Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/100212

Eco-epidemiological Model with Infected Prey: Dynamic Behaviour of a Fractional Order Model Including Prey Refuge and Type II Functional Response

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Received: 11 Jul. 2023, Revised: 13 Jan. 2024, Accepted: 17 Jan. 2024
Published online: 1 Apr. 2024

Abstract: In this study, using the Holling type II response function and prey refuge, we bring fractional order into an eco-epidemiological paradigm with diseased prey in which the predator consumes a vast amount of healthy prey in an excessively disproportionately large amount. We show that there are solutions to the fractional order eco-epidemiological paradigm and that these solutions are distinct, non-negativity, and limited. Additionally, we generated a number of equilibrium points and investigated the local and global stability of the interior equilibrium point. We explore the roles that fractional order and the prey shelter play in maintaining the stability of the proposed system's equilibrium point. Numerous examples are used to illustrate the results, and numerical simulations are used to support our theoretical conclusions.

Keywords: Eco-epidemic paradigm, Fractional-order, Prey refuge, Type II response, Stability.

1 Introduction

The creation of a model that explores the interactions between species and uses that model's analysis to forecast the species' future dynamics is one of the main topics of mathematical biology. Prey-predator interactions are often seen in the majority of organisms. The use of ordinary differential equations to represent the changing aspects of predator-prey systems has a rich and intriguing history. Kot [1], Murray [2], and the sources given therein provide some of those fascinating works. On the other hand, infectious illnesses that affect a species cannot be disregarded. Almost everyone who is alive may experience various infectious illnesses at some point in their lifetime, and these diseases have a huge impact on population size. The main causes of decreased reproductive rates, species mortality, refuge, etc., may be infectious illnesses. Consequently, a biological system that has been stable for a while could become unstable, lose its stability, and ultimately go extinct. Eco-epidemiology, the study of ecology and epidemiology combined, is therefore extremely realistic and difficult from a practical standpoint. The eco-epidemic field has expanded greatly and quickly during the last few decades. Hadeler et al. [3] developed the initial model for disease transmission among the interacting populations. Various predator-prey models in the context of illness were the subject of numerous studies [4–10]. The eco-epidemiology of such systems was applied for the first time by Chattopadhyay et al. [4].

In numerous biological models, non-integer order differential equations are being effectively applied to analyse the inherent changing characteristics of the ecosystems [11]. The non-integer order derivative may be better suited for simulating systems depending on prior experiences since it is a non-local operator in the sense that it considers some processes' histories of earlier states as having an impact on the system's current state [12]. Greater degrees of freedom

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are possible with fractional-order systems, which are also physically explicable as a memory index [13] Du et al. The concept of generic non-integer derivatives, as put forth by Zhao et al. [14], is frequently used to explain the nature of memory phenomena. By selecting a suitable non-integer order derivative which best fits the data and, as a result, more accurately predicts disease progression, it may also be possible to adapt the non-integer order system to actual data, as demonstrated by Almeida et al. [15]. Therefore, noninteger-order differential equations might help to simulate biological occurrences more accurately. Some works on the modelling of differential equations with fractional orders are presented in [16–31]. Our study uses fractional-order differential equations to represent the prey refuge dilemma, which includes contaminated prey. This provides the piece an additional novelty and dimension.

The present paper is organised in the following manner: The presumptions as well as evolution of a model with different parameters and their corresponding descriptions are covered in Section 2 of the study. Details on the existence and uniqueness of the model, as well as the positivity and boundaries of the solutions, are given in Section 3. In Section 4, the equilibrium points are discussed in addition to their presence and the stability characterization of the paradigm at the coexistence equilibrium point. In Section 5, we employ the MATLAB programme to quantitatively validate every one of our important theoretical findings Section 6 includes a brief general overview and addresses the biological implications of our mathematical and theoretical findings.

2 Mathematical model formulations

2.1 Model Assumptions

Recall that and indicate the respective total population amounts of prey and predators. Following are the underlying assumptions of the present fractional eco-epidemiological paradigm.

a) The amount of prey increases logistically given carrying capacity with inherent birth pace while there is no infection.
b) Assuming the presence of disease, the prey population is split as a pair of categories: vulnerable prey (represented by) and infected prey (represented by). The population at time is therefore
c) The only ability of the vulnerable prey is reproduction, and the sick prey is eliminated at a natural rate
d) The disease cannot be transferred vertically and only spreads through contact among the prey population. Prey species that have been affected do not recover or develop immunity. We suppose that the disease spreads at the same rate according to the basic rule of mass action .
e) The prey refuge constant , and the vulnerable prey accessible for predation are expected to take refuge by the susceptible prey species .
f) Both vulnerable along with infected prey are predated by predators at predation coefficients of and , correspondingly, assuming a Holling type-II functional response. With efficiency , the prey is devoured and transformed to a predator.
g) A steady rate of natural mortality losses affect the predator.

