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Reliability Analysis of a System Designed to Withstand a Specified Shock Magnitude Range

S. M. Bakry[1,*], *N. A. Mokhlis*[2], *and A. E. B. Abd Elghany*[3]

[1] Department of Basic Science, Faculty of Computer and Information Science, Ain Shams University, Cairo, Egypt
[2] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
[3] Department of Basic Science, Faculty of Engineering, Benha University, Shoubra, Cairo, Egypt

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Abstract: The system under consideration has n independent components, and its operation relies on the functioning of at least k components, $1 \leq k \leq n$. The system experiences $(n+1)$ distinct shocks. Shock j impacts the j^{th} component, $j = 1, 2, \ldots, n$, while shock $(n+1)$ simultaneously impacts all components. Any shock is lethal if its magnitude is below or above the component-designed thresholds d_1 or d_2, respectively. A shock is characterized by its magnitude and arrival time, forming a bivariate random vector. The bivariate random vectors specifying the magnitudes and the arrival times of the shocks are assumed to be independent and follow non-identical bivariate distributions. The reliability of a k-out-of-n: G system under the influence of this type of shocks is derived. The reliability of parallel and series systems is obtained as special situations. The bivariate Pareto type I distribution is applied as an example of the bivariate distribution of the magnitude and arrival time of the shocks. Furthermore, numerical illustrations are conducted to highlight the theoretical results obtained.

Keywords: System reliability, k-out-of-n: G system, Marshall-Olkin shocks, Bivariate Pareto type I distribution, Order statistics.

Notations

n: The total number of components in the system.

k: The minimum number of components that are sufficient for the system to operate.

D_j: The shock's magnitude that impacts the j^{th} component, $j = 1, \ldots, n$.

T_j: The shock's arrival time that impacts the j^{th} component, $j = 1, \ldots, n$.

D_{n+1}: The shock's magnitude that impacts all components, simultaneously.

T_{n+1}: The shock's arrival time that impacts all components, simultaneously.

(D_j, T_j): Bivariate random vector specifying the magnitude and the arrival time of the j^{th} shock, $j = 1, \ldots, n+1$.

d_1, d_2: The lower and the upper thresholds for the shocks magnitudes that the system components can withstand, respectively. Consequently, the j^{th} component, $j = 1, \ldots, n$ fails only if $D_j < d_1$ or $D_j > d_2$, and simultaneous failure of all components occurs only if $D_{n+1} < d_1$ or $D_{n+1} > d_2$.

Y_j: The arrival time of the j^{th} shock that had magnitude below d_1 or above d_2 (lethal shock), i.e., $Y_j = (T_j | D_j < d_1 \text{ or } D_j > d_2)$, $j = 1, \ldots, n+1$.

$Y_{n-k+1:n}$: It signifies the order statistic of rank $(n-k+1)^{th}$ within the sequence of random variables $Y_1, \ldots, Y_n$.

$R_{k:n}[d_1, d_2; t]$: Is the reliability of the k-out-of-n: G system under study.

*Corresponding author e-mail: soad.bakry@cis.asu.edu.eg

1 Introduction

Many engineering systems can be affected by different shocks that reduce their performance or cause their failure. These shocks can be classified as internal or external. Internal shocks arise from within the system itself, triggered by changes in operating conditions, internal components, or interactions between system parts. External shocks originate from outside the system and are caused by factors such as environmental conditions, external forces, and others. Notably, external shocks can cause damage to electrical devices such as lighting systems, computers, and telecommunication systems, often due to temperature, stress, voltage, and other external factors. Additionally, they can pose detrimental effects on both humans and the environment, as exemplified by the destructive impact of volcanoes, earthquakes, and floods. Numerous studies have been conducted by researchers investigating the lifespan of systems under shock exposure.

In the simplest Marshall-Olkin shocks, the system contains two components exposed to shocks coming randomly from three distinct sources. The shock generated by the j^{th} source impacts the j^{th} component, $j = 1, 2$, while the shock generated by the 3^{rd} source simultaneously impacts both components. [1] derived the joint survival function of this system when the shocks are lethal and non-lethal, as well as three independent Poisson processes that regulate the circumstances of those shocks. [2] obtained the reliability of a k-out-of-n: F system exposed to Marshall-Olkin shocks, without considering the magnitude of the shocks. They assumed that the shocks' arrival times are exchangeable, independent, and identical random variables. The mean residual life and the mean inactivity time functions of the system presented by [2] were investigated by [3]. [4] derived the joint survival function of a two-component system under shocks generated by three separate sources. Failure of each component occurs when it encounters a consecutive sequence of shocks originating from a single source. [5] presented the optimal preventive maintenance policy for a k-out-of-n: F system exposed to Marshall-Olkin shocks. They considered three different types of costs. [6] obtained the reliability, average residual lifespan, and average time until failure of the k-out-of-n: F system under shocks sourced from $(n + 1)$ distinct origins. The occurrence of component failure happens when it is exposed to a consecutive succession of shocks originating from a particular source. [7] conducted a study on the system introduced by [2], investigating the optimal age-based preventive maintenance policies. Under a nonhomogeneous Poisson process governing the arrival of shocks, [8] investigated the optimal replacement policy for the k-out-of-n: F system. [9] studied the reliability of a parallel system experiencing shocks originating from a renewal point process. [10] studied the reliability and the optimal replacement policy of a k-out-of-n: G system exposed to shocks. They focused on the number of components that the shock can destroy.

There are few studies that have focused on the magnitude of the shock. [11] provided the joint survival function for the lifetimes of 2-components exposed to Marshall-Olkin shocks. They assumed that the shock would prove lethal if its magnitude surpassed a predefined upper threshold. [12] derived the reliability of a k-out-of-n: G system exposed to $(n + 1)$ distinct shocks, considering the magnitudes of these shocks. They assumed that the components of the system withstand only a specific upper threshold of the magnitudes of the shocks.

In the present article, the system under study is exposed to $(n + 1)$ distinct shocks. The components of the system are designed to withstand a certain range of magnitudes (between d_1 and d_2). The reliability formulas of k-out-of-n: G, parallel, and series systems are derived. The following are some examples showing the applicability of the system under study in engineering.

(i) Transformers in a voltage transformer system are designed to operate within a certain voltage range, and they may fail if the voltage exceeds a certain level or drops below a certain level. This can happen due to electrical disturbances such as lightning strikes, power surges, and brownouts. The voltage transformer system can be designed as a k-out-of-n: G system, where there are multiple transformers, and at least k must be functional for the system to function.

(ii) In a wind turbine system, there are multiple blades, each with its own pitch control mechanism, that are arranged in a k-out-of-n configuration, such that at least k blades must be functioning properly for the turbine to generate electricity. The system is designed to withstand a certain range of wind speeds and other environmental factors, and if the magnitude of these factors remains within this range, the turbine will continue to function reliably. If the lower threshold for the wind speed is set too low, the turbine blades may not be able to capture enough energy from the wind, leading to a decrease in power output. On the other hand, if the upper threshold for wind speed is set too high, the turbine blades may be subjected to excessive stress, leading to premature failure of the pitch control mechanism or the blades themselves.

The paper is arranged in the following manner: In Section 2, the derivation of the system reliability formula is based on assuming that the magnitudes and arrival times of distinct shocks being independent but non-identical bivariate random variables. The reliability of parallel and series systems is obtained as special situations. In Section 3, the reliability formulas are derived under the assumption of identical shocks. As an application, in Section 4, the bivariate distributions of the shocks' arrival times and magnitudes are assumed to be bivariate Pareto type I distributions with different parameters. In Section 5, numerical illustrations of the theoretical results are performed, demonstrating the effects of time, shock magnitude, and bivariate distribution parameters on system reliabilities. Finally, the conclusion is presented in Section 6. As calculations of the system reliability exposed to non-identical shocks is not easily handled, an algorithm for calculating the reliability is provided in the Appendix.

2 Formula for System Reliability under Non-Identical Shock Exposure

The shocks influencing the system under study are depicted in the Figure 1 below.

