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Applying Conformable Double Sumudu – Elzaki Approach to Solve Nonlinear Fractional Problems

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Abstract: In this study, a conformable double Sumudu-Elzaki (CDSE) transformation and a decomposition method are combined to develop a new method that solves nonlinear problems considering some specific conditions. This combination can be referred to as the CDSE-Decomposition method. Furthermore, we explain and discuss the main features and main results of the presented method. The CDSE-Decomposition method presents analytic series solutions with high convergence to the exact solution in a closed form. The benefit of utilizing the proposed technique is that it presents analytic series solutions to the objective equations without the requirement for any constrained assumptions, transformation or discretization. In addition, various numerical experiments are presented to prove the efficiency of the obtained method. The results show the power and effectiveness of the proposed approach in handling a range of physical and engineering problems that arize in mathematics.

Keywords: Sumudu transform, Elzaki transform, conformable double Sumudu-Elzaki transformation, decomposition method, Conformable partial derivative.

1 Introduction, Motivation and Preliminaries

Fractional partial differential equations are essential for simulating a variety of real-world applications in science, including physics, electrical circuits, fluid dynamics, optics, and mathematical biology [1,2,3]. An interesting definition given by Khalil et al. [4] is called a conformal fractional derivative, which satisfies most of the conventional properties of derivatives.

Numerous mathematicians and researchers have recently created novel techniques for solving conformable fractional partial differential equations, including the conformable Laplace transform method [5], the Exponential rational function method [6], the Simplest Equation Method [7],the reliable method [8], the modified double conformable Laplace transform [9], the Tanh method [10], the conformable double Laplace transform [11,12,13,14], the conformable double Sumudu transform [15,16], the reduced differential transform method [17], the conformable double Laplace decomposition method [18], the conformable double Sumudu decomposition method [19], the conformable triple Laplace transform decomposition method [20], the conformable triple Laplace and Sumudu transforms decomposition method [21],and the conformable double Laplace - Sumudu iterative method [22].

The double Sumudu-Elzaki transform method, a novel double integral transform strategy, has been successfully developed in the recent years to solve variety kinds of partial differential equations [23]. Unfortunately, this transform has difficulty with nonlinear situations, much like other integral transforms do. Additionally, mathematicians have developed novel methods that incorporate transforms with numerical techniques, including variational iteration, decomposition, perturbation, and others [24,25,26,27,28,29,30].

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In this research we consider a conformable linear fractional partial differential equations of the form

$$\frac{\partial^{m\vartheta_2}}{\partial v^{m\vartheta_2}}\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)+\frac{\partial^{n\vartheta_1}}{\partial u^{n\vartheta}}\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)+N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)$$
$$=f\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right),\quad \frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}>0,\quad 0<\vartheta_1,\vartheta_2\le 1,\quad m,n\in\mathbb{N}, \tag{1}$$

with the initial conditions (ICs)

$$\frac{\partial^{j\vartheta_2}\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},0\right)}{\partial v^{j\vartheta_2}}=g_j\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right), j=0,1,...,m-1, \tag{2}$$

and the boundary conditions (BCs)

$$\frac{\partial^{k\vartheta_1}\psi\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)}{\partial u^{k\vartheta_1}}=h_k\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right), k=0,1,...,n-1, \tag{3}$$

where $N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)$ is a nonlinear term, and $f\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ is the source term.
As part of this work, we offer the CDSE-Decomposition method for analyzing the applications presented in the shape (1), taking into consideration the circumstances (2) and (3). The conformable double Sumudu-Elzaki methodology is typically modified, and we do so by combining it with the decomposition method [31,32,33,34] to create the CDSE-Decomposition method. In contrast to other numerical approaches, the CDSE-Decomposition method allows a speedy convergence to the accurate solution without making any constrictive assumptions about the solution, and it requires neither discretization nor differentiation. For more information, see [35]. The goal of this work is to provide a faster method for determining the precise solution of nonlinear conformable partial differential equations using CDSE-Decomposition method.
This study introduces main definitions and characteristics of the CDSE transformation, then we introduce the main idea of finding accurate series solution solutions of the given equations using CDSE transformation coupled with a decomposition method introduced approach, and finally we presents some applications to show its reliability.