In Figure 1, the structure of the model is displayed. Yang et al.'s work [32] presents a set of ordinary differential equations that are produced from the flow chart.

$$\frac{du}{dt} = ru\left(1-\frac{u}{k}\right) - buv - a_1uw, \tag{1}$$
$$\frac{dv}{dt} = buv - a_2vw - c_1v, \tag{2}$$
$$\frac{dw}{dt} = ea_1uw + ea_2vw - c_2w. \tag{3}$$

It is presumed that the parameters are continuous and positive. The following is a biological explanation of each parameter utilised in system (1)-(3). In the current work, we extend the integer order paradigm (1)-(3) to take into account a fractional order eco-epidemiological paradigm that includes prey refuge with type II functional response.

$$^{C}D^{\xi}u(t) = ru\left(1-\frac{u}{k}\right) - buv - \frac{a_1(1-\theta)wu}{1+\gamma(1-\theta)u}, \tag{4}$$
$$^{C}D^{\xi}v(t) = buv - a_2vw - c_1v, \tag{5}$$
$$^{C}D^{\xi}w(t) = \frac{ea_1(1-\theta)uw}{1+\gamma(1-\theta)u} + ea_2vw - c_2w \tag{6}$$

with initial conditions $u(0) = u_0 \geq 0, v(0) = v_0 \geq 0, w(0) = w_0 \geq 0$, where $0 < \xi < 1$, and CD^{ξ} denotes the Caputo fractional derivative. Table 1 provides descriptions of each of the parameters.

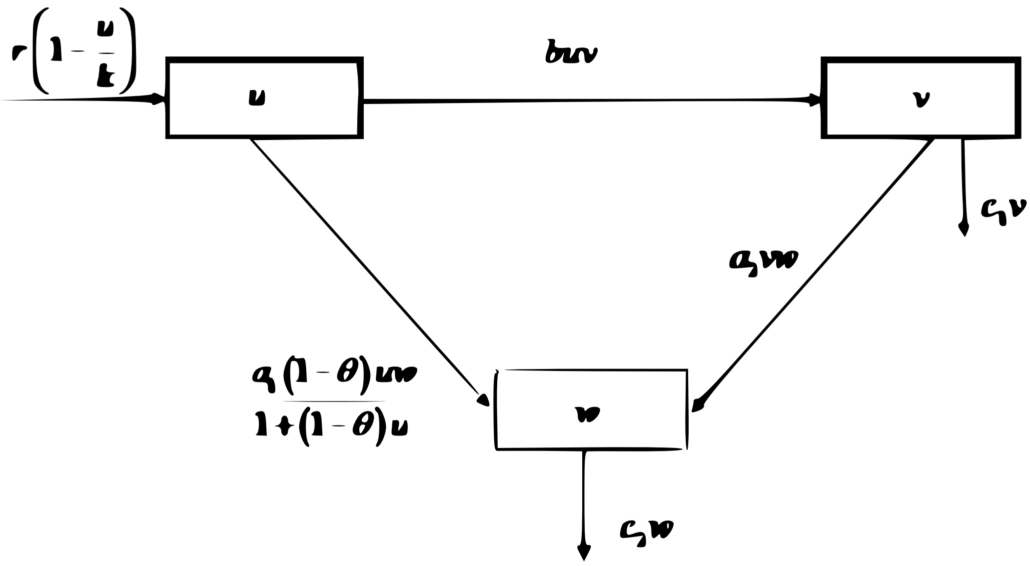

Fig. 1: Eco-epidemiological paradigm flow diagram

Parameters	Description in a biological sense
r	Intrinsic growth amount of prey
k	Vulnerable prey carrying capacity
b	Measure of illness spread among prey
c_1	Measure of mortality of disease-induced
e	Rate of predator conversion
a_1	Predation frequency of susceptible prey
a_2	Rate of predation on contaminated prey
c_2	Mortality rate of predator

Table 1: Definitions of model parameters

2.2 Preliminaries:

The most widely used fractional derivatives of mathematical modelling and engineering applications are Riemann-Liouville and Caputo derivatives, while there are many more types as well [11]. We use Caputo fractional derivatives in this work to create our model. The primary benefit of non-integer order differential equations is that their starting values adopt a similar type as those of integer order differential equations.

1.The fractional integral of order $\xi > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as follows:

$$I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t-\tau)^{\xi-1} f(\tau) d\tau \tag{7}$$

where Γ is the Euler gamma function.

2.The following definition applies to the Caputo fractional order derivative:

$$\frac{d^{\xi} f(t)}{dt^{\xi}} = I^{n-\xi} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\xi)} \int_0^t (t-\tau)^{n-\xi-1} f^{(n)}(\tau) d\tau \tag{8}$$

where Γ is the Euler gamma function, $f(t)$ is a time dependent function and ξ is the order of the derivative $(n-1 < \xi \leq n)$.

