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Medical Data for the Nadaraya-Watson Kernel Method in Nonparametric Regression Models Based on Hyperbolic Secant Function

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Abstract: Nonparametric kernel estimates used in this work aim to compare different treatment options by examining the recorded medical data. The following use of the suggested strategy depends on a kernel function and a parameter called bandwidth. The Nadaraya-Watson kernel (NWK) estimation is a necessary nonparametric kernel estimator used in regression models. A new Nadaraya-Watson regression estimate depends on the hyperbolic secant kernel (HSK) with fixed bandwidth (FNW) and Variable Bandwidth (VNW) is proposed.. We calculated some properties of the unknown regression function estimator, including bias, variance, optimal bandwidth, and a global measure of error criterion mean square error. Finally, simulation and three real data sets are used to evaluate its performance. Results from simulation and real data showed that the VNW using HSK is more effective than the FNW based on Average Mean Square Error Criterion. Also, Nadaraya-Watson using HSK function is more effective than Nadaraya-Watson using the Gaussian kernel density function.

Keywords: Nonparametric estimation, Regression, Nadaraya and watson, Kernel functions, Hyperbolic Secant, Bandwidth, COVID-19.

1 Introduction

In statistics, there are three approaches to estimation in regression analysis: parametric and nonparametric, as well as semi-parametric. A parametric model presumes that the shape of the regression curve, such as linear, quadratic, or cubic, and the form of the function are both known.

Whereas, nonparametric regression is used when the patterns of the relationship are unknown, it is very good in this case because it has high flexibility, then the mixture between parametric and nonparametric regression is semiparametric regression, for more details see [2] and [5]. In different statistical situations, nonparametric regression models (NRM) focus on accurately determining the relationship between dependent variables and independent variables. The selection of the kernel function is important, also the performance and behaviour of the kernel estimators are greatly affected by the bandwidth b, as the choice of bandwidth is of great importance in nonparametric regression. For more details in this regard, see [7, 11, 15, 10] and [17], and references therein.

Let $\{(X_i, Y_i)\}_{i=1}^{n} \in$ R be a bivariate random sample with size n (X_i independent and Y_i dependent variable). The regression equation is given below:

$$Y_i = \beta(X_i) + \varepsilon_i; \quad i = 1,2, \dots, n, \tag{1}$$

where $\beta(X_i)$ is the unknown regression function and ε_i are observation errors, its mean and variance(residual variance) equal to zero and σ_ε^2 respectively. $Cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$. There are many methods for estimating NRM; the most popular of them is Nadaraya-Watson (NW) kernel regression, it is a non-parametric statistical technique for estimating the conditional expectation of a random variable. The NW estimator, which is a nonlinear approximation of a regression model based on experimental data, was developed by researchers Watson and Nadaraya in 1964 [16], [25] and [21].

It is based on the smoothing parameter b, also known as bandwidth. A large value of b results in a smooth density estimate; for further information, see [24] It is possible to select the bandwidth to become fixed or variable. The bandwidth can be calculated using several approaches, such as cross-validation and Silverman's law of thumb [23], or by supposing a variety of initial values.

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1.1 Nonparametric kernel function estimator

The kernel density estimator (KDE) is the simplest non-parametric estimator. It is used to estimate the probability density function (pdf) using a random sample and some kernels K. It is introduced by [20]. Assume that $\{(X_i)\}_{i=1}^{n}$ be a random variables from a distribution. The kernel estimation of $f(x)$ is given by:

$$\hat{f}_n(x) = \frac{1}{nb}\sum_{i=1}^{n} K\left(\frac{x-X_i}{b}\right), \quad x \in \mathbb{R}, \tag{2}$$

where $K(.)$ is a kernel function and b is called a bandwidth. The kernel function that will be used in the research is given by:

$$K(t) = \frac{1}{\pi} sech(t), \quad t \in \mathbb{R},$$

where $sech(u) = 2/(e^{u} + e^{-u})$. Therefore, the hyperbolic secant kernel density estimator (HSKDE) is obtained by:

$$\hat{f}(x) = \frac{1}{nb}\sum_{i=1}^{n} \frac{1}{\pi} sech\left(\frac{x-X_i}{b}\right), \quad x \in \mathbb{R}.$$