2 Conformable Fractional Derivative

In this part of the study, we introduce some basic definitions and theorems related to the conformable fractional derivatives.

Definition 1.[4] *Let $m<\vartheta\le m+1$, $m\in\mathbb{N}$, and $\psi:(0,\infty)\to R$, then the ϑ^{th} order conformable fractional derivatives of ψ is defined by:*

$$D^{\vartheta}\psi(v)=\lim_{\varepsilon\to 0}\frac{\psi^{([\vartheta]-1)}\left(v+\varepsilon v^{[\vartheta]-\vartheta}\right)-\psi^{([\vartheta]-1)}(v)}{\varepsilon},\quad v>0,\quad \vartheta\in(m,m+1]. \tag{4}$$

As a special case, if $\vartheta\in(0,1]$, then we have:

$$D^{\vartheta}\psi(v)=\lim_{\varepsilon\to 0}\frac{\psi\left(v+\varepsilon v^{1-\vartheta}\right)-\psi(v)}{\varepsilon},\quad v>0,\quad \vartheta\in(m,m+1].$$

Definition 2.[36] *Let $m<\vartheta_1,\vartheta_2\le m+1, m\in\mathbb{N}$, and $\psi(u,v):R\times(0,\infty)\to R$, then the conformable partial derivative of order ϑ_1 and ϑ_2 of the function $\psi(u,v)$ is defined by:*

$$D_u^{\vartheta_1}\psi(u,v)=\lim_{\delta\to 0}\frac{\psi^{([\vartheta_1]-1)}\left(u+\delta u^{[\vartheta_1]-\vartheta_1},v\right)-\psi^{([\vartheta_1]-1)}(u,v)}{\delta}, \tag{5}$$

$$D_v^{\vartheta_2}\psi(u,v)=\lim_{\varepsilon\to 0}\frac{\psi^{([\vartheta_2]-1)}\left(u,v+\varepsilon v^{[\vartheta_2]-\vartheta_2}\right)-\psi^{([\vartheta_2]-1)}(u,v)}{\varepsilon}, \tag{6}$$

where $u,v>0$, $D_u^{\vartheta_1}=\frac{\partial^{\vartheta_1}}{\partial u^{\vartheta_1}}$ and $D_v^{\vartheta_2}=\frac{\partial^{\vartheta_2}}{\partial v^{\vartheta_2}}$ are referred to as fractional derivatives of order ϑ_1 and ϑ_2, respectively.

Theorem 1.[14] *Suppose that $\psi(u,v)$ be a differentiable at a point $u,v>0$, $0<\vartheta_1,\ \vartheta_2\leq 1$, then:*

$$\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}=u^{-\vartheta_1+1}\frac{\partial\psi}{\partial u}, \tag{7}$$

$$\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}}=v^{-\vartheta_2+1}\frac{\partial\psi}{\partial v}. \tag{8}$$

In the following example we present some basic properties of the conformable derivative.

Example 1. Suppose that $0<\vartheta_1,\vartheta_2\leq 1$, and c,d,λ, and $\mu\in\mathbb{R}$; then

$$\frac{\partial^{\vartheta_1}}{\partial u^{\vartheta_1}}\left(c\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)+d\xi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)=c\left(\frac{\partial^{\vartheta_1}}{\partial u^{\vartheta_1}}\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)+d\left(\frac{\partial^{\vartheta_1}}{\partial u^{\vartheta_1}}\xi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right),$$

$$\frac{\partial^{\vartheta_2}}{\partial v^{\vartheta_2}}\left(e^{\lambda\frac{u^{\vartheta_1}}{\vartheta_1}+\mu\frac{v^{\vartheta_2}}{\vartheta_2}}\right)=\mu e^{\lambda\frac{u^{\vartheta_1}}{\vartheta_1}+\mu\frac{v^{\vartheta_2}}{\vartheta_2}},$$