No one has, as far as we know, taken into account a fractional-order eco-epidemiological paradigm (4)-(6) that includes prey refuge with type II functional response. The fractional-order eco-epidemiological paradigm that this study discusses has illness in the prey population and a predator that feeds on both susceptible and diseased prey. This study's objectives are to explore the characteristics of the non-integer order eco-epidemiological paradigm that has been proposed (see (4)-(6)) and to provide support for a global stability analysis of such biologically viable equilibria. The above work examines the prosed fractional order eco-epidemiological paradigm (4)-(6), in which disease-infected prey are the prey, to see whether any solutions exist and if they are unique, non-negativity, and limited. At the provincial and universal levels, the asymptotic consistency of the resulting equilibrium points is examined. The resulting theoretical solutions are congruent with the dynamical behaviour of paradigm (4)-(6) as indicated by the numerical simulations.

3 Mathematical analysis

In this part the fractional order eco-epidemiological paradigm (4)-(6) is mathematically explored.

3.1 Existence and uniqueness

It is possible to investigate the presence and uniqueness of the non-integer order paradigm (4)-(6) solutions in the region $\Omega \times (0,T]$,
where $\Omega = \left\{(u,v,w) \in \mathbb{R}^3 : \max(|u|,|v|,|w|) \leq \psi\right\}$.

Theorem 1.*For each $V_0 = (u_0, v_0, w_0) \in \Omega$, can only have one solution $V(t) \in \Omega$ for paradigm (4)-(6) having initial condition V_0, which is specified by every $t \geq 0$.*

*Proof.*The procedure developed by Hong Li et al. [33]. Introduce a function $H(V) = (H_1(V), H_2(V), H_3(V))$, in which

$$H_1(V) = ru\left(1-\frac{u}{k}\right) - buv - \frac{a_1(1-\theta)wu}{1+\gamma(1-\theta)u}, \tag{9}$$

$$H_2(V) = buv - a_2vw - c_1v, \tag{10}$$

$$H_3(V) = \frac{ea_1(1-\theta)uw}{1+\gamma(1-\theta)u} + ea_2vw - c_2w. \tag{11}$$

For any $V, \bar{V} \in \Omega$ it follows from equations (9)-(11) that

$$\begin{aligned}
\|H(V)-H(\bar{V})\| &= |H_1(V)-H_1(\bar{V})| + |H_2(V)-H_2(\bar{V})| + |H_3(V)-H_3(\bar{V})| \\
&= \left| ru\left(1-\frac{u}{k}\right) - buv - \frac{a_1(1-\theta)wu}{1+\gamma(1-\theta)u} - r\bar{u}\left(1-\frac{\bar{u}}{k}\right) + b\bar{u}\bar{v} + \frac{a_1(1-\theta)\overline{wu}}{1+\gamma(1-\theta)\bar{u}} \right| \\
&\quad + |buv - a_2vw - c_1v - b\overline{uv} + a_2\bar{v}\bar{w} + c_1\bar{v}| \\
&\quad + \left| \frac{ea_1(1-\theta)uw}{1+\gamma(1-\theta)u} + ea_2vw - c_2w - \frac{ea_1(1-\theta)\overline{uw}}{1+\gamma(1-\theta)\bar{u}} - ea_2\overline{vw} + c_2\bar{w} \right| \\
&\leq \left| r + \frac{r}{k}(u+\bar{u}) \right| |u-\bar{u}| + b|uv-\overline{uv}| + \frac{a_1(1-\theta)}{(1+\gamma(1-\theta)u)(1+\gamma(1-\theta)\bar{u})} |(uw-\overline{uw}) + \gamma(1-\theta)u\bar{u}(w-\bar{w})| \\
&\quad + b|uv-\overline{uv}| + a_2|vw-\overline{vw}| + c_1|v-\bar{v}| + ea_2|vw-\overline{vw}| + c_2|w-\bar{w}| \\
&\quad + \frac{ea_1(1-\theta)}{(1+\gamma(1-\theta)u)(1+\gamma(1-\theta)\bar{u})} |(uw-\overline{uw}) + \gamma(1-\theta)u\bar{u}(w-\bar{w})| \\
&\leq \left(r + \frac{2r\psi}{k} \right) |u-\bar{u}| + 2b|uv-\overline{uv}| + \frac{a_1(1-\theta)}{(1+\gamma(1-\theta)u)(1+\gamma(1-\theta)\bar{u})} |(uw-\overline{uw}) + \gamma(1-\theta)u\bar{u}(w-\bar{w})| \\
&\quad + a_2|vw-\overline{vw}| + c_1|v-\bar{v}| + ea_2|vw-\overline{vw}| + c_2|w-\bar{w}| \\
&\quad + \frac{ea_1(1-\theta)}{(1+\gamma(1-\theta)u)(1+\gamma(1-\theta)\bar{u})} |(uw-\overline{uw}) + \gamma(1-\theta)u\bar{u}(w-\bar{w})| \\
&\leq \left(r + \frac{2r\psi}{k} + 2b\psi + \frac{a_1(1+e)(1-\theta)\psi}{(1+\gamma(1-\theta)\psi)^2} \right) |u-\bar{u}| + (2b\psi + c_1 + (1+e)a_2\psi)|v-\bar{v}| \\
&\quad + \left((1+e)a_2\psi + c_2 + \frac{(1+e)a_1\gamma(1-\theta)^2\psi^2}{(1+\gamma(1-\theta)\psi)^2} + \frac{a_1(1+e)(1-\theta)\psi}{(1+\gamma(1-\theta)\psi)^2} \right) |w-\bar{w}| \\
&\leq L\|(u,v,w)-(\bar{u},\bar{v},\bar{w})\| \\
&\leq L\|V-\bar{V}\|
\end{aligned}$$

