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Λ-Fractional Forced Van der Pol Oscillator with Horizon

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Abstract: Extending the investigation of the modified Van der Pol equation with its chaotic behaviour into the context of Λ-fractional analysis, the solutions of the Λ-fractional Van der Pol oscillator are studied under the influence of a horizon restricting the region of non-locality. The various solutions to the equation depend upon the horizon, however, the chaotic behaviour of the oscillator mainly depends upon the fractional order. A comparison of the chaotic behaviour of the various Van der Pol oscillator systems is presented.

Keywords: fractional order, fractal dimension, fractional integral, Riemann-Liouville fractional derivative, Λ-fractional derivative, left and right Λ-spaces, horizon.

1 Introduction

Although fractional calculus is a topic introduced almost three centuries ago, its significance has only recently been appreciated, and it has been mainly used to mathematically address non-locality. Experiments, primarily in micromechanics, have demonstrated the importance of the non-locality of particles, thus lifting Noll's axiom of local action [1]. The non-locality principle, applied mostly spatially, can also be applied in time, where phenomena present a persistent "memory". A particular case of a temporally non-local phenomenon is the behaviour of the Van der Pol oscillator, as implemented by Lazopoulos et al. [2]. In the context of that work, the authors demonstrated the utility of the Λ-derivative and of the Λ-space investigation for Van der Pol oscillators.

Fractional calculus and fractional derivatives were first introduced by Leibniz [3], Liouville [4], and Riemann [5], and subsequently were investigated by many other mathematicians [6]. The most well-known fractional derivatives are the Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives. The Riemann-Liouville and Caputo fractional derivatives are the result of successive integration, whereas the Grünwald-Letnikov fractional derivative is the result of successive differentiation.

Fractional derivatives are implemented in a wide range of applications in physics [7,8], mechanics [9,10,11,12,13, 14], chemistry, biology [15], control systems, and economics, among other fields. Although their application in science has become extensive, fractional derivatives present a significant weakness: they do not comply with the prerequisites of a proper derivative according to differential topology. Therefore, they do not correspond to differentials capable of generating differential geometry. In turn, the implementation of fractional calculus in applications of differential geometry, solution of differential equations, and variational approaches can be problematic.

This important theoretical gap was covered by the introduction of the Λ-fractional derivative (Λ-FD) along with the Λ-fractional space (Λ-space) by Lazopoulos et al. [16]. Briefly, the mathematical problem is transferred from the initial space to a virtual space where all quantities are conventional (non-fractional). The transferring is accomplished by means of a particular transform, namely the Λ-transform. In Λ-space all terms are local and derivatives exist. In turn, differentials are properly defined and capable of generating a fractional differential geometry. Then, the various results can be transferred back to the initial space. It is pointed out that only functions can be transferred in this fashion, but not derivatives. Thus, in order to solve a differential equation in Λ-space, terms are transferred back to the initial using the transformation formula. For more details, the interested reader may refer to Lazopoulos et al. [16].

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The chaotic behaviour of the Van der Pol oscillator has been previously discussed in the context of Λ-fractional analysis, with the introduction of the fractional order γ to the Van der Pol equation and its formulation in terms of the Λ-FD [2]. In the present work, the chaotic behaviour of the Van der Pol oscillator is discussed in the context of Λ-fractional analysis with a horizon. The idea has been adopted from the theory of peridynamics [17,18], and it accounts for the influence of particles within an area of distance h from a specific particle x under consideration. In other words, that distance defines the neighbourhood influences on a particular particle in space. Therefore, the horizon expresses non-locality, a phenomenon also addressed by fractional calculus. The use of the horizon is decisive in the study of natural phenomena as it highlights the effect of locality on the behaviour of each point in relation to its effect on the overall system.

2 The Λ-Fractional Derivative

The current section presents a brief outline of fractional calculus. For more information on the subject, the interested reader may refer to the literature [10,11,12].

The left and right fractional integrals for an actual fractional dimension $0 < \gamma \leq 1$ are defined as

$$ {}_aI_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(s)}{(x-s)^{1-\gamma}} ds, \tag{1} $$

$$ {}_xI_b^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_x^b \frac{f(s)}{(s-x)^{1-\gamma}} ds, \tag{2} $$

where $\Gamma(\gamma)$ is Euler's Gamma function. Further, the left and right Riemann-Liouville (RL) fractional derivatives are defined as

$$ {}_a^{RL}D_x^\gamma f(x) = \frac{d}{dx}\left({}_aI_x^{1-\gamma} f(x)\right) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \frac{f(s)}{(x-s)^\gamma} ds, \tag{3} $$

$$ {}_x^{RL}D_b^\gamma f(x) = \frac{d}{dx}\left({}_xI_b^{1-\gamma} f(x)\right) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_x^b \frac{f(s)}{(s-x)^\gamma} ds. \tag{4} $$

For the left fractional integrals and derivatives, the following relation holds:

$$ {}_a^{RL}D_x^\gamma\left({}_aI_x^\gamma f(x)\right) = f(x). \tag{5} $$

A similar relation is true for the right fractional integral and RL-derivative.