$$\frac{\partial^{\vartheta_1}}{\partial u^{\vartheta_1}}\left(e^{\lambda\frac{u^{\vartheta_1}}{\vartheta_1}+\mu\frac{v^{\vartheta_2}}{\vartheta_2}}\right)=\lambda e^{\lambda\frac{u^{\vartheta_1}}{\vartheta_1}+\mu\frac{v^{\vartheta_2}}{\vartheta_2}},$$

$$\frac{\partial^{\vartheta_1}}{\partial u^{\vartheta_1}}\left(\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)=\cos\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right),$$

$$\frac{\partial^{\vartheta_2}}{\partial v^{\vartheta_2}}\left(\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)=\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\cos\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right).$$

3 The Consequences of the CDSE Transformation

In this section we present some fundamental definitions and characteristics about CDSE transformation.

Definition 3. *Let $\psi(u,v)$ is a function of two variables $u,v\in\mathbb{R}^+$, then*

(i) The conformable Sumudu transform of $\psi(u,v)$ with respect to u, denoted by $S_u^{\vartheta_1}[\psi(u,v):w]=\Psi(w,v)$ and defined as:

$$S_u^{\vartheta_1}[\psi(u,v):w]=\Psi(w,v)=\frac{1}{w}\int_0^\infty e^{-\frac{u^{\vartheta_1}}{w\vartheta_1}}\psi(u,v)d_{\vartheta_1}u,\quad u>0. \tag{9}$$

(ii) The conformable Elzaki transform of $\psi(u,v)$ with respect to v, denoted by $E_v^{\vartheta_2}[\psi(u,v):q]=\Psi(u,q)$, and defined as:

$$E_v^{\vartheta_2}[\psi(u,v):q]=\Psi(u,q)=q\int_0^\infty e^{-\frac{v^{\vartheta_2}}{q\vartheta_2}}\psi(u,v)d_{\vartheta_2}v,v>0. \tag{10}$$

(iii) The CDSE transformation of $\psi(u,v)$, denoted by $S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v):(w,q)]=\Psi(w,q)$ and defined as:

$$S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v):(w,q)]=\Psi(w,q)=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}\psi(u,v)d_{\vartheta_1}ud_{\vartheta_2}v, \tag{11}$$

or

$$S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v):(w,q)]=\Psi(w,q)=q^2\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{\vartheta_1}+\frac{v^{\vartheta_2}}{\vartheta_2}\right)}\psi(wu,vq)d_{\vartheta_1}ud_{\vartheta_2}v, \tag{12}$$

where $q,w\in\mathbb{C}$, $d_{\vartheta_1}u=u^{\vartheta_1-1}du$, and $d_{\vartheta_2}v=v^{\vartheta_2-1}dv$.

Recall that: $S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v)]=E_v^{\vartheta_2}S_u^{\vartheta_1}[\psi(u,v)]$, when $\psi(u,v)$ satisfies the necessary conditions [37]. The inverse $\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}[\Psi(w,q)]=\psi(u,v)$ is defined by:

$$\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}[\Psi(w,q)]=\psi(u,v)=\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{1}{w}e^{\frac{u^{\vartheta_1}}{w\vartheta_1}}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}qe^{\frac{v^{\vartheta_2}}{w\vartheta_2}}\Psi(w,q)dq\right]dw. \tag{13}$$

Theorem 2.[15] *Assume that* $\psi:\mathbb{R}\times(0,\infty)\to\mathbb{R}$ *such that* $S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v):(w,q)]=\Psi(w,q)$ *exist, then*

$$S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v):(w,q)]=S_uE_v[\psi(u,v):(w,q)],$$

where

$$S_uE_v[\psi(u,v):(w,q)]=\Psi(w,q)=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u}{w}+\frac{v}{q}\right)}\psi(u,v)\,dudv.$$

Theorem 3. *The CDSE transformation for a few functions is provided below.*

(i)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}[c:(w,q)]=S_uE_v[c:(w,q)]=cq^2, c\in\mathbb{R}.$$

(ii)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^m\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)^n:(w,q)\right]=S_uE_v[u^mv^n:(w,q)]=m!n!w^mq^{n+2},\quad m,n\in\mathbb{Z}^+.$$