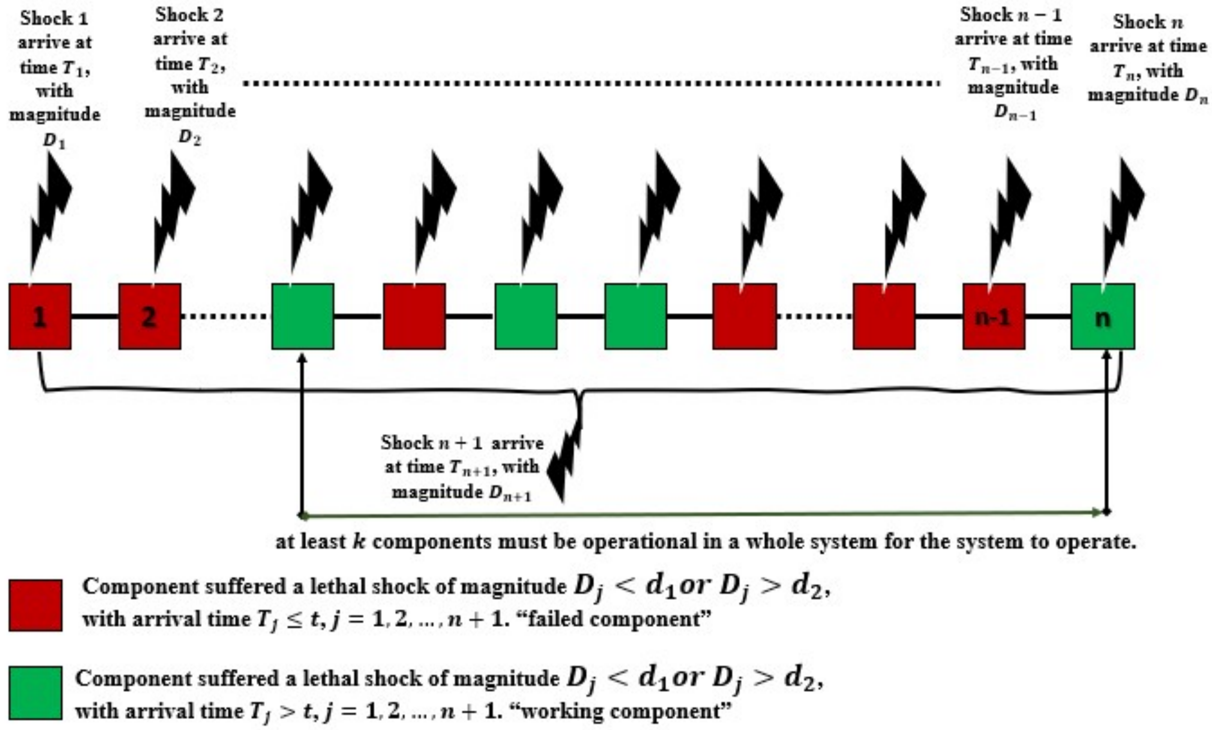


***Fig.1:** A system with n components designed to withstand shock magnitudes limited by two different thresholds.*

The following theorem presents the formula of $R_{k:n}[d_1, d_2; t]$.

Theorem 1

Suppose that D_j and T_j are the magnitude and the arrival time of shock j, $j = 1, 2, \dots, n+1$. Suppose that (D_j, T_j) are independent and non-identically distributed, with the bivariate survival functions $\overline{F}_{T_j,D_j}(.,.)$. Suppose that $\overline{F}_{D_j}(.)$ and $\overline{F}_{T_j}(t)$ are the marginal survival functions of D_j and T_j, respectively. The j^{th} shock impacts the j^{th} component, $j = 1, \dots, n$, while shock $(n+1)$ simultaneously impacts all components. A shock is lethal if its magnitude is below or above the component's designed thresholds d_1 or d_2, respectively. In such a case, $R_{k:n}[d_1, d_2; t]$ is given by,

$$R_{k:n}[d_1, d_2; t] = \frac{\overline{F}_{T_{n+1}}(t) - \overline{F}_{T_{n+1},D_{n+1}}(t, d_1) + \overline{F}_{T_{n+1},D_{n+1}}(t, d_2)}{1 - \overline{F}_{D_{n+1}}(d_1) + \overline{F}_{D_{n+1}}(d_2)} \times \left(\prod_{u=1}^{n} \frac{\overline{F}_{T_u}(t) - \overline{F}_{T_u,D_u}(t, d_1) + \overline{F}_{T_u,D_u}(t, d_2)}{1 - \overline{F}_{D_u}(d_1) + \overline{F}_{D_u}(d_2)} + \sum_{j=1}^{n-k} \sum_{i_1=1}^{n} \dots \sum_{i_j=i_{j-1}+1}^{n} \prod_{L=i_1,\dots,i_j} \left[1 - \frac{\overline{F}_{T_L}(t) - \overline{F}_{T_L,D_L}(t, d_1) + \overline{F}_{T_L,D_L}(t, d_2)}{1 - \overline{F}_{D_L}(d_1) + \overline{F}_{D_L}(d_2)} \right] \times \prod_{\substack{1 \le u \le n \\ u \ne i_1 \ne \dots \ne i_j}} \frac{\overline{F}_{T_u}(t) - \overline{F}_{T_u,D_u}(t, d_1) + \overline{F}_{T_u,D_u}(t, d_2)}{1 - \overline{F}_{D_u}(d_1) + \overline{F}_{D_u}(d_2)} \right), \quad (1)$$

where, $d_1 < d_2, \sum_a^b = 0$, if $a > b$, and $i_0 = 0$.

Proof

Assuming that the components of the system are designed to withstand a specific range of magnitudes limited by lower threshold d_1, and upper threshold d_2, the arrival time of the lethal shock can be defined as follows,

$$Y_j = \left(T_j | D_j < d_1 \cup D_j > d_2\right), j = 1, 2, \ldots, n+1. \tag{2}$$

The system under study fails when the $(n-k+1)^{th}$ component fails or when it receives a lethal shock from the source $(n+1)$. Consequently, we have:

$$R_{k:n}[d_1, d_2; t] = p\{Y_{n-k+1:n} > t\} \ p\{Y_{n+1} > t\}, \tag{3}$$

where, $p\{Y_{n-k+1:n} > t\}$ is defined as the probability that at least k observations in $Y_1, \ldots, Y_n$ are greater than t, or the probability that at most $n-k$ observations are less than or equal to t. Thus, we have

$$p\{Y_{n-k+1:n} > t\} = \sum_{j=0}^{n-k} \sum_{\varphi_{1,2,\ldots,n}} \prod_{L=1}^{j} p\{Y_{i_L} \le t\} \prod_{u=j+1}^{n} p\{Y_{i_u} > t\}, \tag{4}$$

see [13], where $\sum_{\varphi_{1,2,\ldots,n}}$ denotes all possibilities $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ satisfying $i_1 < i_2 < \cdots < i_j$, and $i_{j+1} < i_{j+2} < \cdots < i_n$, $j = 0, \ldots, n-k$. Hence, Equation (4) can be rewritten as follows:

$$p\{Y_{n-k+1:n} > t\} = \prod_{u=1}^{n} p\{Y_u > t\} + \sum_{j=1}^{n-k} \sum_{i_1=1}^{n} \cdots \sum_{i_j=i_{j-1}+1}^{n} \prod_{L=i_1,\ldots,i_j} p\{Y_L \le t\} \prod_{\substack{1\le u\le n \\ u\ne i_1\ne\cdots\ne i_j}} p\{Y_u > t\}, \tag{5}$$

where, $i_0 = 0$, and $\sum_a^b = 0$ if $a > b$. Substituting (5) in (3) we get

$$R_{k:n}[d_1, d_2; t] = p\{Y_{n+1} > t\} \times \left(\prod_{u=1}^{n} p\{Y_u > t\} + \sum_{j=1}^{n-k} \sum_{i_1=1}^{n} \cdots \sum_{i_j=i_{j-1}+1}^{n} \prod_{L=i_1,\ldots,i_j} p\{Y_L \le t\} \prod_{\substack{1\le u\le n \\ u\ne i_1\ne\cdots\ne i_j}} p\{Y_u > t\} \right). \tag{6}$$

Using (2) in (6), we get

$$\begin{aligned} R_{k:n}[d_1, d_2; t] = {} & p\{T_{n+1} > t | D_{n+1} < d_1 \cup D_{n+1} > d_2\} \\ & \times \left(\prod_{u=1}^{n} p\{T_u > t | D_u < d_1 \cup D_u > d_2\} \right. \\ & + \sum_{j=1}^{n-k} \sum_{i_1=1}^{n} \cdots \sum_{i_j=i_{j-1}+1}^{n} \prod_{L=i_1,\ldots,i_j} p\{T_L \le t | D_L < d_1 \cup D_L > d_2\} \\ & \left. \times \prod_{\substack{1\le u\le n \\ u\ne i_1\ne\cdots\ne i_j}} \{T_u > t | D_u < d_1 \cup D_u > d_2\} \right), \end{aligned} \tag{7}$$

where, $p\{T_L \le t | D_L < d_1 \cup D_L > d_2\} = 1 - p\{T_L > t | D_L < d_1 \cup D_L > d_2\}$, and the term $p\{T_j > t | \ D_j < d_1 \cup D_j > d_2\}, j = 1, \ldots, n+1$, in Equation (7) can be written as follows.

$$\begin{aligned} p\{T_j > t | D_j < d_1 \cup D_j > d_2\} &= \frac{p\{T_j > t \ \cap \ (D_j < d_1 \cup D_j > d_2)\}}{p\{D_j < d_1 \ \cup \ D_j > d_2\}} \\ &= \frac{p\{T_j > t, D_j < d_1\} + p\{T_j > t, D_j > d_2\}}{p\{D_j < d_1\} + p\{D_j > d_2\}} \end{aligned} \tag{8}$$

Since $p\{T_j > t, D_j < d_1\} + p\{T_j > t, D_j > d_1\} = p\{T_j > t\}$, the term $p\{T_j > t, D_j < d_1\}$ in Equation (8) can be expressed as

$$p\{T_j > t, D_j < d_1\} = p\{T_j > t\} - p\{T_j > t, D_j > d_1\} \tag{9}$$

Substituting (9) in (8), we get

$$p\{T_j > t | D_j < d_1 \cup D_j > d_2\} = \frac{\overline{F}_{T_j}(t) - \overline{F}_{T_j,D_j}(t,d_1) + \overline{F}_{T_j,D_j}(t,d_2)}{1 - \overline{F}_{D_j}(d_1) + \overline{F}_{D_j}(d_2)}, \tag{10}$$

$j = 1, \dots, n+1.$

By using (10) in (7), we get (1).

