0.46864	0.42643	0.42947	0.45792	0.43244	(40,39*0)	
(0.00003)	(0.00004)	(0.00002)	(0.00002)	(0.00045)	(0.03221)		
0.47892	0.48975	0.47793	0.46986	0.49922	0.45688	(39*0,40)	
(0.00268)	(0.00093)	(0.00242)	(0.00393)	(0.03761)	(0.00474)		
The average estimates of reversed hazard rate function H*(t=2)=0.0852734 (*with the MSEs*)							
0.1287	0.10941	0.19986	0.09873	0.0242	0.10994	(10,18*0,10)	(3,40,20)
(0.02967)	(0.09822)	(0.09947)	(0.06287)	(0.06818)	(0.04872)		
0.0383	0.04892	0.05978	0.06872	0.06682	0.05482	(20,19*0)	
(0.000842)	(0.00711)	(0.00324)	(0.07863)	(0.03422)	(0.00523)		
0.13873	0.11873	0.09986	0.05482	0.03393	0.10023	(19*0,20)	
(0.02312)	(0.02462)	(0.01242)	(0.00943)	(0.01264)	(0.09462)		
0.09947	0.08946	0.07688	0.07682	0.06813	0.09842	(20,38*0,20)	(6,80,40)
(0.00244)	(0.07076)	(0.05946)	(0.00396)	(0.00868)	(0.00621)		
0.08924	0.09322	0.085242	0.085122	0.08686	0.08872	(40,39*0)	
(0.00004)	(0.00872)	(0.00001)	(0.00001)	(0.00021)	(0.00363)		
0.08874	0.09862	0.09784	0.08933	0.07813	0.08932	(39*0,40)	
(0.00253)	(0.01541)	(0.00527)	(0.00543)	(0.00343)	(0.00183)		

By examining Tables 1 and 2, we can observe that both the maximum likelihood estimate (MLE) and Bayes estimates for the parameters λ, η, κ, as well as the reliability, hazard rate, and reversed hazard rate functions, demonstrate excellent performance in terms of mean squared errors ($MSEs$). As the number of groups (n) and the effective sample size (m) increase, we can expect a decrease in the $MSEs$ for all estimates, which aligns with our expectations. Furthermore, when the group size (k) increases, the $MSEs$ show a similar decreasing trend.

In general, the Bayesian estimators exhibit lower $MSEs$ compared to the MLE. This can be attributed to the incorporation of prior information in the Bayesian framework, which enhances the accuracy of the estimates. Specifically, Bayes estimates utilizing a gamma informative prior display superior performance in terms of $MSEs$ when compared to the MLE, primarily due to the inclusion of relevant prior knowledge.

5 Concluding remarks

In this research paper, we focus on the estimation of unknown parameters λ, η, κ, as well as the reliability, hazard rate, and reversed hazard rate functions, assuming a reliable lifetime model. Our approach involves utilizing progressive first-failure censored samples. This censoring scheme offers advantages in terms of reducing test time, where a larger number of items are employed, but only m out of $(k \times n)$ items experience failure.

We have derived both maximum likelihood estimators (MLE) and Bayesian estimators for the parameters λ, η, κ, as well as the reliability, hazard rate, and reversed hazard rate functions. These estimators employ gamma informative priors and cater to both symmetric (squared error) and asymmetric (linex and general entropy) loss functions. While these estimators cannot be obtained in closed form, they can be computed numerically.

Our findings demonstrate that the proposed Bayesian estimators exhibit excellent performance across various values of n and m. Specifically, the Bayes estimators, leveraging gamma informative priors, outperform the MLE in terms of mean squared errors ($MSEs$).

Furthermore, our simulations underscore the significance of using asymmetric loss functions such as linex and general entropy. These loss functions prove valuable in the specific case studied, emphasizing their importance in the estimation process.

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References

[1] Abd AL-Fattah, A. M., EL-Helbawy, A. A., and AL-Dayian, G. R. (2016). Inverted Kumaraswamy distribution: properties and estimation. *Pakistan J. Stat.*, **33**, 37-61.

[2] Balakrishnan, N. and Aggarwala, R. Progressive Censoring: Theory, Methods, Applications. Boston, MA, USA: Birkhäuser (2000).

[3] Balakrishnan, N. and Sandhu, R. A. A simple simulation algorithm for generating progressively type-II generated samples. Am. Stat., **49**, 229-230 (1995).

[4] Balasooriya, U. Failure-censored reliability sampling plans for the exponential distribution. J. Stat. Comput. Simul., **52**, 337-349 (1995).

[5] Barakat, H. M., Nigm, E. M., and Harpy, M. H. Limit theorems of order statistics and record values from the gamma and Kumaraswamy-generated-distributions. Bull. Malays. Math. Sci. Soc., **40**, 1055-1069 (2017).

[6] Cohen, A. C. Progressively censored samples in life testing. Technometrics, **5**, 327-329 (1963).

[7] Dube, M., Garg, R., and Krishna, H. On progressively first failure censored Lindley distribution. Comput. Stat., **31**, 139-163 (2016).

[8] Iqbal, Z., Tahir, M. M., Riaz, N., Ali, S. A., and Ahmad, M. (2017). Generalized inverted Kumaraswamy distribution: properties and application. *Open J. Stat.*, **7**, 645-662.

[9] Johnson, L. G. Theory and technique of variation research. Elsevier, Amsterdam (1964).

[10] Krishna, H. and Kumar, K. Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample. J. Stat. Comput. Simul., **83**(6), 1007-1019 (2013).

[11] Kumar, K., Krishna, H. and Garg, R. Estimation of $P(Y < X)$ in Lindley distribution using progressively first failure censoring. Int. J. Syst. Assur. Eng. Manage., **6**(3), 330-341 (2015).
[12] Kumaraswamy, P. A generalized probability density function for double-bounded random processes. J. Hydrol., **46**, 79-88 (1980).
[13] Lindley, D. V. Approximate Bayesian methods. Trabajos de Estadistica y de Investigacion Operativa, **31**(1), 223-245 (1980).
[14] Lio, Y. and Tsai, T. R. Estimation of $\delta = P(X < Y)$ for Burr XII distribution based on the progressively first failure-censored samples. J. Appl. Stat., **39**, 309-322 (2012).
[15] Mann, N. R., Schafer, R. E. and Singpurwala, N. D. Methods for statistical analysis of reliability and life data. Wiley, New York (1974).
[16] Pradhan, B. and Kundu, D. On progressively censored generalized exponential distribution. Test, **18**, 497-515 (2009).
[17] Sinha, S. K. Reliability and life testing. Wiley Estern Limited, New Delhi (1986).
[18] Wen, D. and Levy, M. S. Admissibility of bayes estimates under blinex loss for the normal mean problem. Commun. Stat. Theory Methods., **30**(1), 155-163 (2001).
[19] Wu, J. W., Hung, W. L. and Tsai, C. H. Estimation of the parameters of the Gompertz distribution under the first failure-censored sampling plan. Statistics, **37**, 517-525 (2003).
[20] Wu, S. J. and Kus, C. On estimation based on progressive first failure censored sampling. Comput. Stat. Data Anal., **53**, 3659-3670 (2009).
[21] Wu, J. W. and Yu, H. Y. Statistical inference about the shape parameter of the Burr type XII distribution under the failure-censored sampling plan. Appl. Math. Comput., **163**, 443-482 (2005).

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