where

$$L=\max\left\{\left(r+\frac{2r\psi}{k}+2b\psi+\frac{a_1(1+e)(1-\theta)\psi}{(1+\gamma(1-\theta)\psi)^2}\right),(2b\psi+c_1+(1+e)a_2\psi),\right.$$
$$\left.\left((1+e)a_2\psi+c_2+\frac{(1+e)a_1\gamma(1-\theta)^2\psi^2}{(1+\gamma(1-\theta)\psi)^2}+\frac{a_1(1+e)(1-\theta)\psi}{(1+\gamma(1-\theta)\psi)^2}\right)\right\}. \quad (12)$$

Therefore, meets Lipschitz criterion with respect to. As a result, the fractional order paradigm (4)-(6) has an only solution with the initial condition $V_0=(v_0,u_0,w_0)$.

3.2 *Non-negativity and boundedness*

We are solely concerned in the non-negative answer due to its biological importance. The data below demonstrate the answers to the fractional order paradigm are not negative (4)-(6). Paradigm (4)-(6) provides us with

$$\begin{aligned} {}^C D^{\xi} u(t)\Big|_{u=0} &=0,\\ {}^C D^{\xi} v(t)\Big|_{v=0} &=0,\\ {}^C D^{\xi} w(t)\Big|_{u=0} &=0. \end{aligned}$$

Therefore, according to Boukhouima et al.'s [34] lemma 5 and 6, the solutions of the non-integer order paradigm (4)-(6) are non-negative. The boundedness of the solutions to the non-integer order paradigm (4)-(6) is examined in the subsequent theorem.

Theorem 2.*The fractional order paradigm* (4)-(6) *solutions that begin in are all uniformly bounded.*

*Proof.*The method developed by Hong-Li et al. [33] is used. In light of the subsequent function, we may demonstrate that all results to paradigm (4)-(6) that begin in are uniformly limited by using the theorem.
We construct a function $\chi(t)=u(t)+v(t)+\frac{1}{e}w(t)$.
If we take the derivative of its fractional time, we obtain

$$\begin{aligned} {}^C D^{\xi}\chi(t) =& {}^C D^{\xi}u(t)+{}^C D^{\xi}v(t)+\frac{1}{e}{}^C D^{\xi}w(t)\\ =& ru\left(1-\frac{u}{k}\right)-buv-\frac{a_1(1-\theta)wu}{1+\gamma(1-\theta)u}+buv-a_2vw-c_1v+\frac{1}{e}\left(\frac{ea_1(1-\theta)uw}{1+\gamma(1-\theta)u}+ea_2vw-c_2w\right)\\ =& ru\left(1-\frac{u}{k}\right)-c_1v-\frac{c_2}{e}w. \end{aligned}$$

Now, for each we have

$${}^C D^{\xi}\chi(t)+\varepsilon\chi(t)=ru\left(1-\frac{u}{k}\right)-c_1v-\frac{c_2}{e}w+\varepsilon\left(u(t)+v(t)+\frac{1}{e}w(t)\right) \quad (13)$$

Taking $\varepsilon<\min\{c_1,c_2\}$, we have

$$\begin{aligned} &\leq -\frac{r}{k}u^2+(r+\varepsilon)u\\ &\leq \frac{k(r+\varepsilon)^2}{4r} \end{aligned}$$

Choi et al. [35]'s Lemma 9 leads to the conclusion that

$$0\leq\chi(t)\leq\chi(0)E_{\xi}\left(-\varepsilon t^{\xi}\right)+\frac{k(r+\varepsilon)^2}{4r}t^{\xi}E_{\xi,\xi+1}\left(-\varepsilon t^{\xi}\right),$$

In above expression, represents the Mittag-Leffler function. It follows from Lemma 5 and Corollary 6 in Choi et al. [35],

$$0\leq\chi(t)\leq\frac{k(r+\varepsilon)^2}{4r\varepsilon},\ \text{as } t\to\infty.$$

As a result, all the results of paradigm (4)-(6) commencing in $\mathbb{R}_+^3$ are uniformly bounded in the region Z, where $Z=\left\{(u,v,w)\in\mathbb{R}_+^3:\chi(t)\leq\frac{k(r+\varepsilon)^2}{4r\varepsilon}+\vartheta,\vartheta>0\right\}$.