The Λ-fractional derivative is defined as

$$ {}_a^\Lambda D_x^\gamma f(x) = \frac{{}_a^{RL}D_x^\gamma f(x)}{{}_a^{RL}D_x^\gamma x}. \tag{6} $$

Recalling the definition of the RL-derivative, Eq. (3), Λ-FD can be expressed as

$$ {}_a^\Lambda D_x^\gamma f(x) = \frac{\frac{d}{dx}\left({}_aI_x^{1-\gamma} f(x)\right)}{\frac{d}{dx}\left({}_aI_x^{1-\gamma} x\right)} = \frac{d\, {}_aI_x^{1-\gamma} f(x)}{d\, {}_aI_x^{1-\gamma} x}. \tag{7} $$

Considering

$$ \begin{aligned} X &= {}_aI_x^{1-\gamma} x, \\ F(X) &= {}_aI_x^{1-\gamma} f(x(X)), \end{aligned} \tag{8} $$

Λ-FD behaves like a conventional derivative with local properties in Λ-space $(X, F(X))$, and a fractional differential geometry can be generated as a conventional differential geometry therein. Subsequently, results can be transferred from Λ-space to the initial space by invoking Eq. (5), in the following fashion:

$$ f(x) = {}_a^{RL}D_x^{1-\gamma} F(X(x)) = {}_a^{RL}D_x^{1-\gamma}\left({}_aI_x^{1-\gamma} f(x)\right). \tag{9} $$

Let us now consider a function $f(x)$ in the initial space $(x, f(x))$. Then, the corresponding function in Λ-space is defined by $(X, F(X))$. Taking in to account the existence of a horizon h, the function $f(x)$ corresponds in Λ-space to $F(X) - F(X-h)$. The Taylor expansion of $F(X-h)$ reads:

$$ F(X-h) \approx F(X) - F'(X)h + o(h). \tag{10} $$

Therefore, the function in Λ-space with horizon h becomes

$$F(X) - F(X-h) \approx F'(X)h. \tag{11}$$

Finally, transferring the result to the initial space yields:

$$f(x) = h\, {}_{a}^{RL}D_x^{1-\gamma} F'(X(x)). \tag{12}$$

3 Van der Pol Oscillator with Horizon

The present study investigates the initial space response $y(t)$ of the Λ-fractional forced Van der Pol oscillator, upon which we have imposed the novel idea of the horizon. The investigation is performed for various values of the fractional order γ. In a previous work, we examined the behaviour of this system without horizon, and highlighted the features and changes of the response $y(t)$ in the initial space, for different γ values [2].

The forced Van der Pol oscillator in Λ-space is described by the following second order differential equation:

$$\frac{d^2Y}{dT^2} - \mu(1-Y^2)\frac{dY}{dT} + Y = F\cos(\omega T), \tag{13}$$

where $T(t)$, $Y(T(t))$ are time and displacement, respectively, in Λ-space, μ is a real-valued parameter expressing non-linearity and strength of damping, F is the amplitude of the driving force and ω its frequency. The time variable in Λ-space, $T(t)$, is related to the time variable in the initial space, t, by means of the relation:

$$T(t) = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}. \tag{14}$$

In order to implement the idea of the horizon in the present context, we apply the differential operator $h\,d\square/dT$ on the left hand side of Eq. (13), and obtain

$$\frac{d^3Y}{dT^3} - \mu(1-Y^2)\frac{d^2Y}{dT^2} + 2\mu Y\left(\frac{dY}{dT}\right)^2 + \frac{dY}{dT} = \frac{F}{h}\cos(\omega T), \tag{15}$$

where h is the extent (value) of the horizon. Differentiation takes place on the left hand side because, according to the theory of peridynamics, only internal forces are considered to change.

The differential equation (15) is solved numerically to provide the solution $Y(T(t))$ in Λ space. Subsequently, the corresponding solution $y(t)$ in real time space is obtained from the relation

$$y(t) = {}_{0}^{RL}D_t^{1-\gamma} Y(t) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t \frac{Y(s)}{(t-s)^{1-\gamma}}ds. \tag{16}$$

In the following, the graphs of the orbit $Y(T)$ versus time T, and the phase space dY/dT versus $Y(T)$, are presented in Λ-space for various values of the parameter μ, along with the corresponding solutions $y(t)$ for various values of fractional order γ in the initial space. In all solutions, the initial conditions were considered as: $Y(0) = 1$, $(dY/dT)(0) = 0$, $(d^2Y/dT^2)(0) = 0$, the frequency was set to $\omega = 3$, and the horizon value to $h = 0.5$.

The responses $Y(T)$ and phase plane graphs dY/dT in Λ-space of the non-damped ($\mu = 0$) forced Van der Pol oscillator of fractional order $\gamma = 1$, for the cases with and without horizon, are shown in Figures 1 and 2, respectively.

Compared to the system without horizon (Fig. 1), the system under the influence of horizon presents a markedly different periodicity pattern, larger $Y(T)$ oscillation amplitude, different phase plane plot, and smaller range of velocity dY/dT values (Fig. 2). For non-unity fractional order, the values $\gamma = 0.8$ and 0.6 were considered. The responses $y(t)$ and phase plane graphs dy/dt in real space of the non-damped forced Van der Pol oscillator of fractional order $\gamma = 0.8$, for the cases with and without horizon, are shown in Figures 3 and 4, respectively. For $\gamma = 0.6$, the same cases are shown in Figures 5 and 6, respectively.