(iii)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[e^{c_1\frac{u^{\vartheta_1}}{\vartheta_1}+c_2\frac{v^{\vartheta_2}}{\vartheta_2}}:(w,q)\right]=S_uE_v\left[e^{c_1u+c_2v}:(w,q)\right]=\frac{q^2}{(1-c_1w)(1-c_2q)}.$$

(iv)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\sin\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}\right)\sin\left(c_2\frac{v^{\vartheta_2}}{\vartheta_2}\right):(w,q)\right]=S_uE_v[\sin(c_1u)\sin(c_2v):(w,q)]=\frac{c_1w}{1+c_1{}^2w^2}\frac{c_2q^3}{1+c_2{}^2q^2}.$$

(v)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[1-e^{c_2\frac{v^{\vartheta_2}}{\vartheta_2}}:(w,q)\right]=S_uE_v\left[1-e^{c_2v}:(w,q)\right]=\frac{-c_2q^3}{1-c_2q}.$$

(vi)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(1-e^{c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\right)\sin\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}\right):(w,q)\right]=S_uE_v\left[(1-e^{c_2v})\sin(c_1u):(w,q)\right]=\frac{-c_1c_2wq^3}{(1-c_2q)(1+c_1{}^2w^2)}.$$

Proof. Here, we provide evidences for the results (i), (ii), and (vi).
(i) gives us

$$S_u^{\vartheta_1}E_v^{\vartheta_2}[c:(w,q)]=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}c\,d_{\vartheta_1}u\,d_{\vartheta_2}v=\left(\frac{1}{w}\int_0^\infty e^{-\frac{u}{w}}c\,du\right)\left(q\int_0^\infty e^{-\frac{v}{q}}(1)\,dv\right)=cq^2.$$

(ii) gives us

$$\begin{aligned}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[e^{a\frac{u^{\vartheta_1}}{\vartheta_1}+b\frac{v^{\vartheta_2}}{\vartheta_2}}:(w,q)\right]&=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}e^{a\frac{u^{\vartheta_1}}{\vartheta_1}+b\frac{v^{\vartheta_2}}{\vartheta_2}}d_{\vartheta_1}u\,d_{\vartheta_2}v\\&=\left(\frac{1}{w}\int_0^\infty e^{-\frac{u}{w}}e^{au}du\right)\left(q\int_0^\infty e^{-\frac{v}{q}}e^{bv}dv\right)=\frac{1}{1-aw}\frac{q^2}{1-bq}.\end{aligned}$$

For (vi), we have

$$\begin{aligned}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(1-e^{c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\right)\sin\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}\right):(w,q)\right]&=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}\left(1-e^{c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\right)\sin\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}\right)d_{\vartheta_1}u\,d_{\vartheta_2}v\\&=\left(\frac{1}{w}\int_0^\infty e^{-\frac{u}{w}}\sin(c_1u)du\right)\left(q\int_0^\infty e^{-\frac{v}{q}}(1-e^{c_2v})\,dv\right)\\&=S_u[\sin(c_1u):w]\,E_v[1-e^{c_2v}:q]=\frac{c_1w}{1+c_1{}^2w^2}\frac{-c_2q^3}{1-c_2q}.\end{aligned}$$

Theorem 4.[38] *Let* $0<\vartheta_1,\vartheta_2\le 1$, *and* $\psi(u,v):[0,\infty)\to R$, *we have*

(i)

$$S_u^{\vartheta_1}\left[\psi(u,v):w\right]=\frac{1}{w}L_u^{\vartheta_1}\left[\psi\left(\left(\vartheta_1 u\right)^{\frac{1}{\vartheta_1}},v\right):\frac{1}{w}\right],E_v^{\vartheta_2}\left[\psi(u,v):q\right]=qL_v^{\vartheta_2}\left[\psi\left(u,\left(\vartheta_2 v\right)^{\frac{1}{\vartheta_2}}\right):\frac{1}{q}\right],$$

(ii)