1.2 Fixed Nadaraya-Watson kernel estimator

The fixed bandwidth b is appropriate when the unknown regression model behaves identically during the estimation interval. The rule-of-thumb method is the most simple method to determine the fixed bandwidth, the NWK estimator of the regression function in the formula (1) is shown as the following

$$\hat{\beta}_{FNW}(x) = \frac{\sum_{i=1}^{n} y_i K_b(x-X_i)}{\sum_{i=1}^{n} K_b(x-X_i)}. \tag{3}$$

In an expanded form, the considered estimator of the regression function by using HSK is thus given by:

$$\hat{\beta}(x) = \frac{\sum_{i=1}^{n} \frac{1}{\pi} sech\left(\frac{x-X_i}{b}\right) y_i}{\sum_{i=1}^{n} \frac{1}{\pi} sech\left(\frac{x-X_i}{b}\right)}. \tag{4}$$

1.3 Variable Nadaraya-Watson kernel estimator

FNW is not always the best choice, when we estimate the density of long-tailed and multi-modal distributions, we use adaptive kernel estimators with varying bandwidths $b(X_i)$, see [14]. The VNW estimator of the regression function in the formula (1) is obtained as

$$\hat{\beta}_{VNW}(x) = \frac{\sum_{i=1}^{n} \frac{y_i}{b(X_i)} K\left(\frac{x-X_i}{b(X_i)}\right)}{\sum_{i=1}^{n} \frac{1}{b(X_i)} K\left(\frac{x-X_i}{b(X_i)}\right)}. \tag{5}$$

In 1982, [1] suggested formula to compute a variable bandwidth (X_i) :

$$b(X_i) = \frac{b}{\sqrt{f(X_i)}} \tag{6}$$

where $f(X_i)$ is the pdf of the variable (X_i) that can be calculated using the kernel function estimator.

1.4 Preliminaries

In this subsection, we state the necessary mathematical properties used in this study. Also, we mention the bias and variance of the HSKDE, for more details see [8].

- $\int_{-\infty}^{\infty} K(\omega)d\omega = 1, \quad \int_{-\infty}^{\infty} \omega K(\omega)d\omega = 0$ and $\int_{-\infty}^{\infty} \omega^2 K(\omega)d\omega = \frac{\pi^2}{4}$.
- $\int_{-\infty}^{\infty} K(\omega)^2 d\omega = \frac{2}{\pi^2}, \quad \int_{-\infty}^{\infty} \omega K(\omega)^2 d\omega = 0$ and $\int_{-\infty}^{\infty} \omega^2 K(\omega)^2 d\omega = \frac{1}{6}$.
- Suppose that $n \to +\infty$, $b \to 0$ and $nb \to +\infty$, then

$$Bias\left(\hat{f}(x)\right) \simeq \frac{b^2\pi^2}{8} f''(x). \quad \text{and} \quad Var\left[\hat{f}(x)\right] \simeq \frac{2}{nb\pi^2} f(x).$$

In this paper, we will discuss a new, mathematically and practical regression model using HSK. The paper's reminder is described in the sections as follows: In Section 2, we investigate the bias, then the variance, and then each of the average mean squared error and optimal bandwidth of the regression function estimator using HSK. In Section 3, evaluation global criteria. In Section 4, the performance of the proposed estimator is verified through a simulation study using two nonlinear regression functions. The three medical real data sets results are presented in Section 5 to verify the performance of the estimator. The conclusion of this paper in Section 6.

2. The Properties of the Proposed Estimator

Here, we presents and discusses the bias, variance, mean squared error, and optimal bandwidth of the regression function estimator $\hat{\beta}(x)$.

2.1 Bias and variance of $\hat{\beta}(x)$

Proposition 2.1 *It is supposed that $n \to +\infty$, $b \to 0$ and $nb \to +\infty$.*

- The bias of $\hat{\beta}(x)$ satisfies

$$Bias\big(\hat{\beta}(x)\big) \simeq \frac{b^2\pi^2}{8}\beta''(x).$$

- The variance of $\hat{\beta}(x)$ satisfies

$$Var\big[\hat{\beta}(x)\big] \simeq \frac{2\sigma_\varepsilon^2}{nb\pi^2 f(x)}.$$

Proof.