In all four cases, oscillation period and amplitude are not constant any more, with the former decreasing and the latter increasing with increasing time t. For $\gamma = 0.8$, oscillation amplitudes for the system without horizon are slightly smaller than the system under the influence of the horizon, while the range of velocity values in the system with horizon is smaller (Figs. 3, 4). For $\gamma = 0.6$, the converse is true regarding amplitudes, however, the range of velocity values in the system with horizon is also smaller (Figs. 5, 6). Although periodicity is qualitatively and quantitatively different between same

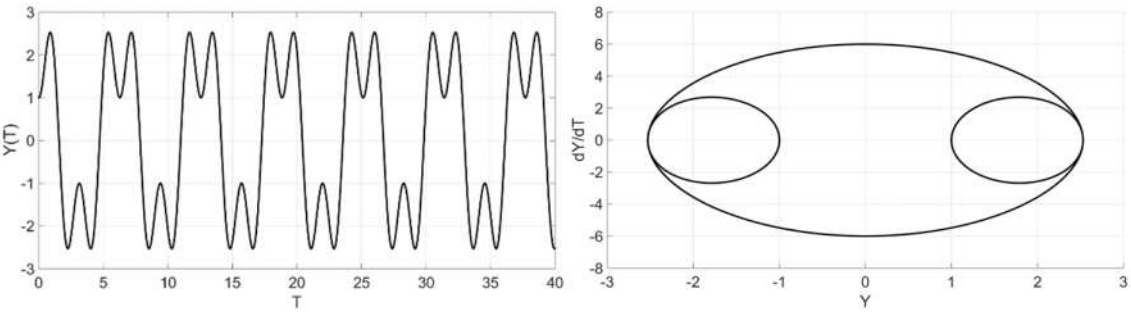

Fig. 1: Solution $Y(T)$ (left) and phase plane (right) in Λ-space without horizon for $\mu = 0$, $F = 10$, and $\gamma = 1$.

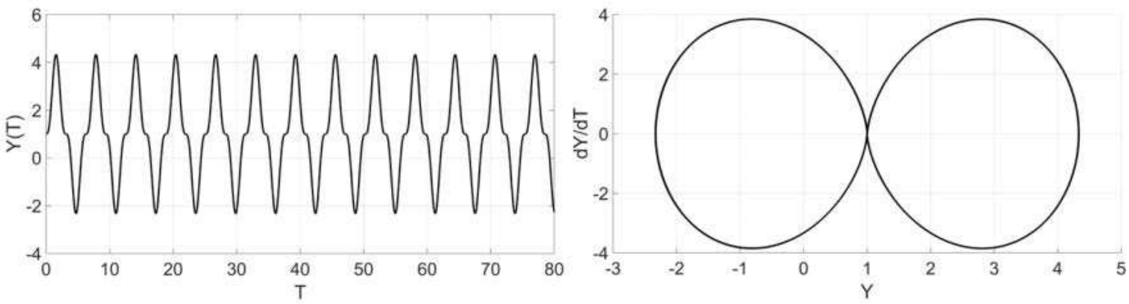

Fig. 2: Solution $Y(T)$ (left) and phase plane (right) in Λ-space with horizon for $\mu = 0$, $F = 10$, and $\gamma = 1$.

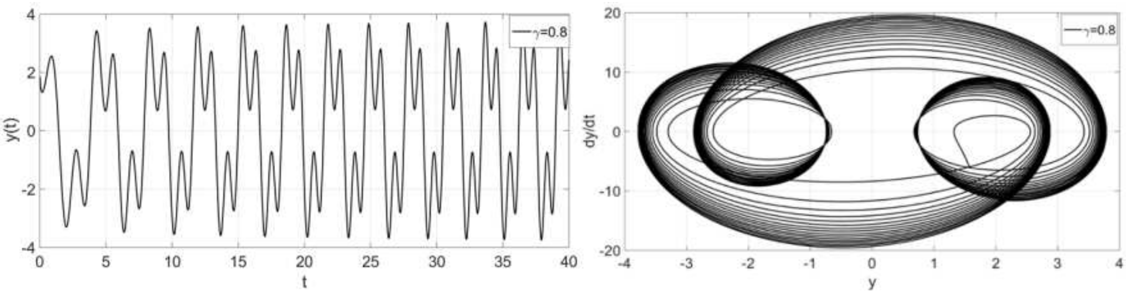

Fig. 3: Solution $y(t)$ (left) and phase plane (right) in real space without horizon for $\mu = 0$, $F = 10$, and $\gamma = 0.8$.