Special Situations

Substituting $k = n$ and $k = 1$ in Equation (1) yields $R_{n:n}[d_1, d_2; t]$ and $R_{1:n}[d_1, d_2; t]$, respectively. Thus, we get.

$$R_{n:n}[d_1, d_2; t] = \frac{\overline{F}_{T_{n+1}}(t) - \overline{F}_{T_{n+1},D_{n+1}}(t,d_1) + \overline{F}_{T_{n+1},D_{n+1}}(t,d_2)}{1 - \overline{F}_{D_{n+1}}(d_1) + \overline{F}_{D_{n+1}}(d_2)} \times \prod_{u=1}^{n} \frac{\overline{F}_{T_u}(t) - \overline{F}_{T_u,D_u}(t,d_1) + \overline{F}_{T_u,D_u}(t,d_2)}{1 - \overline{F}_{D_u}(d_1) + \overline{F}_{D_u}(d_2)}, \tag{11}$$

and

$$R_{1:n}[d_1, d_2; t] = \frac{\overline{F}_{T_{n+1}}(t) - \overline{F}_{T_{n+1},D_{n+1}}(t,d_1) + \overline{F}_{T_{n+1},D_{n+1}}(t,d_2)}{1 - \overline{F}_{D_{n+1}}(d_1) + \overline{F}_{D_{n+1}}(d_2)} \left(1 - \prod_{L=1}^{n} \left[1 - \frac{\overline{F}_{T_L}(t) - \overline{F}_{T_L,D_L}(t,d_1) + \overline{F}_{T_L,D_L}(t,d_2)}{1 - \overline{F}_{D_L}(d_1) + \overline{F}_{D_L}(d_2)}\right]\right) \tag{12}$$

The computation of $R_{k:n}[d_1, d_2; t]$ in (1) requires calculating all combinations that satisfy the condition $1 \le i_1 < \dots < i_j \le n$, $j = 1, \dots, n-k$. However, dealing with these combinations directly is complicative. To address this issue, we present an algorithm in the Appendix.

3 Formula for System Reliability under Identical Shock Exposure

Theorem 2

Suppose that (D_j, T_j), $j = 1, 2, \dots, n+1$. Suppose that $(D_j, T_j), j = 1, \dots, n$ are identically distributed with a bivariate survival function $\overline{F}_{T,D}(.,.)$. Suppose that the bivariate survival function of (D_{n+1}, T_{n+1}) is $\overline{F}_{T_{n+1},D_{n+1}}(.,.)$. Suppose that $\overline{F}_D(.)$ and $\overline{F}_T(t)$ are the marginal survival functions of the magnitude and the arrival time, respectively of shocks $j, j = 1, 2, \dots, n$, while $\overline{F}_{D_{n+1}}(.)$ and, $\overline{F}_{T_{n+1}}(t)$ are the corresponding marginals of the shock $(n+1)$. The j^{th} shock impacts the j^{th}, $j = 1, \dots, n$ component, while shock $(n+1)$ simultaneously impacts all components. A shock is lethal if its magnitude is below or above the component's designed thresholds d_1 or d_2, respectively. Then, $R_{k:n}[d_1, d_2; t]$ is given by,

$$R_{k:n}[d_1, d_2; t] = \frac{\overline{F}_{T_{n+1}}(t) - \overline{F}_{T_{n+1},D_{n+1}}(t,d_1) + \overline{F}_{T_{n+1},D_{n+1}}(t,d_2)}{1 - \overline{F}_{D_{n+1}}(d_1) + \overline{F}_{D_{n+1}}(d_2)} \sum_{j=0}^{n-k} \binom{n}{j} \sum_{w=0}^{j} (-1)^w \binom{j}{w} \times \left[\frac{\overline{F}_T(t) - \overline{F}_{T,D}(t,d_1) + \overline{F}_{T,D}(t,d_2)}{1 - \overline{F}_D(d_1) + \overline{F}_D(d_2)}\right]^{n+w-j}, \tag{13}$$

where, $d_1 < d_2$.

Proof

Since $(D_j, T_j), j = 1, 2, \dots, n$ are independent and identically distributed random vectors, it follows that $Y_j's, j = 1, 2, \dots, n$ are also independent and identically distributed random variables with a common cumulative function $F_{Y_j}(t) = F_Y(t) = p\{Y \le t\}, j = 1,2, \dots, n$. Therefore, we have:

$$p\{Y_{n-k+1:n} > t\} = \sum_{j=0}^{n-k} \binom{n}{j} [p\{Y \le t\}]^j [p\{Y > t\}]^{n-j}.$$

Then, we have

$$R_{k:n}[d_1, d_2; t] = p\{Y_{n+1} > t\} \sum_{j=0}^{n-k} \binom{n}{j} [\, p\{Y \le t\}\,]^j \; [\, p\{Y > t\}\,]^{n-j}. \tag{14}$$

Using (2) in (14), we get

$$R_{k:n}[d_1, d_2; t] = p\{T_{n+1} > t |\, D_{n+1} < d_1 \cup D_{n+1} > d_2\} \times \sum_{j=0}^{n-k} \binom{n}{j} [\, 1 - p\{\, T > t \mid D < d_1 \cup D > d_2\}]^j \; [\, p\{\, T > t |\, D < d_1 \cup D > d_2\}]^{n-j}$$

The above equation can be rewritten as follows,

$$R_{k:n}[d_1, d_2; t] = p\{\, T_{n+1} > t |\, D_{n+1} < d_1 \cup D_{n+1} > d_2\} \times \sum_{j=0}^{n-k} \binom{n}{j} \sum_{w=0}^{j} (-1)^w \binom{j}{w} \; [\, p\{T > t |\, D < d_1 \cup D > d_2\}]^{n+w-j} \tag{15}$$

Using (10) with

$p\{T_j > t | D_j < d_1 \cup D_j > d_2\} = \frac{\overline{F}_T(t) - \overline{F}_{T,D}(t,d_1) + \overline{F}_{T,D}(t,d_2)}{1 - \overline{F}_D(d_1) + \overline{F}_D(d_2)}$, $j = 1, \dots, n$, in (15), we get (13).

Special Situations

Substituting with $k = n$ and $k = 1$ in (13), we obtain $R_{n:n}[d_1, d_2; t]$ and $R_{1:n}[d_1, d_2; t]$ in the case where the shocks impacting single components are identical, respectively as follows.

$$R_{n:n}[d_1, d_2; t] = \frac{\overline{F}_{T_{n+1}}(t) - \overline{F}_{T_{n+1},D_{n+1}}(t, d_1) + \overline{F}_{T_{n+1},D_{n+1}}(t, d_2)}{1 - \overline{F}_{D_{n+1}}(d_1) + \overline{F}_{D_{n+1}}(d_2)} \times \left[\frac{\overline{F}_T(t) - \overline{F}_{T,D}(t, d_1) + \overline{F}_{T,D}(t, d_2)}{1 - \overline{F}_D(d_1) + \overline{F}_D(d_2)}\right]^n, \tag{16}$$

and

$$R_{1:n}[d_1, d_2; t] = \frac{\overline{F}_{T_{n+1}}(t) - \overline{F}_{T_{n+1},D_{n+1}}(t, d_1) + \overline{F}_{T_{n+1},D_{n+1}}(t, d_2)}{1 - \overline{F}_{D_{n+1}}(d_1) + \overline{F}_{D_{n+1}}(d_2)} \times \left(1 - \left[1 - \frac{\overline{F}_T(t) - \overline{F}_{T,D}(t, d_1) + \overline{F}_{T,D}(t, d_2)}{1 - \overline{F}_D(d_1) + \overline{F}_D(d_2)}\right]^n\right). \tag{17}$$

Remark

The result in (13) may be obtained directly from Equation (1), by replacing $\sum_{i_1=1}^{n} \dots \sum_{i_j=i_{j-1}+1}^{n} \prod_{L=i_1,\dots,i_j} [\, 1 - p\{T_j > t | D_j < d_1 \cup D_j > d_2\}] \prod_{\substack{1 \le u \le n \\ u \ne i_1 \ne \dots \ne i_j}} p\{T_j > t | D_j < d_1 \cup D_j > d_2\}$, with $\binom{n}{j} \prod_{L=1}^{j} [\, 1 - p\{T > t |\, D < d_1 \cup D > d_2\}] \prod_{u=j+1}^{n} p\{T > t | D < d_1 \cup D > d_2\}$.