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi(u,v):(w,q)\right]=S_uE_v\left[\psi\left(\left(\vartheta_1 u\right)^{\frac{1}{\vartheta_1}},\left(\vartheta_2 v\right)^{\frac{1}{\vartheta_2}}\right):(w,q)\right].$$

Theorem 5.[39,40] *If the CDSE transformation of $\psi(u,v)$ exists, then*

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi(u,v):(w,q)\right]=\frac{q}{w}L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\psi(u,v):\left(\frac{1}{w},\frac{1}{q}\right)\right],$$

where

$$L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\psi(u,v):(w,q)\right]=\Psi(w,q)=\int_0^\infty\int_0^\infty e^{-\left(w\frac{u^{\vartheta_1}}{\vartheta_1}+q\frac{v^{\vartheta_2}}{\vartheta_2}\right)}\psi(u,v)\,d_{\vartheta_1}u\,d_{\vartheta_2}v,$$

is the conformable double Laplace transformation.

Theorem 6. *Assume that $\psi(u,v)$ and $\xi(u,v)$ are two functions with the CDSE transformation. Then,*

i.

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[c_1\psi(u,v)+c_2\xi(u,v)\right]=c_1S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi(u,v):(w,q)\right]+c_2S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\xi(u,v):(w,q)\right],$$

ii.

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[e^{-c_1\frac{u^{\vartheta_1}}{\vartheta_1}-c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\psi(u,v):(w,q)\right]=\frac{1+c_2q}{1+c_1w}\Psi\left(\frac{w}{1+c_1w},\frac{q}{1+c_2q}\right),$$

iii.

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi(\lambda u,\mu v):(w,q)\right]=\frac{1}{r}\Psi\left(\frac{w}{\lambda^{\vartheta_1}},\frac{q}{\mu^{\vartheta_2}}\right);r=\lambda^{\vartheta_1}\mu^{\vartheta_2},$$

iv.

$$(-1)^{m+n}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{u^{m\vartheta_1}}{\vartheta_1^m}\frac{v^{n\vartheta_2}}{\vartheta_2^n}\psi(u,v):(w,q)\right]=\frac{q}{w}\frac{\partial^{m+n}}{\partial w^m\partial q^n}\left(\frac{w}{q}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi(u,v):(w,q)\right]\right).$$

Proof.

i.The use of the CDSE transformation specification makes the proof of (i) simple to demonstrate.
ii.

$$\begin{aligned}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[e^{-c_1\frac{u^{\vartheta_1}}{\vartheta_1}-c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\psi(u,v):(w,q)\right]&=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}e^{-c_1\frac{u^{\vartheta_1}}{\vartheta_1}-c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\psi(u,v)\,d_{\vartheta_1}u\,d_{\vartheta_2}v\\&=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{w}+c_1\right)\frac{u^{\vartheta_1}}{\vartheta_1}-\left(\frac{1}{q}+c_2\right)\frac{v^{\vartheta_2}}{\vartheta_2}}\psi(u,v)\,d_{\vartheta_1}u\,d_{\vartheta_2}v\\&=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{1+c_1w}{w}\right)\frac{u^{\vartheta_1}}{\vartheta_1}-\left(\frac{1+c_2q}{q}\right)\frac{v^{\vartheta_2}}{\vartheta_2}}\psi(u,v)\,d_{\vartheta_1}u\,d_{\vartheta_2}.\end{aligned}$$

Putting $p=\frac{w}{1+c_1w}$, $s=\frac{q}{1+c_2q}$, then

$$\begin{aligned}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[e^{-c_1\frac{u^{\vartheta_1}}{\vartheta_1}-c_2\frac{v^{\vartheta_2}}{\vartheta_2}}\psi(u,v):(w,q)\right]&=\frac{1+c_2q}{1+c_1w}\left(\frac{s}{p}\int_0^\infty\int_0^\infty e^{-\frac{u^{\vartheta_1}}{p\vartheta_1}-\frac{v^{\vartheta_2}}{s\vartheta_2}}\psi(u,v)\,d_{\vartheta_1}u\,d_{\vartheta_2}v\right)\\&=\frac{1+c_2q}{1+c_1w}\Psi(p,s)=\frac{1+c_2q}{1+c_1w}\Omega\left(\frac{w}{1+c_1w},\frac{q}{1+c_2q}\right).\end{aligned}$$