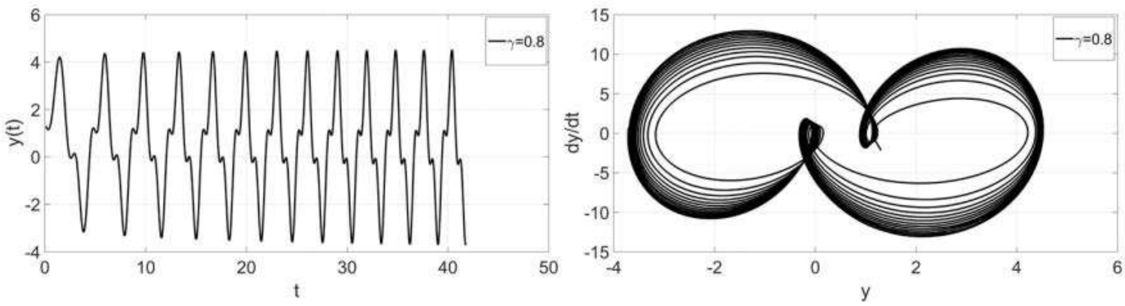

Fig. 4: Solution $y(t)$ (left) and phase plane (right) in real space with horizon for $\mu = 0$, $F = 10$, $\omega = 3$, and $\gamma = 0.8$.

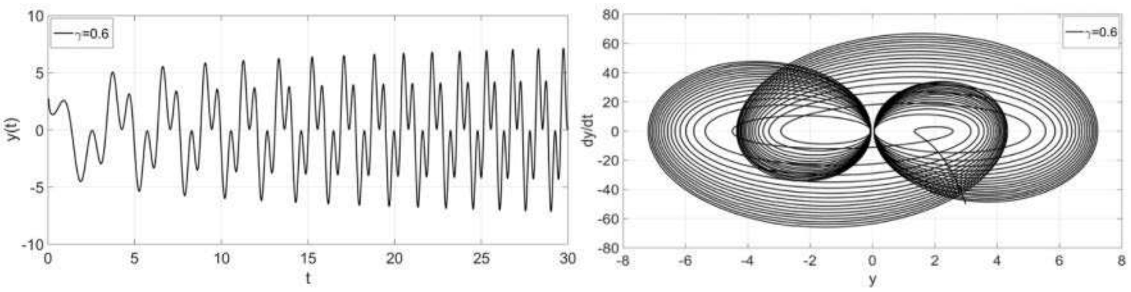

Fig. 5: Solution $y(t)$ (left) and phase plane (right) in real space without horizon for $\mu = 0$, $F = 10$, and $\gamma = 0.6$.

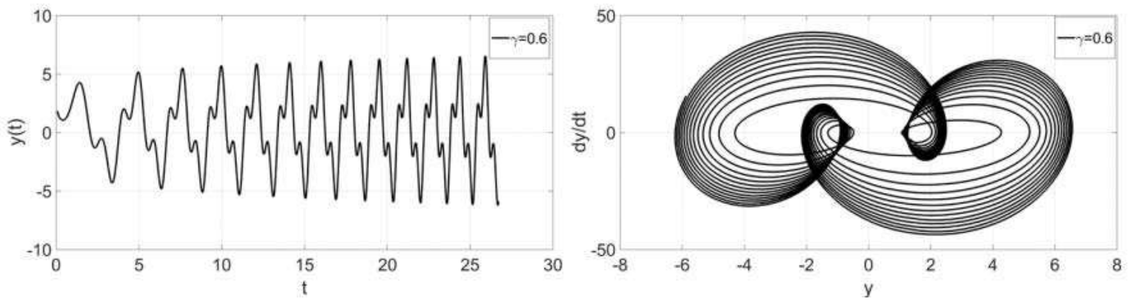

Fig. 6: Solution $y(t)$ (left) and phase plane (right) in real space with horizon for $\mu = 0$, $F = 10$, and $\gamma = 0.6$.

fractional order systems with and without horizon (compare Fig. 3 with 4, and 5 with 6), it is only quantitatively different between different fractional order systems with or without horizon (compare Fig. 3 with 5, and 4 with 6). In particular, decreasing the fractional order from $\gamma = 0.8$ to 0.6 resulted in decrease of oscillation period, increase of amplitude, and increase of the range of velocity values.

Subsequently, the behaviour of the forced Van der Pol oscillator was investigated in the presence of damping ($\mu \neq 0$). The responses $Y(T)$ and phase plane graphs dY/dT in Λ-space of the oscillator for $\mu = 1$ and $\gamma = 1$, for the cases with and without horizon, are shown in Figures 7 and 8, respectively.

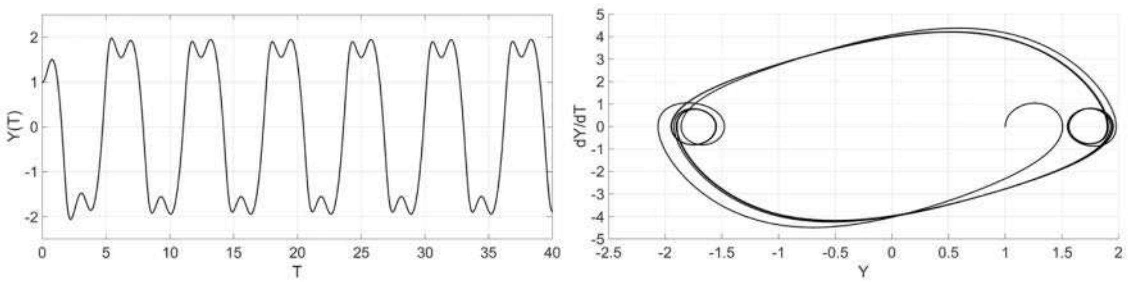

Fig. 7: Solution $Y(T)$ (left) and phase plane (right) in Λ-space without horizon for $\mu = 1$, $F = 5$, and $\gamma = 1$.