Special Cases

- Substituting with $d_1 = d_2 = d$ in Equations (1), and (13) for non-identical, and identical cases, respectively, we obtain the reliability of the k-out-of-n: G system exposed to Marshall-Olkin shocks, focusing only on their arrival times as follows.

$$R_{k:n}[d,d;t]=R_{k:n}[t]$$
$$=\bar{F}_{T_{n+1}}(t)\left(\prod_{u=1}^{n}\bar{F}_{T_u}(t)+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\dots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}\left[1-\bar{F}_{T_L}(t)\right]\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\bar{F}_{T_u}(t)\right), \tag{18}$$

for non-identical shocks, and

$$R_{k:n}[d,d;t]=R_{k:n}[t]=\bar{F}_{T_{n+1}}(t)\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\bar{F}_T(t)\right]^{n+w-j}\text{, for identical shocks.} \tag{19}$$

- Substituting with $d_1=-\infty$, and $d_2=d$ in Equations (1), and (13) for non-identical, and identical cases, respectively, we obtain the reliability of the k-out-of-n: G system exposed to Marshall-Olkin shocks, with components designed to withstand a certain range of magnitude specified by only an upper threshold d as follows.

$$R_{k:n}[-\infty,d;t]=\frac{\bar{F}_{T_{n+1},D_{n+1}}(t,d)}{\bar{F}_{D_{n+1}}(d)}\left(\prod_{u=1}^{n}\frac{\bar{F}_{T_u,D_u}(t,d)}{\bar{F}_{D_u}(d)}+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\dots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}\left[1-\frac{\bar{F}_{T_L,D_L}(t,d)}{\bar{F}_{D_L}(d)}\right]\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\frac{\bar{F}_{T_u,D_u}(t,d)}{\bar{F}_{D_u}(d)}\right), \tag{20}$$

for non-identical shocks, and

$$R_{k:n}[-\infty,d;t]=\frac{\bar{F}_{T_{n+1},D_{n+1}}(t,d)}{\bar{F}_{D_{n+1}}(t)}\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\frac{\bar{F}_{T,D}(t,d)}{\bar{F}_D(d)}\right]^{n+w-j}, \tag{21}$$

for identical shocks, see, [12].

- Substituting with $d_2=\infty$, and $d_1=d$ in Equations (1), and (13) for non-identical, and identical cases respectively, we obtain the reliability of the k-out-of-n: G system exposed to Marshall-Olkin shocks, with components designed to withstand a certain range of magnitudes specified by only a lower threshold d as follows.

$$R_{k:n}[d,\infty;t]=\frac{\bar{F}_{T_{n+1}}(t)-\bar{F}_{T_{n+1},D_{n+1}}(t,d)}{1-\bar{F}_{D_{n+1}}(d)}\left(\prod_{u=1}^{n}\frac{\bar{F}_{T_u}(t)-\bar{F}_{T_u,D_u}(t,d)}{1-\bar{F}_{D_u}(d)}+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\dots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}\left[1-\frac{\bar{F}_{T_L}(t)-\bar{F}_{T_L,D_L}(t,d)}{1-\bar{F}_{D_L}(d)}\right]\times\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\frac{\bar{F}_{T_u}(t)-\bar{F}_{T_u,D_u}(t,d)}{1-\bar{F}_{D_u}(d)}\right), \tag{22}$$

for non-identical shocks, and

$$R_{k:n}[d,\infty;t]=\frac{\overline{F}_{T_{n+1}}(t)-\overline{F}_{T_{n+1},D_{n+1}}(t,d)}{1-\overline{F}_{D_{n+1}}(d)}\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^{w}\binom{j}{w}\left[\frac{\overline{F}_{T}(t)-\overline{F}_{T,D}(t,d)}{1-\overline{F}_{D}(d)}\right]^{n+w-j}, \quad (23)$$

for identical shocks.

4 Application Using Bivariate Pareto Type I Distribution

4.1. Exact Reliability Formulas of Systems when Exposed to Non-Identical Shocks with Bivariate Pareto Type I Distribution

[14] introduced the bivariate Pareto distributions of type I and II. The bivariate Pareto distributions have various applications in different fields such as reliability engineering, risk analysis, quality control, hydrology, environmental studies, modeling the lifetime of the units in a system, and others, see [15], and [16]. The joint survival distribution function of the bivariate Pareto type I distribution is represented by,

$$\bar{F}_{X,Y}(x,y)=\left(\frac{x}{\theta}+\frac{y}{\beta}-1\right)^{-\alpha}, x>\theta; y>\beta;\ \theta,\beta,\alpha>0. \quad (24)$$

X and Y have Pareto type I distributions as their marginals, with survival distribution functions,

$$\bar{F}_{X}(x)=\left(\frac{x}{\theta}\right)^{-\alpha}, x>\theta;\ \theta,\alpha>0, \quad (25)$$

and

$$\bar{F}_{Y}(y)=\left(\frac{y}{\beta}\right)^{-\alpha}, y>\beta;\ \beta,\alpha>0, \text{ respectively.} \quad (26)$$

The means of X and Y are given respectively by.

$$E(X)=\frac{\alpha\theta}{\alpha-1}, \quad (27)$$

and

$$E(Y)=\frac{\alpha\beta}{\alpha-1}. \quad (28)$$

The positive correlation coefficient between X and Y is $\rho=\frac{1}{\alpha},\ \alpha>2$.

Suppose that (D_j,T_j), $j=1,2,\dots,n+1$ are independent and non-identical random vectors with joint survival distribution functions in (24), with parameters $\theta_j,\beta_j,\alpha_j$, $j=1,2,\dots,n+1$. Using (24), (25), and (26), in (1), (11) and (12), we get

$$R_{k:n}[d_1,d_2;t]=\varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})\left(\prod_{u=1}^{n}\varphi(d_1,d_2;t;\theta_u,\beta_u,\alpha_u)\right.$$
$$+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\dots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}[1-\varphi(d_1,d_2;t;\theta_L,\beta_L,\alpha_L)]$$
$$\left.\times\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\dots\ne i_j}}\varphi(d_1,d_2;t;\theta_u,\beta_u,\alpha_u)\right), \quad (29)$$

$$R_{n:n}[d_1,d_2;t]=\varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})\prod_{u=1}^{n}\varphi(d_1,d_2;t;\theta_u,\beta_u,\alpha_u), \quad (30)$$

and

$$R_{1:n}[d_1,d_2;t]=\varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})\left(1-\prod_{L=1}^{n}[1-\varphi(d_1,d_2;t;\theta_L,\beta_L,\alpha_L)]\right), \tag{31}$$

where,

$$\varphi(d_1,d_2;t;\theta_j,\beta_j,\alpha_j)=\frac{(\beta_j t)^{-\alpha_j}+\sum_{i=1}^{2}(-1)^i(\beta_j t+\theta_j d_i-\theta_j\beta_j)^{-\alpha_j}}{\theta_j^{-\alpha_j}\left(\beta_j^{-\alpha_j}-d_1^{-\alpha_j}+d_2^{-\alpha_j}\right)}, \tag{32}$$

$j=1,\ldots,n+1$.

Using the algorithm given in the Appendix, (29), (30) and (31) can be computed easily.

4.2. Exact Reliability Formulas of Systems when Exposed to Identical Shocks with Bivariate Pareto Type I Distribution

Suppose that (D_j,T_j), $j=1,2,\ldots,n$, are identical and independent random vectors with common joint survival distribution function as in (24), with parameters θ, β, α, while (D_{n+1},T_{n+1}) has joint survival distribution function as in (24), with parameters θ_{n+1}, β_{n+1}, α_{n+1}. Suppose that (D_j,T_j), $j=1,2,\ldots,n$, and (D_{n+1},T_{n+1}) are independent. Using (24), (25), and (26), in (13), (16), and (17), we get

$$R_{k:n}[d_1,d_2;t] = \varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})\sum_{j=0}^{n-k}\binom{n}{j} \times\sum_{w=0}^{j}(-1)^w\binom{j}{w}[\varphi(d_1,d_2;t;\theta,\beta,\alpha)]^{n+w-j}, \tag{33}$$

$$R_{n:n}[d_1,d_2;t] = \varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})[\varphi(d_1,d_2;t;\theta,\beta,\alpha)]^n, \tag{34}$$

and

$$R_{1:n}[d_1,d_2;t] = \varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})(1-[1-\varphi(d_1,d_2;t;\theta,\beta,\alpha)]^n), \tag{35}$$

where,

$\varphi(d_1,d_2;t;\theta,\beta,\alpha)$, and $\varphi(d_1,d_2;t;\theta_{n+1},\beta_{n+1},\alpha_{n+1})$ are given by (32), with $\theta_j=\theta$, $\beta_j=\beta$, and $\alpha_j=\alpha$ for $j=1,\ldots,n$.

Special Cases

- Let $\bar{F}_{T_j}(t)$, $j=1,\ldots,n+1$, and $\bar{F}_T(t)$ have survival functions as in (25) with parameters θ_j, $j=1,\ldots,n+1$, and θ, respectively. Using (25) in Equations (18), and (19), we get

$$R_{k:n}[t]=\left(\frac{t}{\theta_{n+1}}\right)^{-\alpha_{n+1}}\left(\prod_{u=1}^{n}\left(\frac{t}{\theta_u}\right)^{-\alpha_u} +\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\cdots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\ldots,i_j}\left[1-\left(\frac{t}{\theta_L}\right)^{-\alpha_L}\right]\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\left(\frac{t}{\theta_u}\right)^{-\alpha_u}\right), \tag{36}$$

for non-identical shocks, and

$$R_{k:n}[t]=\left(\frac{t}{\theta_{n+1}}\right)^{-\alpha_{n+1}}\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\left(\frac{t}{\theta}\right)^{-\alpha}\right]^{n+w-j}, \text{ for identical shocks.} \tag{37}$$

- Let $\bar{F}_{T_j,D_j}(t,d)$, $j=1,\ldots,n+1$, and $\bar{F}_{T,D}(t,d)$ have joint survival functions as in (24) with parameters θ_j, β_j, α_j, $j=1,\ldots,n+1$, and θ, β, α, respectively. Using (24), and (26) in Equations (20), and (21), we get

$$R_{k:n}[\beta_j,d;t]=\frac{(\beta_{n+1}t+\theta_{n+1}d-\theta_{n+1}\beta_{n+1})^{-\alpha_{n+1}}}{(\theta_{n+1}d)^{-\alpha_{n+1}}}\left(\prod_{u=1}^{n}\frac{(\beta_u t+\theta_u d-\theta_u\beta_u)^{-\alpha_u}}{(\theta_u d)^{-\alpha_u}}\right.$$
$$\left.+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\cdots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}\left[1-\frac{(\beta_L t+\theta_L d-\theta_L\beta_L)^{-\alpha_L}}{(\theta_L d)^{-\alpha_L}}\right]\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\frac{(\beta_u t+\theta_u d-\theta_u\beta_u)^{-\alpha_u}}{(\theta_u d)^{-\alpha_u}}\right),$$

for non-identical shocks, and

$$R_{k:n}[\beta_j,d;t]=\frac{(\beta_{n+1}t+\theta_{n+1}d-\theta_{n+1}\beta_{n+1})^{-\alpha_{n+1}}}{(\theta_{n+1}d)^{-\alpha_{n+1}}}\times\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\frac{(\beta t+\theta d-\theta\beta)^{-\alpha}}{(\theta d)^{-\alpha}}\right]^{n+w-j}, \tag{38}$$

for identical shocks.