iii.Suppose $\gamma=\lambda w$ and $\eta=\mu v$, then

$$\begin{aligned}S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(\lambda u,\mu v):(w,q)] &= \frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}\psi(\lambda u,\mu v)\,d_{\vartheta_1}u\,d_{\vartheta_2}v\\ &=\frac{1}{w}\int_0^\infty e^{-\frac{u^{\vartheta_1}}{w\vartheta_1}}\left(q\int_0^\infty e^{-\frac{v^{\vartheta_2}}{q\vartheta_2}}\psi(\lambda u,\mu v)\,d_{\vartheta_2}v\right)d_{\vartheta_1}u\\ &=\frac{1}{\mu^{\vartheta_2}w}\int_0^\infty e^{-\frac{u^{\vartheta_1}}{q\vartheta_1}}\left(q\int_0^\infty e^{-\frac{\eta^{\vartheta_2}}{q\mu^{\vartheta_2}\vartheta_2}}\psi(\lambda u,\eta)\,d_{\vartheta_2}\eta\right)d_{\vartheta_1}u\\ &=\frac{1}{\mu^{\vartheta_2}w}\int_0^\infty e^{-\frac{u^{\vartheta_1}}{w\vartheta_1}}\Psi\left(\lambda u,\frac{q}{\mu^{\vartheta_2}}\right)d_{\vartheta_1}u=\frac{1}{\mu^{\vartheta_2}\lambda^{\vartheta_1}}\int_0^\infty\frac{1}{w}e^{-\frac{\gamma^{\vartheta_1}}{w\lambda^{\vartheta_1}\vartheta_1}}\Psi\left(\gamma,\frac{q}{\mu^{\vartheta_2}}\right)d_{\vartheta_1}\gamma\\ &=\frac{1}{\mu^{\vartheta_2}\lambda^{\vartheta_1}}\Psi\left(\frac{w}{\lambda^{\vartheta_1}},\frac{q}{\mu^{\vartheta_2}}\right)\end{aligned}$$

iv.Here, by combining Theorem 5 with Theorem 2.1 from [12], we obtain,

$$\begin{aligned}(-1)^{m+n}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{u^{m\vartheta_1}}{\vartheta_1^m}\frac{v^{n\vartheta_2}}{\vartheta_2^n}\psi(u,v):(w,q)\right] &= \frac{q}{w}(-1)^{m+n}L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\frac{u^{m\vartheta_1}}{\vartheta_1^m}\frac{v^{n\vartheta_2}}{\vartheta_2^n}\psi(u,v):\left(\frac{1}{w},\frac{1}{q}\right)\right]\\ &=\frac{q}{w}\frac{\partial^{m+n}}{\partial w^m\partial q^n}\left(L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\psi(u,v):\left(\frac{1}{w},\frac{1}{q}\right)\right]\right)\\ &=\frac{q}{w}\frac{\partial^{m+n}}{\partial w^m\partial q^n}\left(\frac{w}{q}S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v):(w,q)]\right).\end{aligned}$$

Theorem 7.[22] *Suppose that $\psi(u,v)$ is a function of exponential order a and b defined on the interval $(0,U)$ and $(0,V)$, then CDSE transformation of $\psi(u,v)$ well-defined for all $\frac{1}{w}$ and $\frac{1}{q}$ provided that Re$\left[\frac{1}{w}\right]>a$ and Re$\left[\frac{1}{q}\right]>b$.*

Theorem 8.[22] *If $S_u^{\vartheta_1}E_v^{\vartheta_2}[\psi(u,v)]=\Psi(w,q)$, then the CDSE transformation of the conformable fractional partial derivatives $\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}$ and $\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}}$ can be represented as follows:* $0<\vartheta_1,\ \vartheta_2\leq 1$

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right]=\frac{1}{w}\Psi(w,q)-\frac{1}{w}\Psi(0,q),\tag{14}$$