Contrary to previous cases, the behaviours of the systems with and without the influence of horizon are no longer directly comparable. In the case of the oscillator with horizon, a purely chaotic behaviour emerged where response $Y(T)$ values are only positive and oscillations take place around an increasing median line (Fig. 8).

The responses $y(t)$ and phase plane graphs dy/dt in real space of the forced Van der Pol oscillator with damping $\mu = 1$ and fractional order $\gamma = 0.8$, for the cases with and without horizon, are shown in Figures 9 and 10, respectively. For $\gamma = 0.6$, the same cases are shown in Figures 11 and 12, respectively.

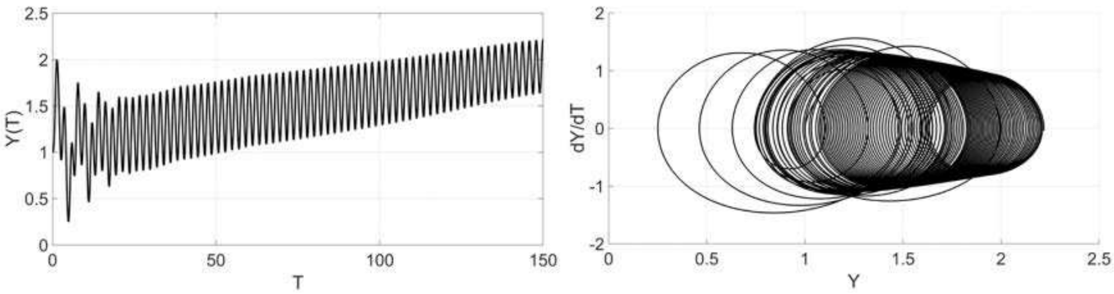

Fig. 8: Solution $Y(T)$ (left) and phase plane (right) in Λ-space with horizon for $\mu = 1$, $F = 5$, and $\gamma = 1$.

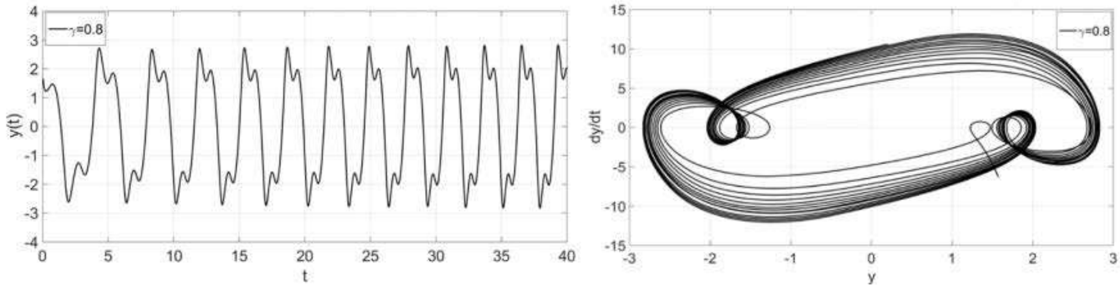

Fig. 9: Solution $y(t)$ (left) and phase plane (right) in real space without horizon for $\mu = 1$, $F = 5$, and $\gamma = 0.8$.

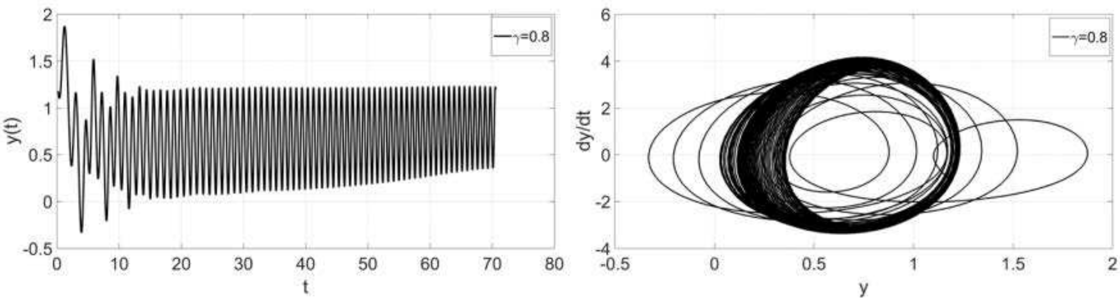

Fig. 10: Solution $y(t)$ (left) and phase plane (right) in real space with horizon for $\mu = 1$, $F = 5$, and $\gamma = 0.8$.