- Start with

$$E\big[\hat{\beta}(x)\big] = E\left[\frac{\int \hat{f}(x,y)ydy}{\hat{f}(x)}\right],$$

where $E\big[\hat{f}(x)\big]$ is given from the following equation and for more details see [8]

$$E\big[\hat{f}(x)\big] = f(x) + \tfrac{\pi^2}{8}b^2 f''(x) + o(b^2). \tag{7}$$

Hence, we need to find $E\big(\int \hat{f}(x,y)ydy\big)$,

$$E\left[\int \hat{f}(x,y)ydy\right] = \frac{1}{n}\sum_{i=1}^{n} K_{b_x}(x-X_i)y_i = \frac{1}{nb_x}\sum_{i=1}^{n} K_{b_x}\left(\frac{x-X_i}{b_x}\right)y_i$$
$$= \iint \frac{1}{b} vK\left(\frac{x-u}{b}\right) f(u,v)dudv.$$

By using the change of variable $\omega = (x-u)/b$, so $du = -bd\omega$, then

$$E\left[\iint \hat{f}(x,y)ydy\right] = \iint vK(\omega)f(x-b\omega,v)d\omega dv$$
$$= \iint vK(\omega)f(v|x-b\omega)f(x-b\omega)d\omega dv$$
$$= \iint vK(\omega)f(v|x-b\omega)f(x-b\omega)d\omega dv$$
$$= \int K(\omega)f(x-b\omega)\underbrace{\int vf(v|x-b\omega)dv}_{\beta(x-b\omega)}\,d\omega.$$

$$E\big[\int \hat{f}(x,y)ydy\big] = \int K(\omega)f(x-b\omega)\beta(x-b\omega)d\omega. \tag{8}$$

This integration can be approximated using Taylor's expansion, then deduce that

$$f(x-b\omega) = f(x) - b\omega f'(x) + \tfrac{1}{2}b^2\omega^2 f''(x) + o(b^2). \tag{9}$$

$$\beta(x-b\omega)=\beta(x)-b\omega\beta'(x)+\tfrac{1}{2}b^2\omega^2\beta''(x)+o(b^2). \quad (10)$$

In addition to, standard techniques also lead to the following integrated outcomes:

$$\int_{-\infty}^{\infty}K(\omega)d\omega=1 \text{ and } \int_{-\infty}^{\infty}\omega^nK(\omega)d\omega=0; \text{n odd number.} \quad (11)$$

From Equations (8), (9), (10) and (11), we get

$$E\left[\int \hat{f}(x,y)ydy\right]=f(x)\beta(x)+b_x^2\sigma_k^2\left[f'(x)\beta'(x)+\frac{f''(x)\beta(x)}{2}+\frac{f(x)\beta''(x)}{2}\right]+o(b^2). \quad (12)$$

Divide the equation (12) by the equation (7), we get:

$$E\left[\hat{\beta}(x)\right]\simeq\beta(x)+\frac{b_x^2\pi^2}{8}\beta''(x).$$

Recall that,

$$Bias\left(\hat{\beta}(x)\right)=E\left[\hat{\beta}(x)\right]-\beta(x)$$

Hence,

$$Bias\left(\hat{\beta}(x)\right)\simeq\frac{{b_x}^2\pi^2}{8}\beta''(x).$$

The bias of $\hat{\beta}(x)$ formula is obtained.