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}}\right]=\frac{1}{q}\Psi(w,q)-q\Psi(w,0).\tag{15}$$

Proof. Here, we proceed with the result's proof (14),

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right]=\frac{q}{w}\int_0^\infty\int_0^\infty e^{-\left(\frac{u^{\vartheta_1}}{w\vartheta_1}+\frac{v^{\vartheta_2}}{q\vartheta_2}\right)}\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}d_{\vartheta_1}u\,d_{\vartheta_2}v=q\int_0^\infty e^{-\frac{v^{\vartheta_2}}{q\vartheta_2}}v^{\theta_2-1}\left(\int_0^\infty\frac{1}{w}e^{-\frac{u^{\vartheta_1}}{w\vartheta_1}}\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}u^{\theta_1-1}du\right)dv.\tag{16}$$

Since we have from Theorem 1, $\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}=u^{-\vartheta_1+1}\frac{\partial\psi}{\partial u}$. We plug this outcome into Eq. (16). Therefore, Eq. (16) becomes

$$\begin{aligned}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right]&=q\int_0^\infty e^{-\frac{v^{\vartheta_2}}{q\vartheta_2}}\left(\int_0^\infty\frac{1}{w}e^{-\frac{u^{\vartheta_1}}{w\vartheta_1}}\frac{\partial\psi}{\partial u}du\right)d_{\vartheta_2}v=q\int_0^\infty e^{-\frac{v^{\vartheta_2}}{q\vartheta_2}}\left(-\frac{1}{w}\psi(0,v)+\frac{1}{w}\Psi(w,v)\right)d_{\vartheta_2}v\\ &=\frac{1}{w}\Psi(w,q)-\frac{1}{w}\Psi(0,q).\end{aligned}\tag{17}$$

The same method can also be used to demonstrate the final result (15).
The results mentioned above can be generally expanded as follows:

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{\partial^{n\vartheta_1}\psi}{\partial u^{n\vartheta_1}}\right]=w^{-n}\Psi(w,q)-\sum_{k=0}^{n-1}w^{-n+k}E_v^{\vartheta_2}\left[\frac{\partial^{k\vartheta_1}}{\partial u^{k\vartheta_1}}\psi(0,v)\right],\tag{18}$$

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\frac{\partial^{m\vartheta_2}\psi}{\partial v^{m\vartheta_2}}\right]=q^{-m}\Psi(w,q)-\sum_{j=0}^{m-1}q^{2-m+j}S_u^{\vartheta_1}\left[\frac{\partial^{j\vartheta_2}}{\partial v^{j\vartheta_2}}\psi(u,0)\right].\tag{19}$$

Theorem 9.(Convolution Theorem) *Assume that $\psi(u,v)$ and $\xi(u,v)$ are two functions with the CDSE transformation, then,*

$$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[(\psi**\xi)(u,v):(w,q)\right]=\frac{w}{q}\Psi(w,q)\,\Omega(w,q),$$

where

$$(\psi**\xi)(u,v)=\int_0^u\int_0^v\psi(u-\eta,v-\gamma)\,\xi(\eta,\gamma)\,d\eta d\gamma.$$

Proof. Using Theorems 5 and 2.2 in [12], we obtain,

$$\begin{aligned}
S_u^{\vartheta_1}E_v^{\vartheta_2}\left[(\psi**\xi)(u,v):(w,q)\right]&=\frac{q}{w}L_u^{\vartheta_1}L_v^{\vartheta_2}\left[(\psi**\xi)(u,v):\left(\frac{1}{w},\frac{1}{q}\right)\right]\\
&=\frac{q}{w}L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\psi(u,v):\left(\frac{1}{w},\frac{1}{q}\right)\right]L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\xi(u,v):\left(\frac{1}{w},\frac{1}{q}\right)\right]\\
&=\frac{w}{q}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi(u,v):(w,q)\right]S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\xi(u,v):(w,q)\right]\\
&=\frac{w}{q}\Psi(w,q)\,\Omega(w,q).
\end{aligned}$$

Table 1, below introduces CDSE transformation for some basic functions

Table 1: CDSE transformation for some functions of two variables.