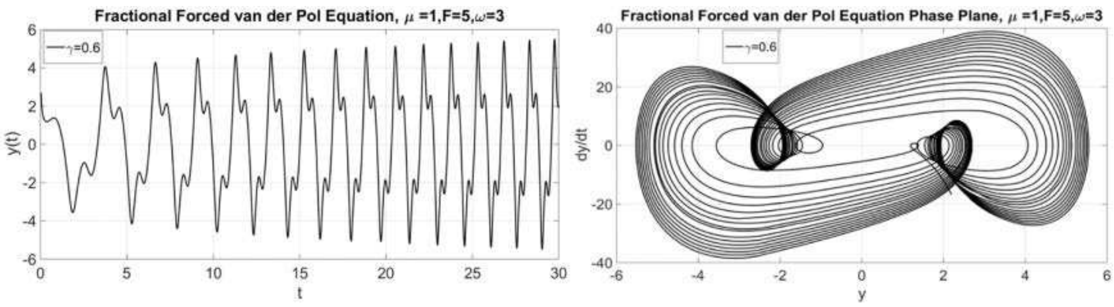

Fig. 11: Solution $y(t)$ (left) and phase plane (right) in real space without horizon for $\mu = 1$, $F = 5$, and $\gamma = 0.6$.

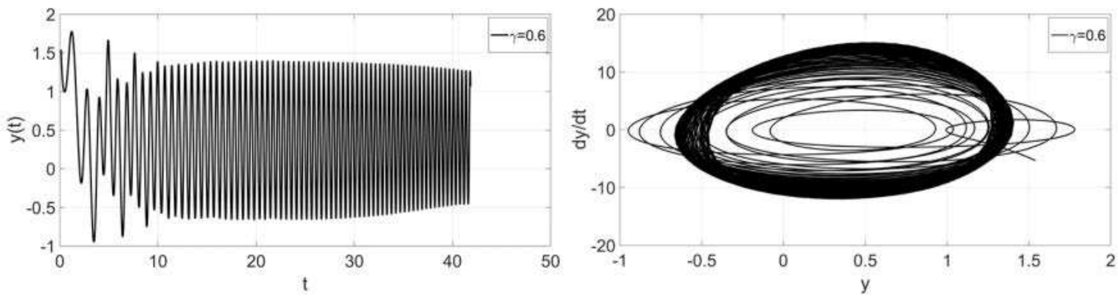

Fig. 12: Solution $y(t)$ (left) and phase plane (right) in real space with horizon for $\mu = 1$, $F = 5$, and $\gamma = 0.6$.

Again, although the responses are markedly different between same fractional order systems with and without horizon (compare Fig. 9 with 10, and 11 with 12), there are generally only quantitatively different between different fractional order systems with or without horizon (compare Fig. 9 with 11, and 10 with 12).

Further, increasing the value of damping to $\mu = 10$, the responses $Y(T)$ and phase plane graphs dY/dT in Λ-space of the oscillator for $\gamma = 1$, for the cases with and without horizon, are shown in Figures 13 and 14, respectively.

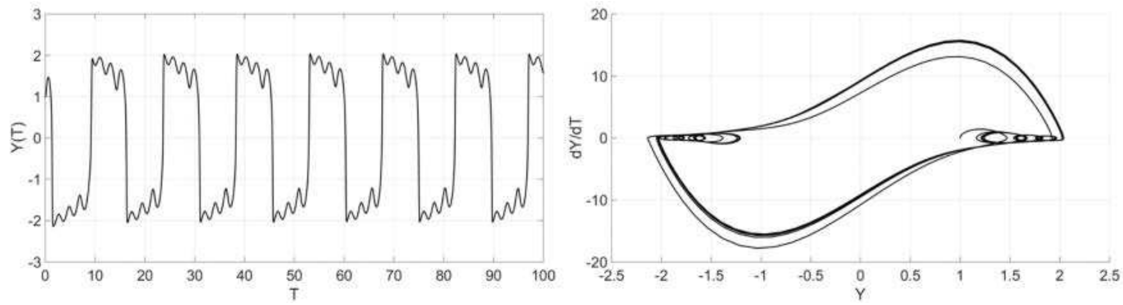

Fig. 13: Solution $Y(T)$ (left) and phase plane (right) in Λ-space without horizon for $\mu = 10$, $F = 10$, and $\gamma = 1$.

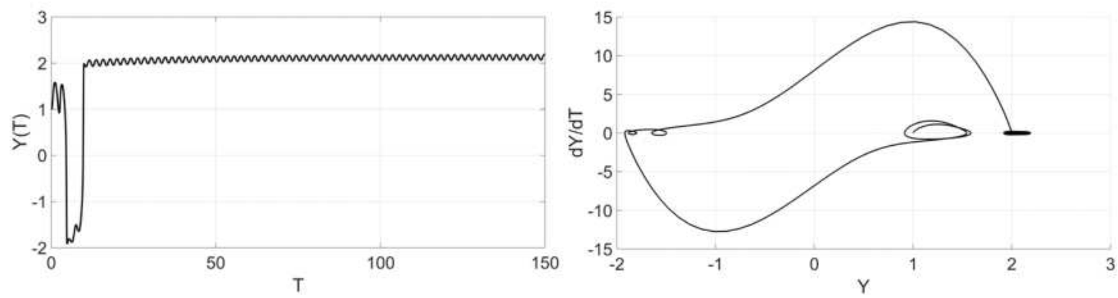

Fig. 14: Solution $Y(T)$ (left) and phase plane (right) in Λ-space with horizon for $\mu = 10$, $F = 10$, and $\gamma = 1$.