- Let's now concentrate on the variance of $\hat{\beta}(x)$.

$$Var\left[\hat{\beta}(x)\right]=Var\left[\frac{\int \hat{f}(x,y)ydy}{\hat{f}(x)}\right]=Var\left[\frac{\sum_{i=1}^{n}K_b(x-X_i)y_i}{\sum_{i=1}^{n}K_b(x-X_i)}\right]=Var\left[\frac{A}{B}\right].$$

The $Var\left[\hat{\beta}(x)\right]$ can be determined by applying the below formula, It is an approximation of the proportion of variance between two random variables, as shown in [18].

$$V\left(\frac{A}{B}\right)\approx\left(\frac{EA}{EB}\right)^2\left[\frac{V(A)}{(EA)^2}+\frac{V(B)}{(EB)^2}-\frac{2Cov(A,B)}{(EA)(EB)}\right]. \quad (13)$$

In view of the equations (7) and (12), we get $E(B)$ and $E(A)$ respectively, also from paper [8] we get $V(B)$, now we want to calculate $V(A)$ and $Cov(A,B)$:

Firstly, find $V(A)$

$$V[A]=V\left[\frac{1}{n}\sum_{i=1}^{n}K_b(x-X_i)y_i\right]=\frac{1}{nb}E\left[\sum_{i=1}^{n}K_b^2(x-X_i)y_i^2\right]=\iint\frac{1}{b}vK\left(\frac{x-u}{b}\right)f(u,v)dudv.$$

By using the change of variable $\omega=(x-u)/b$, so $du=-bd\omega$, then

$$\begin{aligned}V[A]&=\frac{1}{nb}\iint v^2K^2(\omega)f(x-b\omega,v)d\omega dv\\&=\frac{1}{nb}\iint v^2K^2(\omega)f(v|x-b\omega)f(x-b\omega)d\omega dv\\&=\frac{1}{nb}\int K^2(\omega)f(x-b\omega)\underbrace{\int v^2f(v|x-b\omega)dv}_{\sigma_\varepsilon^2(x-b\omega)\beta(x-b\omega)}d\omega.\end{aligned}$$

$$V[A]=\frac{1}{nb}\int K^2(\omega)f(x-b\omega)d\omega[\sigma_\varepsilon^2(x)+\beta(x)^2].$$

From the Taylor expansion in equation (9), we get

$$V[A]=\frac{1}{nb}\left[f(x)\underbrace{\int_{-\infty}^{\infty}K^2(\omega)d\omega}_{R(K)=\frac{2}{\pi^2}}-bf'(x)\underbrace{\int_{-\infty}^{\infty}\omega K^2(\omega)d\omega}_{0}+\frac{1}{2}b^2f'(x)\underbrace{\int_{-\infty}^{\infty}\omega^2K^2(\omega)d\omega}_{\frac{1}{6}}\right](\sigma_\varepsilon^2(x)+\beta.(x)^2). \quad (14)$$

Therefore, we get

$$V[A]\simeq\frac{2f(x)\sigma_\varepsilon^2(x)}{nb\pi^2}.$$

Secondly, find $Cov(A,B)$

$$\begin{aligned}Cov(A,B)&=Cov\left[\frac{1}{n}\sum_{i=1}^{n}K_b(x-X_i)y_i,\frac{1}{n}\sum_{i=1}^{n}K_b(x-X_i)\right]\\&=E\left[\frac{1}{n}\sum_{i=1}^{n}K_b(x-X_i)^2y_i\right]\approx\frac{R(K)f(x)\beta(x)}{nb}\approx\frac{2f(x)\beta(x)}{\pi^2nb}\end{aligned}$$

By calculating each term, we observe that

$$\left(\frac{E(A)}{E(B)}\right)^2=\left[\frac{f(x)\beta(x)}{f(x)+\frac{\pi^2}{8}b^2f''(x)}\right]^2=\frac{f(x)^2\beta(x)^2}{f(x)^2}\simeq\beta(x)^2. \quad (15)$$

$$\frac{V(A)}{(EA)^2}=\frac{2f(x)\sigma_\varepsilon^2}{\pi^2nb\ \ f(x)^2\beta(x)^2}\simeq\frac{2\sigma_\varepsilon^2}{nb\pi^2f(x)\beta(x)^2}. \quad (16)$$

$$\frac{V(B)}{(E(B))^2}=\frac{2f(x)}{nb\pi^2f(x)^2}\simeq\frac{2}{nb\pi^2f(x)}. \quad (17)$$

$$\frac{2Cov(A,B)}{(EA)(EB)} = \frac{4f(x)\beta(x)}{\pi^2 nbf(x)\beta(x)f(x)} \simeq \frac{4}{\pi^2 nbf(x)}. \quad (18)$$