4 Dynamical behaviour

Now we study the stability analysis of each equilibrium point of the proposed paradigm (4)-(6). The inner, or a state of harmonious coexistence, is what interests us. Positive(interior) equilibrium only exists for a certain constrained area of parameters because the refuge parameter is a system parameter. One can find the all-conceivable equilibrium by thinking about ${}^{C}D^{\xi}u(t)\Big|_{u=0}=0$, ${}^{C}D^{\xi}v(t)\Big|_{v=0}=0$ and ${}^{C}D^{\xi}w(t)\Big|_{u=0}=0$. Following is a list of all equilibrium points.

1.The trivial equilibrium point $E_0(0,0,0)$, which always exists.
2.The axial equilibrium point $E_1(k,0,0)$, which always exists.
3.The predator extinction equilibrium $E_2(u_1,v_1,0)$,
where $u_1=\frac{c_1}{b}$ and $v_1=\frac{r(bk-c_1)}{b^2k}$,
4.The disease-free equilibrium point $E_3(u_2,0,w_2)$,
where $u_2=\frac{c_2}{(ea_1-c_2\gamma)(1-\theta)}$ and $w_2=\frac{re}{(ea_1-c_2\gamma)(1-\theta)}\left[1-\frac{c_2}{(ea_1-c_2\gamma)(1-\theta)}\right]$
5.The coexisting equilibrium point $E_4(u^*,v^*,w^*)$ where
From equation (5) $\quad bu^*-a_2w^*-c_1=0\Rightarrow w^*=\frac{1}{a_2}(bu^*-c_1)$
From equation (6) $\quad v^*=\frac{1}{ea_2}\left[c_2-\frac{ea_1(1-\theta)u^*}{1+\gamma(1-\theta)u^*}\right]$

By inserting the findings from (5) and (6) into (4), we get the quadratic equation shown below. $A_1u^2+A_2u+A_3=0$ where $A_1=\frac{r\gamma}{k}(1-\theta)$; $A_2=\frac{r}{k}+\frac{bc_2\gamma}{ea_2}(1-\theta)-r\gamma(1-\theta)$; $A_3=\frac{bc_2}{ea_2}-r-\frac{a_1}{a_2}c_1(1-\theta)$
Then $u^*=\frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}$ The fractional order paradigm (4)-(6)'s Jacobian matrix is shown in the table below.

$$J=\begin{pmatrix} r\left(1-\frac{2u}{k}\right)-bv-\frac{a_1(1-\theta)w}{(1+\gamma(1-\theta)u)^2} & -bu & \frac{-a_1(1-\theta)u}{1+\gamma(1-\theta)u} \\ bv & bu-a_2w-c_1 & -a_2v \\ \frac{ea_1(1-\theta)w}{(1+\gamma(1-\theta)u)^2} & ea_2w & \frac{ea_1(1-\theta)u}{1+\gamma(1-\theta)u}+ea_2v-c_2 \end{pmatrix} \tag{14}$$

4.1 Stability analysis

This section examines the proposed model's local stability analysis in light of several potential equilibrium locations (excluding exterior equilibrium point). The type of the latent values in the appropriate Jacobian matrix near the point (u,v,w) determines the stability of the equilibrium state.

Theorem 3.*The interior equilibrium point $E_4(u^*,v^*,w^*)$ is locally asymptotic stable if $\zeta_1<1$and $\zeta_1+\zeta_2<1$ otherwise not stable.*

*Proof.*The corresponding variational matrix can be obtained by

$$J=\begin{pmatrix} -\frac{ru}{k}+\frac{a_1\gamma(1-\theta)^2uw}{(1+\gamma(1-\theta)u)^2} & -bu & \frac{-a_1(1-\theta)u}{1+\gamma(1-\theta)u} \\ bv & 0 & -a_2v \\ \frac{ea_1(1-\theta)w}{(1+\gamma(1-\theta)u)^2} & ea_2w & 0 \end{pmatrix} \tag{15}$$