As demonstrated in these figures, the effect of the horizon on the system in Λ-space produces oscillations of excessively small amplitude, keeping the system around a possible attractor (Fig. 14), as opposed to a limit cycle behaviour in the case without horizon (Fig. 13).

The responses $y(t)$ and phase plane graphs dy/dt in real space of the forced Van der Pol oscillator with damping $\mu = 10$ and fractional order $\gamma = 0.8$, for the cases with and without horizon, are shown in Figures 15 and 16, respectively. For $\gamma = 0.6$, the same cases are shown in Figures 17 and 18, respectively.

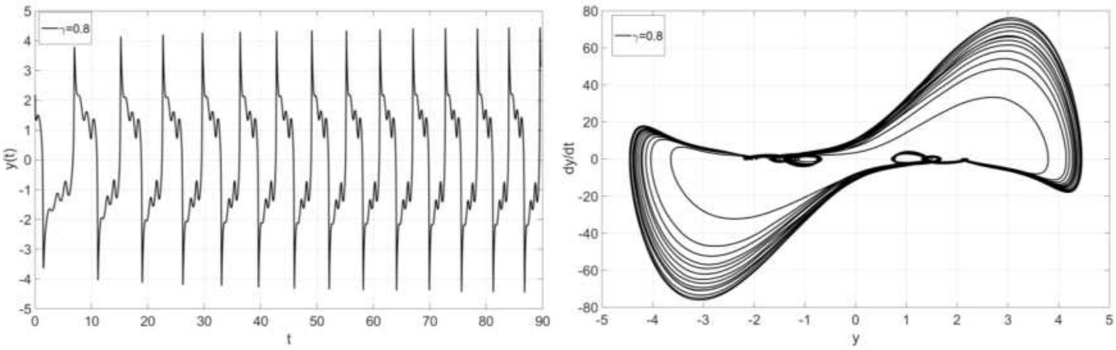

Fig. 15: Solution $y(t)$ (left) and phase plane (right) in real space without horizon for $\mu = 10$, $F = 10$, and $\gamma = 0.8$.

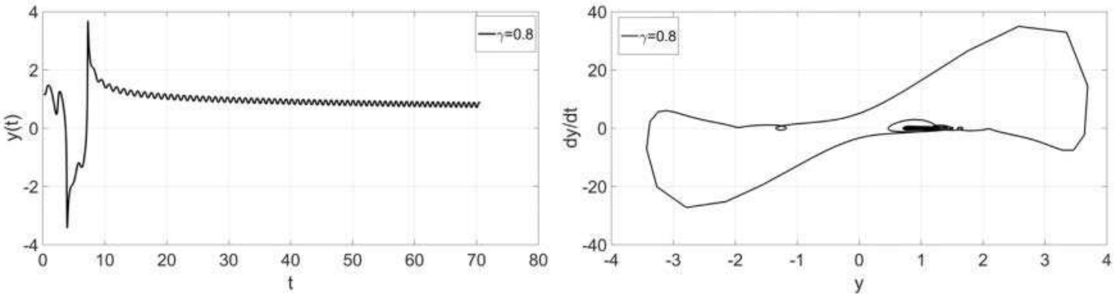

Fig. 16: Solution $y(t)$ (left) and phase plane (right) in real space with horizon for $\mu = 10$, $F = 10$, and $\gamma = 0.8$.

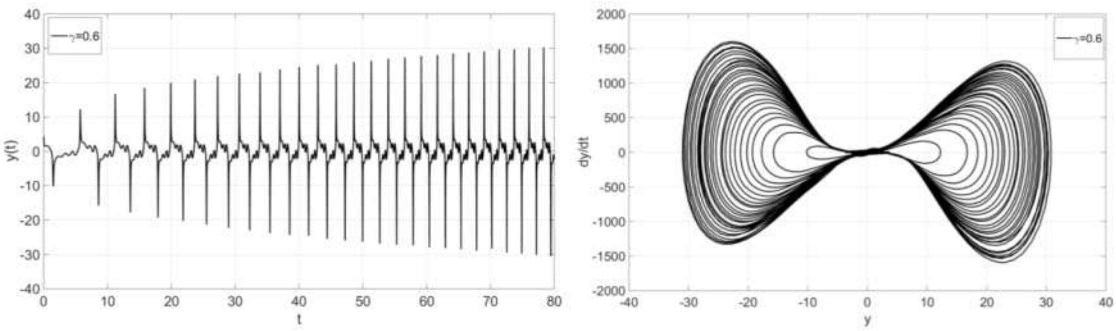

Fig. 17: Solution $y(t)$ (left) and phase plane (right) in real space without horizon for $\mu = 10$, $F = 10$, and $\gamma = 0.6$.