In view of equations (15, 16, 17 and 18). The approximation formula becomes:

$$Var[\hat{\beta}(x)] = \beta(x)^2\left[\frac{2\sigma_\varepsilon^2}{nb\pi^2 f(x)\beta(x)^2} + \frac{2}{nb\pi^2 f(x)} - \frac{4}{nb\pi^2 f(x)}\right] \simeq \frac{2\sigma_\varepsilon^2}{nb\pi^2 f(x)}. \quad \square$$

2.2 Optimal bandwidth

The less distance between two points is called optimal bandwidth (b_{opt}). It has been extensively studied in the literature, see [12], [19] and [26]. b_{opt} is obtained by minimizing the Average Mean Squared Error (AMSE). Due to Proposition 2.1, (AMSE) $= M_b$ is given by

$$M_b\big(\hat{\beta}(x)\big) = \big[Bias\big(\hat{\beta}(x)\big)\big]^2 + Var\big[\hat{\beta}(x)\big] \simeq \frac{b^4\pi^4}{64}\left[\beta''(x) + 2\beta'(x)\frac{f'(x)}{f(x)}\right]^2 + \frac{2\sigma_\varepsilon^2}{nb\pi^2 f(x)}$$

$$\simeq \frac{b^4\pi^4}{64}[\beta''(x)]^2 + \frac{2\sigma_\varepsilon^2}{nb\pi^2 f(x)}.$$

Now, we can determine the optimal bandwidth when we minimize M_b with respect to b, we need to solve the equation $dM_b/db = 0$ where

$$\frac{dM_b}{db} = \frac{4b^3\pi^4}{64}\left[\beta''(x) + 2\beta'(x)\frac{f'(x)}{f(x)}\right]^2 - \frac{2\sigma_\varepsilon^2}{nb^2\pi^2 f(x)}.$$

After some elementary developments, we obtain

$$b_{\text{opt}} = \left[\frac{n\pi^6[\beta''(x)f(x) + 2\beta'(x)f'(x)]^2}{32\sigma_\varepsilon{}^2}\right]^{-1/5}. \quad (19)$$

which relies on the following: sample size n, unknown regression function, unknown pdf, and error variance $\sigma_\varepsilon{}^2$.

3. Evaluation Global Criteria

In statistics, the mean squared difference between actual and estimated values is known as the mean squared error, or MSE. It is a measure of the amount of error in statistical models. Accordingly, it is used to know the estimated quality and we will use it to compare the methods (FNW and VNW), see [4], [9], [22] and [13]. When a model does not contain errors, the MSE equals zero, but when the error of the model increases, its value increases. A lower value of MSE indicates the best estimator. MSE is determined by

$$MSE = \frac{1}{n}\sum_{j=1}^{n}(y_j - \hat{y}_j)^2. \quad (20)$$

where the number of observations is represented by the symbol n, y_j is the actual values and $\hat{y}_j$ is the estimated values.

In the case of repetition several times (r), the average MSE [3] is determined as:

$$AMSE = \frac{1}{r}\sum_{j=1}^{r} MSE_j. \quad (21)$$

4. Simulation Results

In this section, we can evaluate the performances of the FNW and VNW estimators using hyperbolic secant kernel function by Mathematica program. For the simulation, we utilised two non-linear regression functions as follows:

$$y_i = x_i + 2e^{(-16x_i{}^2)} + \varepsilon_i, \quad i = 1,2,3,\ldots,n. \quad (22)$$

$$y_i = \frac{Sin(2.5\pi x_i)}{1+3x_i{}^2} + \varepsilon_i, \quad i = 1,2,3,\ldots,n. \quad (23)$$

where x_i was chosen randomly from $U[0,1]$ and $\varepsilon_i \sim N(0,1)$. Simulation experiments were conducted using four sample sizes (n = 25, 100, 250, 500) and with a repetition of 2000 for each experiment. The fixed bandwidth b was computed using improve Silverman's thumb rule [8] which is given by:

$$b_{pract} = \frac{0.9}{n^{1/5}}A, \quad A = \min\left(S, \frac{IQR}{1.34 + k}\right).$$

where, n is the number of observations.