- Let $\overline{F}_{T_j,D_j}(t,d)$, $j=1,\dots,n+1$, and $\overline{F}_{T,D}(t,d)$ have joint survival functions as in (24) with parameters θ_j, β_j, α_j, $j=1,\dots,n+1$, and θ, β, α, respectively. Using (24), (25), and (26) in Equations (22), and (23), we get

$$R_{k:n}[d,\infty;t]=\left(\frac{(\beta_{n+1}t)^{-\alpha_{n+1}}-(\beta_{n+1}t+\theta_{n+1}d-\theta_{n+1}\beta_{n+1})^{-\alpha_{n+1}}}{\theta_{n+1}^{-\alpha_{n+1}}(\beta_{n+1}^{-\alpha_{n+1}}-d^{-\alpha_{n+1}})}\right)$$
$$\times\left(\prod_{u=1}^{n}\left(\frac{(\beta_u t)^{-\alpha_u}-(\beta_u t+\theta_u d-\theta_u\beta_u)^{-\alpha_u}}{\theta_u^{-\alpha_u}(\beta_u^{-\alpha_u}-d^{-\alpha_u})}\right)\right.$$
$$+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\cdots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}\left[1-\frac{(\beta_L t)^{-\alpha_L}-(\beta_L t+\theta_L d-\theta_L\beta_L)^{-\alpha_L}}{\theta_L^{-\alpha_L}(\beta_L^{-\alpha_L}-d^{-\alpha_L})}\right]$$
$$\left.\times\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\left(\frac{(\beta_u t)^{-\alpha_u}-(\beta_u t+\theta_u d-\theta_u\beta_u)^{-\alpha_u}}{\theta_u^{-\alpha_u}(\beta_u^{-\alpha_u}-d^{-\alpha_u})}\right)\right),$$

for non-identical shocks, and

$$R_{k:n}[d,\infty;t]=\frac{(\beta_{n+1}t)^{-\alpha_{n+1}}-(\beta_{n+1}t+\theta_{n+1}d-\theta_{n+1}\beta_{n+1})^{-\alpha_{n+1}}}{\theta_{n+1}^{-\alpha_{n+1}}(\beta_{n+1}^{-\alpha_{n+1}}-d^{-\alpha_{n+1}})}\times\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\frac{(\beta t)^{-\alpha}-(\beta t+\theta d-\theta\beta)^{-\alpha}}{\theta^{-\alpha}(\beta^{-\alpha}-d^{-\alpha})}\right]^{n+w-j}, \tag{39}$$

for identical shocks.

- By substituting $\alpha_{n+1}=0$ in Equations (29), and (33), we can get $R_{k:n}[d_1,d_2;t]$ without the effect of the shock $(n+1)$ that affects all components simultaneously, for non-identical, and identical shocks, respectively as follows:

$$R_{k:n}[d_1,d_2;t]=\left(\prod_{u=1}^{n}\varphi(d_1,d_2;t;\theta_u,\beta_u,\alpha_u)+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\dots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}[1-\varphi(d_1,d_2;t;\theta_L,\beta_L,\alpha_L)]\times\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\varphi(d_1,d_2;t;\theta_u,\beta_u,\alpha_u)\right), \quad (40)$$

for non-identical shocks, where $\varphi(d_1,d_2;t;\theta_j,\beta_j,\alpha_j)$, $j=1,\dots,n$ is given by (32), and

$$R_{k:n}[d_1,d_2;t]=\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\frac{(\beta t)^{-\alpha}+\sum_{i=1}^{2}(-1)^i(\beta t+\theta d_i-\theta\beta)^{-\alpha}}{\theta^{-\alpha}(\beta^{-\alpha}-d_1^{-\alpha}+d_2^{-\alpha})}\right]^{n+w-j}, \quad (41)$$

for identical shocks.

- By substituting, $\alpha_{n+1}=0$, in Equations (36), and (37), we can get $R_{k:n}[t]$ without the effect of the shock $(n+1)$ that affects all components simultaneously, and the magnitudes of the shocks $1,\dots,n$ are not considered, for non-identical, and identical cases, respectively, as follows,

$$R_{k:n}[t]=\prod_{u=1}^{n}\left(\frac{t}{\theta_u}\right)^{-\alpha_u}+\sum_{j=1}^{n-k}\sum_{i_1=1}^{n}\dots\sum_{i_j=i_{j-1}+1}^{n}\prod_{L=i_1,\dots,i_j}\left[1-\left(\frac{t}{\theta_L}\right)^{-\alpha_L}\right]\prod_{\substack{1\le u\le n\\ u\ne i_1\ne\cdots\ne i_j}}\left(\frac{t}{\theta_u}\right)^{-\alpha_u}, \quad (42)$$

for non-identical shocks, and

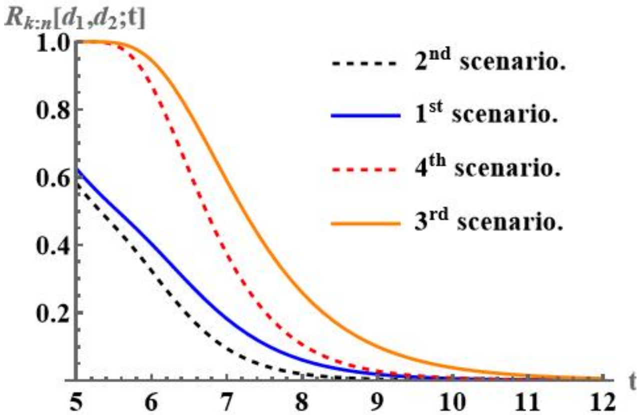
$$R_{k:n}[t]=\sum_{j=0}^{n-k}\binom{n}{j}\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left[\left(\frac{t}{\theta}\right)^{-\alpha}\right]^{n+w-j}, \text{for identical shocks.} \quad (43)$$

5 Numerical Illustration

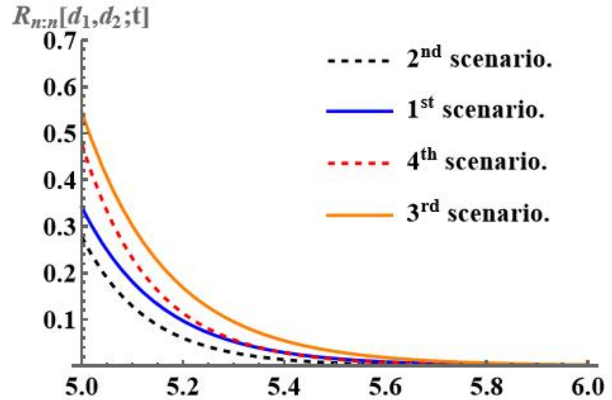
Taking $n=12$, and $k=5$, $R_{k:n}[d_1,d_2;t]$, $R_{n:n}[d_1,d_2;t]$, and $R_{1:n}[d_1,d_2;t]$ in (33), (34), and (35) are studied, to observe the impact of time, shocks' magnitude, and the distributions parameters θ, β, α, θ_{n+1}, β_{n+1}, and α_{n+1}.

In Figure 2, we conduct a comparison between four scenarios: The first is when the system is exposed to Marshel-Olkin shocks while considering only their arrival times (Equation (37)). The second is when the system is exposed to Marshel-Olkin shocks while considering both their arrival times and magnitudes (Equations (33), (34), (35) for $R_{k:n}[d_1,d_2;t]$, $R_{n:n}[d_1,d_2;t]$, $R_{1:n}[d_1,d_2;t]$, respectively). The third is when the system is not exposed to shock $(n+1)$, and only the arrival times of the shocks $1,\dots,n$ that affect components $1,\dots,n$ are considered (Equation (43)). The fourth is when the system is not exposed to shock $(n+1)$, and both the magnitudes and the arrival times of the shocks $1,\dots,n$ that affect components $1,\dots,n$ are considered (Equation (41)). The purpose of these comparisons is to assess the effect of the shock magnitude over time on the system performance, when the system components are designed to withstand a certain range of the magnitudes specified by lower and upper thresholds d_1 and d_2. Substituting $k=1$, and $k=n$ in Equations (37), (41), and (43), we get the results for the parallel and series systems, respectively. The reliabilities are computed for specific values of $d_1=20$, $d_2=100$, $\theta=4.9$, $\theta_{n+1}=4$, $\beta=15$, $\beta_{n+1}=10$, $\alpha=2.5$, and $\alpha_{n+1}=2.1$.