The following latent equation can be obtained from (15),

$$\lambda^3+L_1\lambda^2+L_2\lambda+L_3=0$$

where

$$\begin{aligned} L_1 &=\frac{ru}{k}-\frac{a_1\gamma(1-\theta)^2uw}{(1+\gamma(1-\theta)u)^2}, \\ L_2 &=a_2^2evw+\frac{a_1^2(1-\theta)^2euw}{(1+\gamma(1-\theta)u)^3}+b^2uv \\ L_3 &=ea_2^2vw\left(\frac{ru}{k}-\frac{a_1\gamma(1-\theta)^2uw}{(1+\gamma(1-\theta)u)^2}\right)+\frac{b\gamma(1-\theta)^2ea_1a_2u^2vw}{(1+\gamma(1-\theta)u)^2} \end{aligned}$$

Clearly, $L_1>0, L_2>0, L_3>0$ and $L_1L_2-L_3=.\left(\frac{ru}{k}-\frac{a_1\gamma(1-\theta)^2uw}{(1+\gamma(1-\theta)u)^2}\right)+\left[b^2uv+\frac{a_1^2(1-\theta)^2euw}{(1+\gamma(1-\theta)u)^3}\right]-\frac{b\gamma(1-\theta)^2ea_1a_2u^2vw}{(1+\gamma(1-\theta)u)^2}$

If $\zeta_1 < 1$ and $\zeta_1 + \zeta_2 < 1$ where

$$\zeta_1 = \frac{ka_1\gamma w(1-\theta)^2}{r(1+\gamma(1-\theta)u)^2}, \quad \zeta_2 = \frac{ka_1(1-\theta)^2\gamma w(1+\gamma(1-\theta)u)}{b^2ruv(1+\gamma(1-\theta)u)^3 + a_1^2(1-\theta)^2 eruw}$$

The coexistence equilibrium point of paradigm (4)-(6) is locally asymptotically stable according to the Routh-Hurwitz criterion.

4.2 Global stability

The global stability of the paradigm (4)-(6) near the interior equilibrium point is being investigated in this section using a suitable Lyapunov function.

Theorem 4. *Around the coexistence equilibrium point, the suggested system is expected to be globally asymptotically stable if* $\left\{ \frac{r}{k} > \frac{a_1(1-\theta)^2\gamma}{(1+\gamma(1-\theta)u_4)(1+\gamma(1-\theta)u)} \right\}$ *holds.*

Proof. The global asymptotic stability of the coexistence point is explored using the subsequent positive definite Lyapunov function

$$Q(u,v,w) = u - u_4 - u_4 \ln\left(\frac{u}{u_4}\right) + v - v_4 - v_4 \ln\left(\frac{v}{v_4}\right) + \frac{1}{e}\left(w - w_4 - w_4 \ln\left(\frac{w}{w_4}\right)\right) \tag{16}$$

Lemma 1. *From Vargas-De-Leon [36] and the time derivative of along the scheme (4)-(6) solution is used.*

$$\begin{aligned}
CD^{\xi}Q(u,v,w) &\le \left(\frac{u-u_4}{u}\right)^C D^{\xi}u(t) + \left(\frac{v-v_4}{v}\right)^C D^{\xi}v(t) + \frac{1}{e}\left(\frac{w-w_4}{w}\right)^C D^{\xi}w(t) \\
&\le (u-u_4)\left(r\left(1-\frac{u}{k}\right) - bv - \frac{a_1(1-\theta)w}{1+\gamma(1-\theta)u}\right) \\
&+ (v-v_4)(bu - a_2w - c_1) + (w-w_4)\left(\frac{a_1(1-\theta)u}{1+\gamma(1-\theta)u} + a_2v - \frac{c_2}{e}\right) \\
&\le (u-u_4)\left\{\begin{array}{l} -\frac{r}{k}(u-u_4) - b(v-v_4) + \frac{a_1(1-\theta)w}{(1+\gamma(1-\theta)u_4)(1+\gamma(1-\theta)u)} \\ (-(w-w_4) + \gamma(1-\theta)[w_4(u-u_4) - u_4(w-w_4)]) \end{array}\right\} \\
&+ (v-v_4)\{b(u-u_4) - a_2(w-w_4)\} \\
&+ (w-w_4)\left\{ea_2(v-v_4) + \frac{ea_1(1-\theta)(u-u_4)(w-w_4)}{(1+\gamma(1-\theta)u_4)(1+\gamma(1-\theta)u)}\right\} \\
&\le -\left\{\frac{r}{k} - \frac{a_1(1-\theta)^2\gamma}{(1+\gamma(1-\theta)u_4)(1+\gamma(1-\theta)u)}\right\}(u-u_4)^2
\end{aligned}$$

Hence at the coexistence equilibrium point the system is globally asymptotically stable if

$$\frac{r}{k} > \frac{a_1(1-\theta)^2\gamma}{(1+\gamma(1-\theta)u_4)(1+\gamma(1-\theta)u)}$$

holds.