S is the sample standard deviation.

IQR is the value of Interquartile Range in the data sample.

k is smoothing coefficient, $k = 0,1,2,3, \ldots$.

From Tables 1 and 2, it is clear that fixed and variable Nadaraya-Watson kernel estimators are non-parametric approaches that can be used to estimate a nonlinear regression model. They are a more flexible approach than other non-parametric approaches and provide accurate prediction results. The proposed estimator based on HSK has been compared in two cases: FNW and VNW based on AMSE criteria. We note that the proposed method using the variable bandwidth is more efficiency than the fixed bandwidth. VNW is more efficient than FNW for all cases of k, where the $AMSE$ value in VNW is less than FNW. We notice that all estimators are improved by increasing the sample size.

The $AMSE$ criteria of FNW and VNW kernel estimators are given in Tables 1 and 2.

Table 1: The $AMSE$ *criteria for the FNW and VNW kernel estimators using* HSK *for the first model.*

k	n	FNW	VNW
0	25	1.25061	1.22934
	100	1.21651	1.21244
	250	1.20673	1.20587
	500	1.20521	1.20511
1	25	1.26335	1.23734
	100	1.22667	1.21683
	250	1.21219	1.20721
	500	1.20381	1.20121
2	25	1.31735	1.28506
	100	1.24893	1.2332
	250	1.222	1.21263
	500	1.21088	1.20552
3	25	1.34069	1.31207
	100	1.26469	1.245
	250	1.22606	1.21465
	500	1.22039	1.21231
4	25	1.39808	1.36323
	100	1.28209	1.25624
	250	1.23988	1.22588
	500	1.22411	1.21468

Table 2: The $AMSE$ *criteria for the FNW and VNW kernel estimators using* HSK *for the second model.*

k	n	FNW	VNW
0	25	1.23237	1.21433
	100	1.20942	1.20529
	250	1.20765	1.20348
	500	1.20083	1.19622
1	25	1.27225	1.24691
	100	1.22833	1.21628
	250	1.21888	1.20872
	500	1.20996	1.20149
2	25	1.32242	1.28462
	100	1.24872	1.22975
	250	1.23176	1.21808
	500	1.22172	1.21264
3	25	1.36708	1.33373
	100	1.26635	1.24371
	250	1.24156	1.22725
	500	1.23034	1.22005

4	25	1.37692	1.34446
	100	1.29554	1.26913
	250	1.25216	1.23548
	500	1.23653	1.22691

5. Real Data Examples in Medicine with Analysis

In this section, three real-life datasets in medicine are used to evaluate the performance of our proposed methods.

The first data set represents a COVID-19 mortality rates data belongs to Italy of 59 days, which is recorded from 27 February to 27 April 2020, see [6].

The second data set represents a COVID-19 mortality rate data belongs to Mexico of 108 days, which is recorded from 4 March to 20 July 2020. This data formed of rough mortality rate, see [6].

The third data set is Leukemia cancer data which collected from January 2015 to December 2020 at Nanakali Hospital for Blood in Erbil City of Iraq, as shown in [9]. Moreover, the CD45 outcome as an explanatory variable and Platelet (PLT) as a response variable in AML type of Leukemia cancer from 30 patients.

5.1 Analysis

The formula suggested by Silverman's rule of thumb [23] for calculating kernel bandwidth is given by

$$b_{pract} = \frac{0.9}{n^{1/5}} A, \quad A = \min\left(S, \frac{IQR}{1.34}\right),$$

From the real data results, note that Tables 3, 5 and 7 compares between the FNW and VNW by using the Hyperbolic secant and Gaussian kernels. The VNW of Hyperbolic Secant kernel has the smallest MSE, so Hyperbolic Secant Kernel regression is more efficiency than Gaussian Kernel regression. In addition, Tables 2, 4 and 6 represent the results between the FNW and VNW methods at b=(0.5, 0.7, 1, 5, 10, and 15) for the first and second real data, respectively. The results show that the VNW of HSK has the smallest MSE than the FNW in various bandwidths, this means that VNW is more efficient than FNW. Figures 1, 2 and 3 represent the results between FNW and VNW at b=(0.5, 1, 5, 10, 15 and 20) for real data.