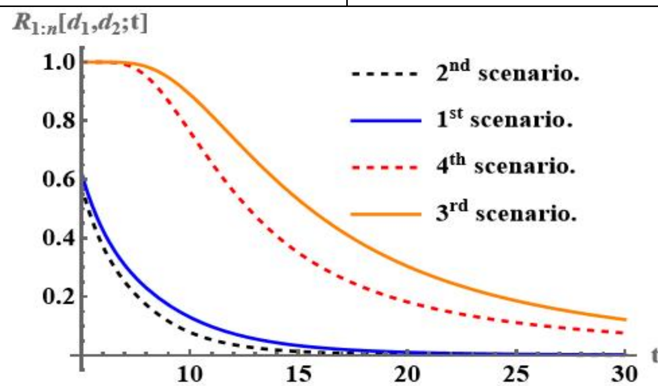
From Figure 2, we conclude that the reliability of all systems decreases faster with time when the system is exposed to Marshall-Olkin shocks, and the magnitudes of the shocks is considered. The reliability reaches its maximum value when eliminating the effect of the shock that comes from $(n+1)^{th}$ source, and only the arrival time of the shock is considered. This means that the magnitude of the shock affects the value of reliability not only its arrival time. Also, we can see that the reliability of the series system decreases faster over time for all four scenarios, and then the reliability of the k-out-of-n system, but the reliability of the parallel system decreases slower over time, and this is according to the structure of the system (this means that as k increases, the reliability of the system decreases faster with time).



(a) Reliability of the k-out-of-n: G system.



(b) Reliability of the series system.

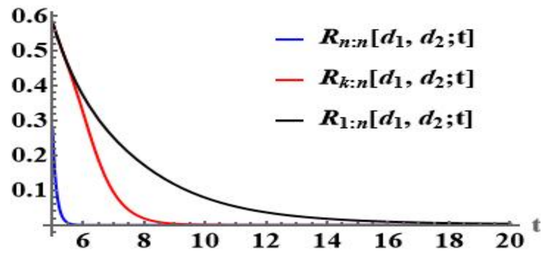


(c) Reliability of the parallel system.

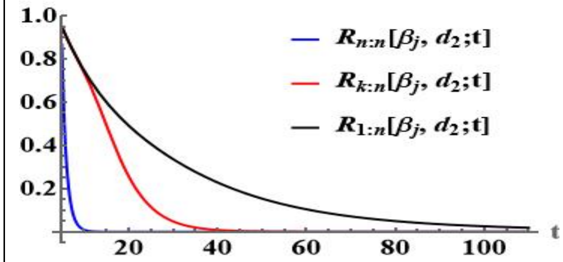
Fig. 2: The comparison between the four scenarios.

In Figure 3, we conduct a comparison to see the effect of the threshold on the system reliability. Using Equations (33), (34), (35) for $R_{k:n}[d_1, d_2; t]$, $R_{n:n}[d_1, d_2; t]$, $R_{1:n}[d_1, d_2; t]$, respectively in Figure (3, a), for the case when the components are designed to withstand a certain range of magnitudes specified by two thresholds (d_1 "lower" and d_2 "upper"). In Figure (3, b) Equation (38) is used, for the case when the components are designed to withstand a certain range of magnitudes specified by only an upper threshold d_2. Using Equation (39) in Figure (3, c), for the case when the components are designed to withstand a certain range of magnitudes specified by only a lower threshold d_1. Substituting $k = 1$, and $k = n$ in Equations (38), and (39), we get the results for the parallel and series systems, respectively, in Figures (3, b), and (3, c). The reliabilities are computed for specific values of $\theta = 4.9$, $\theta_{n+1} = 4$, $\beta = 15$, $\beta_{n+1} = 10$, $\alpha = 2.5$, and $\alpha_{n+1} = 2.1$.

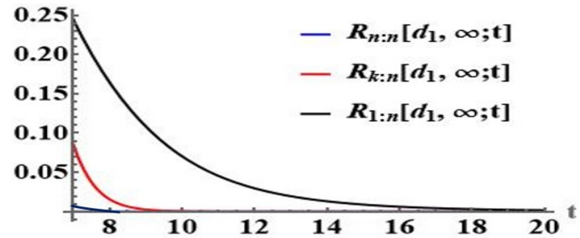
From Figure 3, we can see that $R_{k:n}[\beta_j, d_2; t] > R_{k:n}[d_1, d_2; t] > R_{k:n}[d_1, \infty; t]$. The reliability of the system reaches its highest value when the system contains only an upper threshold, on the other hand the reliability of the system reaches its lowest value when the system contains only a lower threshold. Also, the values of $R_{k:n}[d_1, d_2; t]$, $R_{k:n}[d_1, \infty; t]$ decreases faster than $R_{k:n}[\beta_j, d_2; t]$ over time. The same results are observed for the parallel and the series systems.



(a) $R_{k:n}[d_1, d_2; t]$ versus t, $d_1 = 20$, $d_2 = 100$.



(b) $R_{k:n}[\beta_j, d_2; t]$ versus t, $d_2 = 100$.



(c) $R_{k:n}[d_1, \infty; t]$ versus t, $d_1 = 20$.

Fig. 3: The effect of the threshold on the system reliability.

In Figure 4, we show the effect of the scale parameters θ, and θ_{n+1} on the system reliability. The effect of θ is studied for specific values of $\theta_{n+1} = 4.9$, $t = 5$; $d_1 = 20$, $d_2 = 100$; $\beta = 15$, $\beta_{n+1} = 10$, $\alpha = 2.5$, and $\alpha_{n+1} = 2.1$. The effect of θ_{n+1} is studied for specific values of $\theta = 4.9$, $t = 5$; $d_1 = 20$, $d_2 = 100$; $\beta = 15$, $\beta_{n+1} = 10$, $\alpha = 2.5$, and $\alpha_{n+1} = 2.1$.

From Figure 4, we see that as $\theta(\theta_{n+1})$ increases, the reliability of the system increases. This is expected since we see from Equation (27) that $E(T) \propto \theta$. The effect of θ_{n+1} in increasing the reliability is faster than θ, in the case of $R_{k:n}[d_1, d_2; t]$, $R_{n:n}[d_1, d_2; t]$, while the effect of both θ, and θ_{n+1} in increasing $R_{1:n}[d_1, d_2; t]$ appears quickly. Also, we see that at a certain value of θ, $R_{k:n}[d_1, d_2; t]$, and $R_{1:n}[d_1, d_2; t]$ reaches its maximum and stabilizes at this value, and $R_{1:n}[d_1, d_2; t]$ stabilizes faster than $R_{k:n}[d_1, d_2; t]$, while the effect of θ in increasing $R_{n:n}[d_1, d_2; t]$ appears slower and increases suddenly at a certain value of θ.

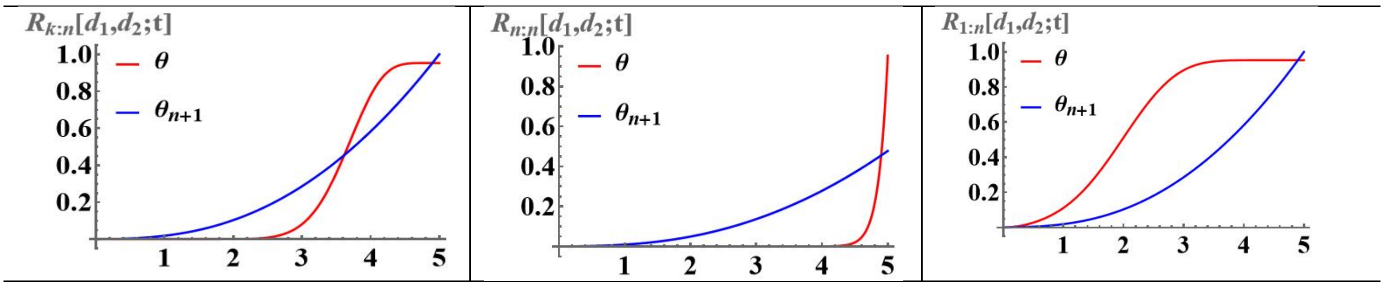


Fig. 4: The effect of θ, and θ_{n+1} on the system reliability.

Figure 5 shows the effect of the scale parameters β, and β_{n+1} on the system reliability. The effect of β is studied for specific values of $\beta_{n+1} = 10$, $t = 5$; $d_1 = 20$, $d_2 = 100$; $\theta = 4$, $\theta_{n+1} = 3.5$, $\alpha = 2.5$, and $\alpha_{n+1} = 2.1$. The effect of β_{n+1} is studied for specific values of $\beta = 10$, $t = 5$; $d_1 = 20$, $d_2 = 100$; $\theta = 4$, $\theta_{n+1} = 3.5$, $\alpha = 2.5$, and $\alpha_{n+1} = 2.1$.

From Figure 5, we can see that the system reliability decreases as β (β_{n+1}) increases until the value of β (β_{n+1}) reaches $d_1(\beta = d_1(\beta_{n+1} = d_1))$, then the reliability increases. This is because when the value of β reaches d_1, the effect of the lower threshold d_1 is cancelled, and the system is affected only by the upper threshold d_2.