5 Numerical simulations

To demonstrate the theoretical conclusion about the non integral order and global stability areas for the equilibrium points of the system, numerical simulations of the fractional order eco-epidemiological paradigm (4)-(6) are carried out under this part. For the numerical simulations of the scheme (4)-(6) based on the following fractional differential equation, the Adams-Bashforth-Moulton scheme is utilised (Diethelm et al. [37,38], Li et al. [39], Garrappa, [40]).

These simulations are extremely helpful from an eco-epidemiological point of view since they also demonstrate the impact of non-integral order and prey refuge with type II functional response on the stability of the equilibrium points. The parameter settings shown in the figure captions have been used in all numerical runs to approximation the answer. Using varied choices of the model parameters and beginning circumstances indicated in the respective captions, Figure 2 displays the numerical simulations of models (4)-(6).

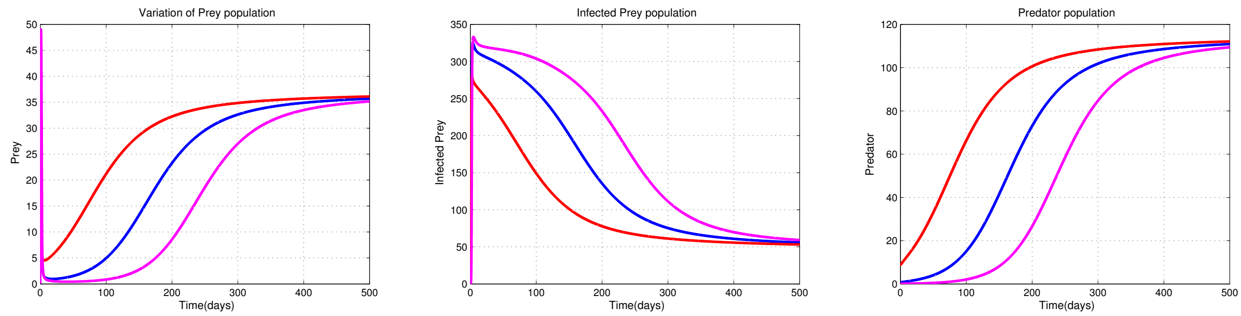

Fig. 2: The figures represent the trajectories of the system. (4)-(6) for $r = 20, k = 50, b = 0.0623, a_1 = 0.6, a_2 = 0.02, \theta = 0.08, \gamma = 0.8, c_1 = 0.00847, c_2 = 0.02, e = 0.0123$ and $\xi = 0.8$ with different initial conditions $(0.363, 0.139, 0.329), (5.95, 8.6, 6.9), (5, 10, 15)$.

6 Discussions and Conclusion

In this study, a non-integer order eco-epidemiological scheme with disease prey and prey refuge with type II functional responses was investigated. Analysis has been done on the behaviour of the suggested non integer order eco-epidemiological scheme (4)-(6). The explorations of the interior equipoise point of the non-integer order scheme (4)-(6) have demonstrated both provincial and universal stability. The features of the scheme's equilibrium points with regard to fractional order (ξ) and global stability are as have been demonstrated by numerical simulations (4)-(6).

By imposing multiple necessary conditions on the system parameters, we employed stability analysis of non-integer order scheme to demonstrate the local stability of the coexistence equilibrium. On the other hand, the inner equilibrium is consistently and globally asymptotically stable for any ξ. Although the qualitative characteristics of the solutions are identical from those of integer order systems, our numerical findings indicate that solutions of non-integer order systems approach to the corresponding equilibrium in a slower manner as the order of the differential equation decreases. It will be very interesting to see how harvesting is included into the system in the future and what effects that has on real-world scenarios.

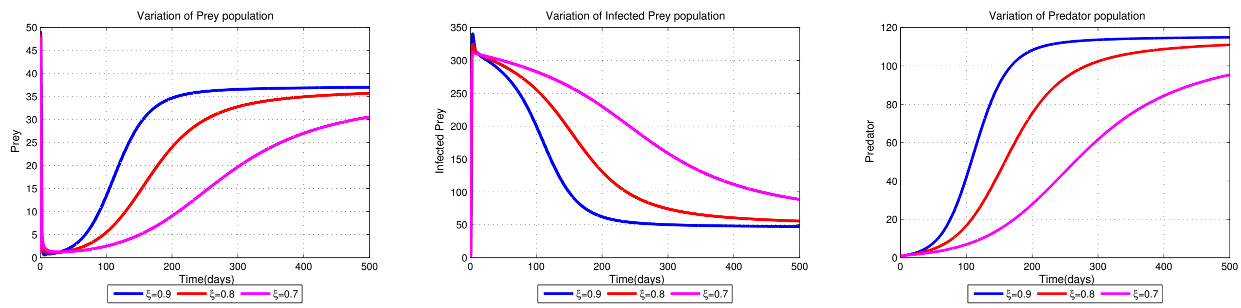

Fig. 3: The figures represent the trajectories of the system. (4)-(6) for $r = 20, k = 50, b = 0.0623, a_1 = 0.6, a_2 = 0.02, \theta = 0.08, \gamma = 0.8, c_1 = 0.00847, c_2 = 0.02, e = 0.0123$ and for different values of $\xi =$ with initial conditions $(5, 10, 15)$.