Table 3: MSE criteria of FNW and VNW methods for data set 1 (n=59).

kernel function	Fixed NW	Variable NW
HSK	42.3522	20.2063
GK	47.2447	20.214

Table 4: MSE criteria of the FNW and VNW kernel estimators using HSK for data set 1.

b	Fixed NW	Variable NW
0.5	54.2981	48.8984
0.7	54.6247	45.2613
1	54.4197	39.1235
5	46.4854	20.3626
10	35.4457	20.1571
15	27.9533	20.1464

Table 5: MSE criteria of FNW and VNW methods for data set 2 (n=108).

kernel function	Fixed NW	Variable NW
HSK	19.0117	10.5107
GK	20.8948	10.513

Table 6: MSE criteria of the FNW and VNW kernel estimators using HSK for data set 2.

b	Fixed NW	Variable NW
0.5	21.9841	22.0085
0.7	22.3359	20.8481
1	22.5734	18.9517
5	22.0933	10.8311
10	19.5473	10.5197
15	17.1738	10.4973

Table 7: MSE criteria of FNW and VNW methods for data set 3 (n=30).

kernel function	Fixed NW	Variable NW
HSK	2650.96	2429.16
GK	2873.05	2429.17

Table 8: MSE criteria of the FNW and VNW kernel estimators using HSK for data set 3.

b	Fixed NW	Variable NW
0.5	4013.17649	3432.87376
0.7	4053.94697	2975.44439
1	4098.43535	2663.6992
5	2996.85262	2429.95172
10	2549.91906	2429.065606
15	2463.80458	2429.01636

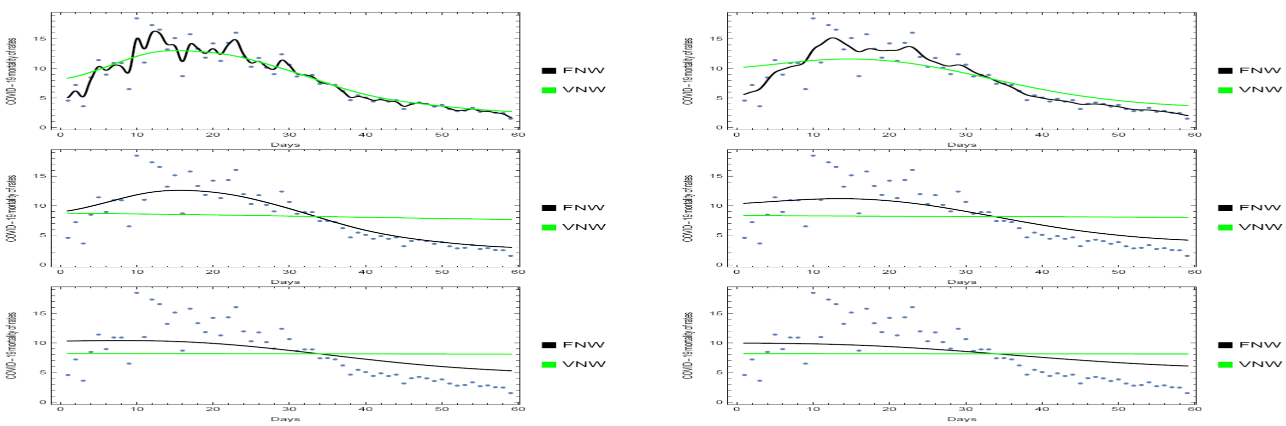

Fig.1: The FNW and VNW for COVID-19 mortality rates versus days for data set 1 at b = 0.5, 1, 5, 10, 15 and 20, respectively.