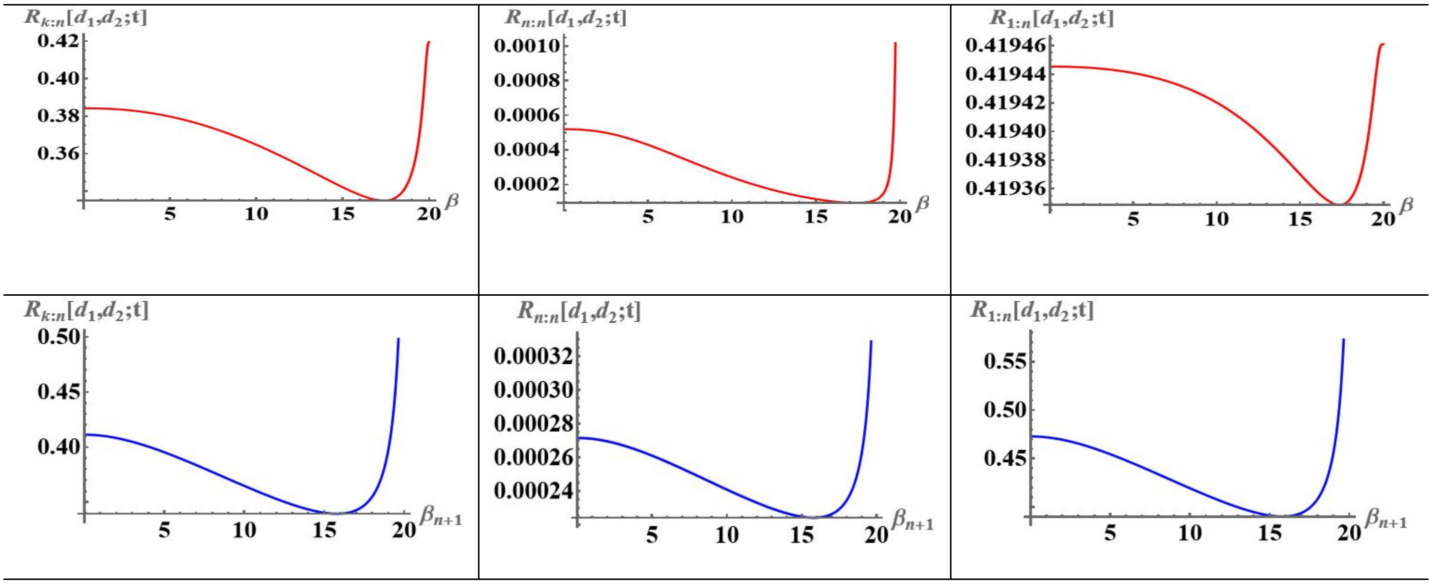


Fig. 5: The effect of β, and β_{n+1} on the system reliability.

In Figure 6, we show the effect of the shape parameters α, and α_{n+1} on the system reliability. The effect of α is studied for specific values of $\alpha_{n+1} = 2.5$, $t = 5$; $d_1 = 20$, $d_2 = 100$; $\theta = 4$, $\theta_{n+1} = 3.5$, $\beta = 15$, and $\beta_{n+1} = 10$. The effect of α_{n+1} is studied for specific values of $\alpha = 2.5$, $t = 5$; $d_1 = 20$, $d_2 = 100$; $\theta = 4$, $\theta_{n+1} = 3.5$, $\beta = 15$, and $\beta_{n+1} = 10$.

From Figure 6, we see that as α (α_{n+1}) increases, the value of the system reliability decreases. This is expected since we see from Equations (27), and (28), that $E(T)$ and $E(D)$ are decreasing in α. The effect of α in decreasing the reliability is faster than α_{n+1}, in the case of $R_{k:n}[d_1, d_2; t]$, $R_{n:n}[d_1, d_2; t]$, while the effect of α_{n+1} in decreasing $R_{1:n}[d_1, d_2; t]$ is faster than α.

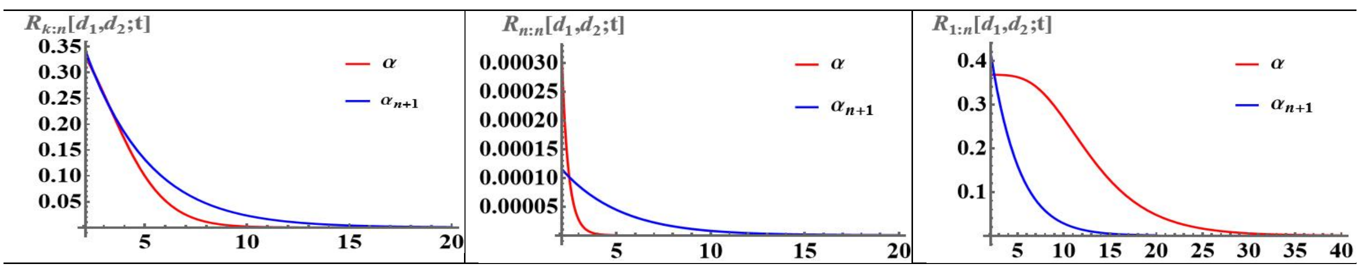


Fig.6: The effect of α and α_{n+1} on the system reliability.

Table 1 shows the implementation time of the algorithm provided in the appendix, for computing $R_{k:n}[d_1, d_2; t]$, $R_{n:n}[d_1, d_2; t]$, $R_{1:n}[d_1, d_2; t]$, and $R_{k:n}[t]$ given in Equations (29), (30), (31), and (36), respectively. Also, for computing $R_{k:n}[d_1, d_2; t]$ and $R_{k:n}[t]$, when not exposed to the shock $(n + 1)$ (Equations (40) and (42), respectively). The reliabilities are computed for different values of n, and k. The results are computed using R-programming.

From Table 1, we can see that the implementation time of the algorithm does not exceed few minutes even for large values of n and k. Also, it is noticed that the implementation time of the algorithm decreases with the increasing the value of k, where the number of possibilities satisfying $1 \leq i_1 < \ldots < i_j \leq n$ for $j = 1, \ldots, n - k$, decreases.

Table 1: The implementation time of the algorithm computing the reliability with non-identical shocks.

		$t = 7, d_1 = 200, d_2 = 500$							
		$R_{k:n}[d_1, d_2; t]$				implementation time			
		$\theta = [6.9, 5, 6, 6.5, 5.5]$, $\beta = [20, 50, 80, 10, 30]$, $\alpha = [2.1, 3.5, 3, 3.1, 2.8]$, $\theta_{n+1} = 5.9$, $\beta_{n+1} = 70$, $\alpha_{n+1} = 2.5$							
n	k	With Marshall-Olkin, with magnitude	With Marshall-Olkin, without magnitude	Without $(n+1)^{th}$, with magnitude	Without $(n+1)^{th}$, without magnitude	With Marshall-Olkin, with magnitude	With Marshall-Olkin, without magnitude	Without $(n+1)^{th}$, with magnitude	Without $(n+1)^{th}$, without magnitude
5	1	0.6383017	0.651705	0.9991955	0.9992315	0.169982 secs	0.116076 secs	0.159764 secs	0.1223972 secs
	3	0.4946693	0.5088916	0.7743538	0.7802618	0.136452 secs	0.0846839 secs	0.1324241 secs	0.0920789 secs
	4	0.2427931	0.2526416	0.3800676	0.3873646	0.1157091 secs	0.0839109 secs	0.1124251 secs	0.0756741 secs
	5	0.0470700	0.0496526	0.0736833	0.0761302	0.060941 secs	0.0164809 secs	0.0607791 secs	0.0157671 secs
		$\theta = [6.9, 5, 6, 6.5, 5.5, 6.8, 5.7, 4, 4.9, 5.6]$, $\beta = [20, 50, 80, 10, 30, 40, 60, 4, 110, 100]$ $\alpha = [2.1, 3.5, 3, 3.1, 2.8, 4.5, 4, 3, 3.7, 2.8]$, $\theta_{n+1} = 5.9$, $\beta_{n+1} = 70$, $\alpha_{n+1} = 2.5$							
10	1	0.638805	0.6521967	0.9999833	0.9999854	3.678222 secs	3.422815 secs	3.639734 secs	3.379294 secs
	4	0.593810	0.6102148	0.9295485	0.9356164	3.030626 secs	3.095459 secs	2.958277 secs	2.917818 secs
	7	0.1385986	0.1517597	0.2169618	0.2326867	0.790718 secs	0.641908 secs	0.742292 secs	0.641892 secs
	10	0.0004218	0.0005115	0.0006602	0.0007842	0.062266 secs	0.016026 secs	0.059339 secs	0.0147779 secs
		$\theta = [6.9, 5, 6, 6.5, 5.5, 6.8, 5.7, 4, 4.9, 5.6, 6.9, 5, 5.5, 4.5, 3]$ $\beta = [20, 50, 80, 10, 30, 40, 60, 4, 110, 100, 30, 40, 90, 20, 5]$ $\alpha = [2.1, 3.5, 3, 3.1, 2.8, 4.5, 4, 3, 3.7, 2.8, 5, 2.5, 2.9, 3.1, 2.1]$, $\theta_{n+1} = 5.9$, $\beta_{n+1} = 70$, $\alpha_{n+1} = 2.5$							
15	1	0.6388155	0.6522061	0.9999998	0.9999999	2.724808 mins	2.588358 mins	2.705587 mins	2.581221 mins
	5	0.6246283	0.6394762	0.9777912	0.9804816	2.605901 mins	2.425495 mins	2.598685 mins	2.453441 mins
	10	0.08728927	0.0978385	0.1366424	0.1500116	26.5475 secs	25.28616 secs	25.52465 secs	24.66994 secs
	15	0.00000339	0.00000437	0.0000053133	0.00000671	0.0642641 secs	0.015699 secs	0.0609760 secs	0.01484203 secs

6 Conclusion

In this paper, we obtain the reliability of a k-out-of-n: G system exposed to Marshall-Olkin shocks, and its components are designed to withstand a certain range of shocks' magnitudes specified by two thresholds (lower "d_1", and upper " d_2"). The reliability of the system is obtained when the shocks are non-identical and identical. The following situations are obtained as special cases:

- The system's reliability when exposed to Marshall-Olkin shocks, and only their arrival times are considered.
- The system's reliability when not exposed to a shock $(n + 1)$, while the magnitudes and arrival times of shocks $1, \dots, n$ are considered.