Figure 3 presents time series solutions for the system for various fractional orders to show how the system behaves. Higher order fractional time assessments indicate that solutions reach equilibrium more quickly.

The impacts of fractional order ξ and prey refuge on population densities are also shown in figures 4 and 5 using the simulation outcomes. This demonstrates how the fractional order and prey refuge have a significant impact on the system's dynamic behaviour. Our figures primarily show that fractional order has a greater impact on the system than does prey refuge.

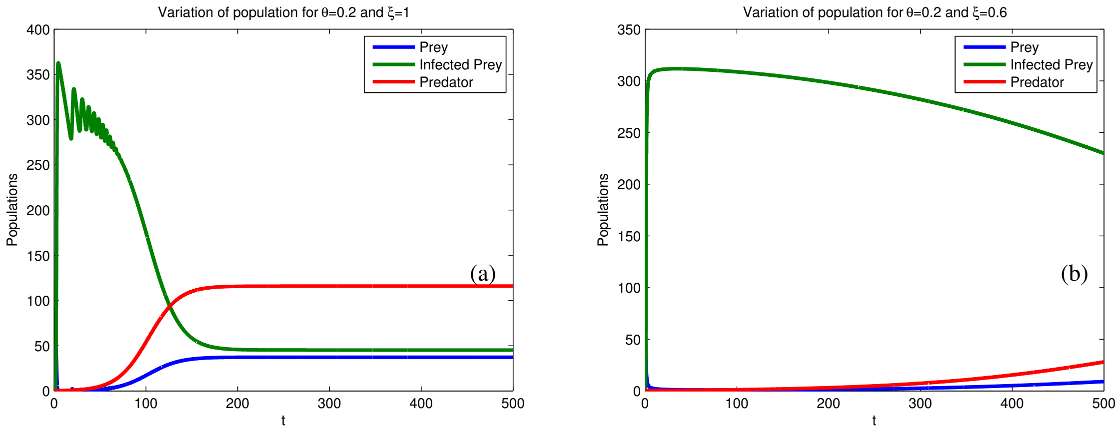

Fig. 4: The figures represent the trajectories of the system (4)-(6) for $a)\theta = 0.2, \xi = 1,\;\; b)\theta = 0.2, \xi = 0.6$ and remaining parameter values are same as figure 1.

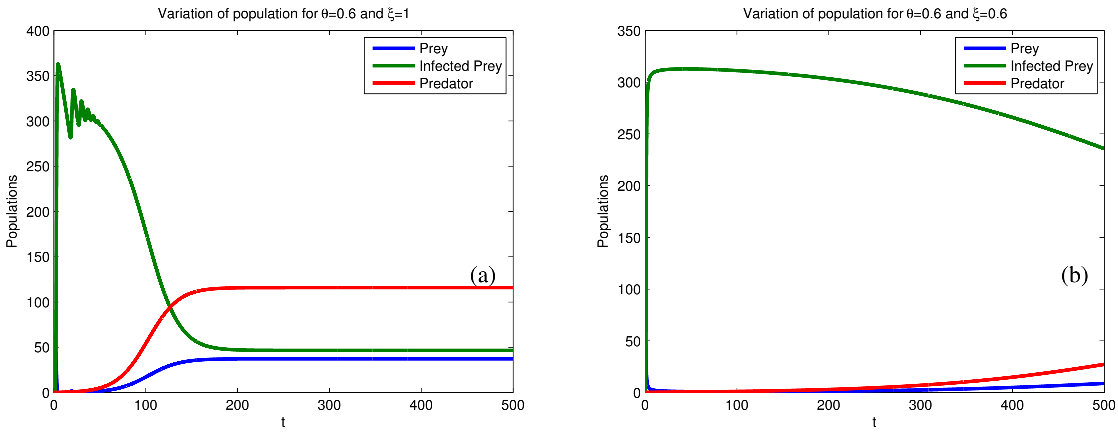

Fig. 5: The figures represent the trajectories of the system (4)-(6) for $a)\theta = 0.6, \xi = 1,\;\; b)\theta = 0.6, \xi = 0.6$ and remaining parameter values are same as figure 1.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgment

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445). The authors are thankful to the Deanship of Graduate Studies and Scientific Research at University of Bisha for supporting this work through the Fast-Track Research Support Program.

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