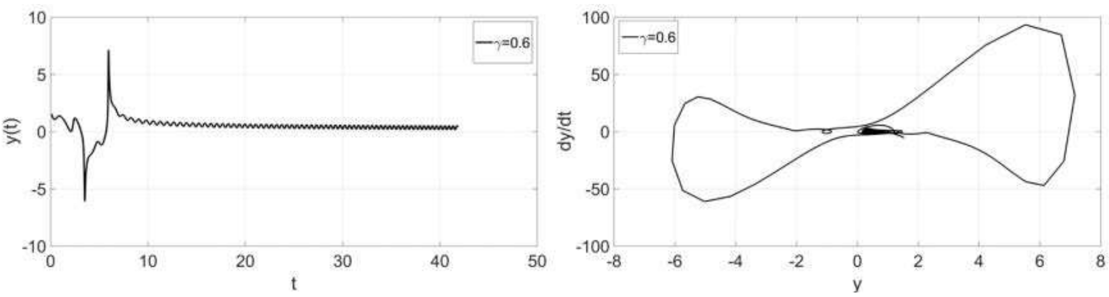

Fig. 18: Solution $y(t)$ (left) and phase plane (right) in real space with horizon for $\mu = 10$, $F = 10$, and $\gamma = 0.6$.

Once more, systems without horizon yet different fractional order values generally present only quantitative differences in response (compare Fig. 15 with 17), and the same is true between different fractional order systems with horizon (compare Fig. 16 with 18). For the employed fractional order values, the existence of a possible attractor for the systems with horizon is implied from the figures (Figs. 16 and 18).

4 Conclusion

The chaotic behaviour of the Van der Pol oscillator in the context of the Λ-fractional analysis with horizon is quite different not only from the classic van der Pol model, but also from the one considered in the Λ-fractional analysis with global fractional derivatives. The Van der Pol oscillator was discussed in terms of the recently introduced Λ-fractional analysis with horizon, restricting the region of global influence.

References

[1] C. Truesdell, *A first course in rational ccontinuum mechanics*, Vol. **1**, Academic Press, New York, etc., 1977.
[2] K. A. Lazopoulos, D. Karaoulanis, A. K. Lazopoulos and A. P. Pirentis, On the chaotic response of the Λ-fractional Van der Pol oscillators, *Asia Math.* **5** (2), 81–95 (2021).
[3] G. W. Leibnitz, Letter to G. A. L'Hôpital, *Leibnitzen's Mathematishe Schriften* Vol. **2**, Georg Olm, Hildesheim, 1962, first published 1849.
[4] J. Liouville, Sur le calcul des différentielles à indices quelconques, *Journ. Ec. Polytech.* **13**, 71–162 (1832).
[5] B. Riemann, Versuch einer allgemeinen Auffassung der Integration and Differentiation, in H. Weber (Ed.), *Gesammelte Mathematische Werke*, Leipzig, 1876.
[6] B. Ross, The development of fractional calculus 1695-1900, *Hist. Math.* **4** (1), 75–89 (1977).
[7] R. Hilfer (Ed.), *Applications of fractional calculus in physics*, World Scientific, Singapore, etc., 2000.
[8] B. J. West, M. Bologna and P. Grigolini, *Physics of fractal operators*, Springer, New York, etc., 2003.
[9] C. S. Drapaca and S. Sivaloganathan, A fractional model of continuum mechanics, *J. Elast.* **107** (2), 105–123 (2012).
[10] M. Di Paola, G. Failla and M. Zingales, Physically-based approach to the mechanics of strong non-local linear elasticity theory, *J. Elast.* **97** (2), 103–130 (2009).
[11] A. Carpinteri, P. Cornetti and A. Sapora, A fractional calculus approach to nonlocal elasticity, *Eur. Phys. J. Spec. Top.* **193** (1), 193–204 (2011).
[12] K. A. Lazopoulos, Fractional vector calculus and fractional continuum mechanics, in *Mechanics through mathematical modelling, celebrating the 70th Birthday of Prof. T. Atanackovic, Novi Sad Serbia 6–11 Sept. 2015*, Book of Abstracts, p. 40, 2015.
[13] D. Baleanu, S. I. Muslih and E. M. Rabei, On fractional Euler–Lagrange and Hamilton equations and the fractional generalization of total time derivative, *Nonlin. Dyn.* **53** (1-2), 67–74 (2008).
[14] W. Sumelka, Fractional viscoplasticity, *Mech. Res. Comm.* **56**, 31-36 (2014).
[15] R. L. Magin, Fractional calculus in bioengineering, Parts 1-3, *Crit. Rev. Biomed. Eng.* **32**, 1–377 (2004).
[16] K. A. Lazopoulos and A. K. Lazopoulos, On the mathematical formulation of fractional derivatives, *Prog. Fract. Differ. Appl.* **5** (4), 261-267 (2019).
[17] S. A. Silling, M. Zimmermann and R. Abeyaratne, Deformation of a peridynamic bar, *J. Elast.* **73** (1-3), 173–190 (2003).
[18] S. A. Silling and R. B. Lehoucq, Peridynamic theory of solid mechanics. *Adv. Appl. Mech.* **44**, 73–168 (2010).