- The system's reliability when not exposed to a shock $(n+1)$, while only the arrival times of shocks $1, \ldots, n$ are considered.
- The reliability of the system when its components are designed to withstand a certain range of magnitudes specified by an upper threshold only.
- The reliability of the system when its components are designed to withstand a certain range of magnitudes specified by a lower threshold only.

The reliability of the parallel and the series systems are presented for all situations. As an application, the bivariate Pareto type I distribution is applied to highlight the theoretical results. An algorithm is introduced to calculate $R_{k:n}[d_1, d_2; t]$ where the shocks are non-identical, which is not straightforward to handle directly. From the numerical illustration we see that, the magnitude of the shock affects the value of reliability not only its arrival time as well as the presence of the $(n+1)^{th}$ source. It is observed that $R_{k:n}[\beta_j, d_2; t] > R_{k:n}[d_1, d_2; t] > R_{k:n}[d_1, \infty; t]$. Also, we see that the system reliability increases with increasing θ, θ_{n+1}, as well as decreasing β, β_{n+1}, α, and α_{n+1}. The implementation time of the algorithm that calculates $R_{k:n}[d_1, d_2; t]$ when exposed to non-identical shocks does not exceed few minutes. Our forthcoming work involves a study on the reliability of the consecutive k-out-of-n system, considering different assumptions in the presence of shocks.

Appendix

In constructing the algorithm, the following notations are utilized:

j: Denotes the count of defective components, $j = 1, \ldots, n-k$.

M: Represents the total count of available combinations satisfying the condition $1 \le i_1 < \ldots < i_j \le n$, $j = 1, \ldots, n-k$.

P: Represents a matrix with j columns, and M rows. Each row in this matrix represents one of the possible combinations satisfying $1 \le i_1 < \ldots < i_j \le n$, $j = 1, \ldots, n-k$.

Algorithm: Compute the value of system reliability where the shocks are non-identical.	
1:	**Input:** *Insert the following set of values.* $\boldsymbol{k}, \boldsymbol{n}$: *Integers* $\boldsymbol{d_1}, \boldsymbol{d_2}$; $\boldsymbol{t}$: *Numeric*
2:	**Output:** *The value of the system reliability*
3:	**Initialize:** *Set all product terms and all sum terms in Equation (1) by 1 and 0, respectively.* **Set** *the product term L* ←1 **Set** *the product term u* ←1 **Set** *the product term L by term u* ←1 **Set** *the sum over all possibilities* ← 0 **Set** *the sum over all j* ← 0
4:	**Function** Conditional Probability (j) // Now we define function that compute the term $p(T_j > t \mid D_j < d_1 \cup D_j > d_2), j = 1, \ldots, n+1$ // $SurFD1[j]$ ← *survival distribution function of the magnitude of the shock j at* d_1 $SurFD2\,[j]$ ← *survival distribution function of the magnitude of the shock j at* d_2 $SurFT\,[j]$ ← *survival distribution function of the time of the shock j at t* $JsurF1\,[j]$ ← *joint survival distribution function of the magnitude and the time of the shock j at* d_1 *and t.* $JsurF2\,[j]$ ← *joint survival distribution function of the magnitude and the time of the shock j at* d_2 *and t.* $conditional\,[j] \leftarrow \frac{SurFT\,[j] - JsurF1\,[j] + JsurF2\,[j]}{1 - SurFD1[j] + SurFD2\,[j]}$ **return** $conditional\,[j]$ **End Function** *Conditional probability of shock* $(n+1)$ ← Conditional Probability $(n+1)$
5:	**Set** *no component failed product* ← 1 // Compute the first product in (1) //

6: **For** $u \leftarrow 1$ **to** n **do**
at current u the component is survived ← Conditional Probability (u)
no component failed product ← *no component failed product* * *at current u the component is survived*
End For

7: **vector:** *vector* (n) // Generate a vector with size n //

8: **For** $j \leftarrow 1$ **to** $n - k$ **do** // Given a vector containing n elements, produce and display all feasible combinations with size j within matrix P //

9: **Function** Combinations $(n$, r, vec, repeats. Allowed = False)
$P \leftarrow$ *Combinations* $(n = length(vector), r = j, vec = vector,$ *repeats. Allowed*= $False)$
// Generate matrix 'P' to store all outputs sequentially //
$M \leftarrow num\_rows(P)$

10: **For** $w \leftarrow 1$ **to** M **do** // Perform a step-by-step movement across each element within every row of matrix P, calculating the product term L as defined in Equation (1)."//

11: **For** $i \leftarrow 1$ **to** j **do**
$L \leftarrow P[w, i]$
at current L the component is failed ← $1 -$ Conditional Probability (L)
$productL \leftarrow productL *$ *at current L the component is failed*
End For

12: **For** $u \leftarrow 1$ **to** n **do** // Verify each row of matrix P for the existence of values between 1 to n. If any of these values are not found in a row, they are considered survival components. //

13: **If** $(u$ NOT IN $P[w,])$ **then**
at current u the component is survived ← Conditional Probability (u)
$productu \leftarrow productu$* *at current u the component is survived*
End IF
End For
$productLu \leftarrow productL * productu$
$sump \leftarrow sump + productLu$

14: **Set** $productL \leftarrow 1$
Set $productu \leftarrow 1$
End For
total sum ← *total sum* + *sump*
Set *sump* ← 0
End For

15: **Obtain the reliability value for the system:**
The reliability value for the system ← *Conditional probability of shock (n+1)* * (*no component failed product* + *total sum*)

16: **Print** (*System reliability*)

References

[1] Marshall, A. W, and Olkin, I, A multivariate exponential distribution, *Journal of the American Statistical Association,* **62**, 30-44 (1967).

[2] I. Bayramoglu and M. Ozkut, The reliability of coherent systems subjected to Marshall–Olkin type shocks, *IEEE Transactions on Reliability,* **64**, 435-443 (2014).

[3] I. Bayramoglu and M. Ozkut, Mean residual life and inactivity time of a coherent system subjected to Marshall–Olkin type shocks, *Journal of Computational and Applied Mathematics,* **298**, 190-200 (2016).

[4] M. Ozkut and S. Eryilmaz, Reliability analysis under Marshall–Olkin run shock model, *Journal of Computational and Applied Mathematics,* **349**, 52-59 (2019).

[5] S. Ashrafi, R. Rostami, and M. Asadi, Preventive Maintenance for Systems Subject to Marshal-Olkin Type Shock Models, in *Proceeding of Reliability Seminar on th 5 The Theory and its Applications*, 38, (2019).

[6] M. Ozkut and K. Cihangir, On the coherent systems subject to Marshall-Olkin type shocks, *Mathematical Sciences and Applications E-Notes,* **8**, 185-192 (2020).

[7] S. Ashrafi, M. Asadi, and R. Rostami, On preventive maintenance of k-out-of-n systems subject to fatal shocks, *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, (2023).

[8] S.-H. Sheu and C.-T. Liou, "Optimal replacement of a k-out-of-n system subject to shocks, *Microelectronics Reliability,* **32**, 649-655 (1992).

[9] G. Skoulakis, A general shock model for a reliability system, *Journal of applied probability,* **37**, 925-935, (2000).

[10] S. Eryilmaz and Y. Devrim, Reliability and optimal replacement policy for a k-out-of-n system subject to shocks, *Reliability Engineering & System Safety,* **188**, 393-397 (2019).

[11] M. Ozkut and I. Bayramoglu, On Marshall–Olkin type distribution with effect of shock magnitude, *Journal of computational and applied mathematics,* **271**, 150-162 (2014).

[12] S. Bakry and N. Mokhlis, Reliability of a k-out-of-n: G System Subjected to Marshall-Olkin Type Shocks Concerning Magnitude, *Journal of Statistics Applications & Probability,* **12**, no. 3, 1143-1153 (2023).

[13] H. A. David and H. N. Nagaraja, *Order statistics*. John Wiley & Sons, (2004).

[14] K. V. Mardia, Multivariate pareto distributions, *The Annals of Mathematical Statistics,* 1008-1015 (1962).

[15] P. Sankaran and N. U. Nair, A bivariate Pareto model and its applications to reliability, *Naval Research Logistics (NRL),* **40**, 1013-1020 (1993).

[16] J. Navarro, J. M. Ruiz, and C. J. Sandoval, Properties of systems with two exchangeable Pareto components, *Statistical Papers,* **49**, 177-190 (2008).