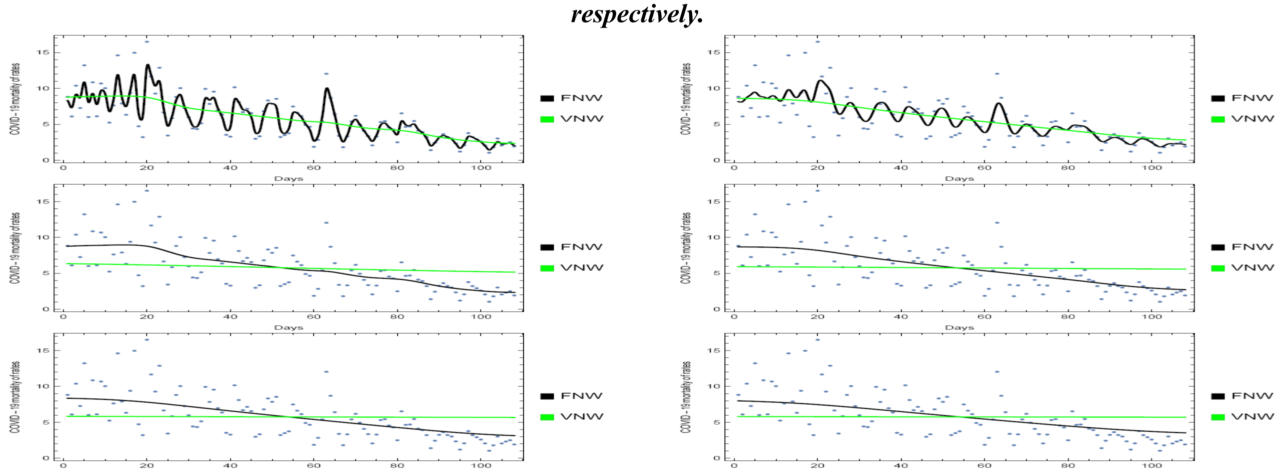

Fig.2: The FNW and VNW for COVID-19 mortality rates versus days for data set 2 at b= 0.5, 1, 5, 10, 15 and 20, respectively

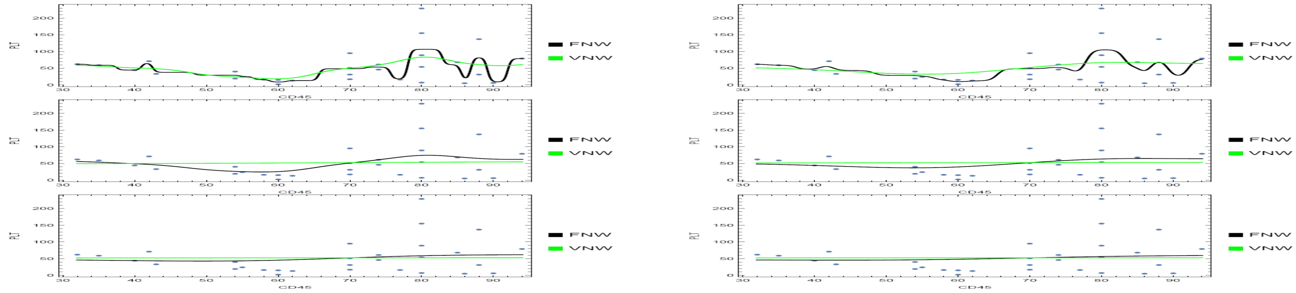

Fig.3: The FNW and VNW for CD45 versus PLT for data set 3 at b = 0.5, 1, 5, 10, 15 and 20, respectively.

6 . Conclusion

The new proposed regression estimator in this paper is depend on the hyperbolic secant kernel. We investigated some of statistical properties such as bias, variance, and average mean squared error. These properties are used to determine the optimal bandwidth. A new proposed regression estimator for VNW is more efficient than FNW in both simulation and three medical practical data sets because the MSE in VNW gives a lower value than MSE in FNW. All estimators are improved by increasing the sample size. Finally, applications are noticed that hyperbolic secant kernel regression is more efficiency than Gaussian kernel regression. We hope that the findings of this study may find wider application in a variety of disciplines.

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