Sr. No	$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$	$S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\zeta\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right]=\Psi(w,q)$
1	c	$cq^2, c\in\mathbb{R}$
2	$\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^m\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)^n, m,n\in\mathbb{Z}^+$	$m!n!w^mq^{n+2}$
3	$e^{c_1\frac{u^{\vartheta_1}}{\vartheta_1}+c_2\frac{v^{\vartheta_2}}{\vartheta_2}}$	$\frac{q^2}{(1-c_1w)(1-c_2q)}$
4	$\sin\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}+c_2\frac{v^{\vartheta_2}}{\vartheta_2}\right)$	$\frac{q^2(c_1w+c_2q)}{(1+c_1^2w^2)(1+c_2^2q^2)}$
5	$\cos\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}+c_2\frac{v^{\vartheta_2}}{\vartheta_2}\right)$	$\frac{q^2(1-c_1c_2wq)}{(1+c_1^2w^2)(1+c_2^2q^2)}$
6	$\sinh\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}+c_2\frac{v^{\vartheta_2}}{\vartheta_2}\right)$	$\frac{q^2(c_1w+c_2q)}{(1-c_1^2w^2)(1-c_2^2q^2)}$
7	$\cosh\left(c_1\frac{u^{\vartheta_1}}{\vartheta_1}+c_2\frac{v^{\vartheta_2}}{\vartheta_2}\right)$	$\frac{au}{1-a^2u^2}$
8	$\sqrt{\frac{u^{\vartheta_1}}{\vartheta_1}\frac{v^{\vartheta_2}}{\vartheta_2}}$	$\frac{\pi\sqrt{wq^5}}{4}$
9	$J_0\left(b\sqrt{\frac{u^{\vartheta_1}}{\vartheta_1}\frac{v^{\vartheta_2}}{\vartheta_2}}\right)$	$\frac{4q^2}{4+b^2wq}$
10	$f\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)g\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)$	$S_u^{\vartheta_1}\left[f\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\right]E_v^{\vartheta_2}\left[g\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right]$

4 Principle of the CDSE Transformation Combined with Decomposition Method

In this section, we used CDSE transformation combined with decomposition method to look for the solution of Eq. (1). Applying the CDLE transformation to Eq. (1), we get

$$\begin{aligned}
&q^{-m}\Psi(w,q)-\sum_{j=0}^{m-1}q^{2-m+j}S_u^{\vartheta_1}\left[\frac{\partial^{j\vartheta_1}}{\partial v^{j\vartheta_1}}\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},0\right)\right]+w^{-n}\Psi(w,q)-\sum_{k=0}^{n-1}w^{-n+k}E_v^{\vartheta_2}\left[\frac{\partial^{k\vartheta_1}}{\partial u^{k\vartheta_1}}\psi\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right]\\
&+S_u^{\vartheta_1}E_v^{\vartheta_2}\left[N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)\right]=F(w,q).
\end{aligned}\tag{20}$$

Using the conformable Sumudu transform for the ICs (2) and the conformable Elzaki transform for the BCs (3), one can obtain,

$$\begin{aligned} S_u^{\vartheta_1}\left[\frac{\partial^{j\vartheta_2}\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},0\right)}{\partial v^{j\vartheta_2}}\right] &= G_j(w), j=0,1,...,m-1,\\ E_v^{\vartheta_2}\left[\frac{\partial^{k\vartheta_1}\psi\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)}{\partial u^{k\vartheta_1}}\right] &= H_k(q), k=0,1,...,n-1. \end{aligned} \tag{21}$$

Using Eqs. (21) and (20), we can say that:

$$\begin{aligned} &q^{-m}\Psi(w,q)-\sum_{j=0}^{m-1}q^{2-m+j}G_j(w)+w^{-n}\Psi(w,q)-\sum_{k=0}^{n-1}w^{-n+k}H_k(q)\\ &=F(w,q)-S_u^{\vartheta_1}E_v^{\vartheta_2}\left[N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)\right]. \end{aligned} \tag{22}$$

Simplify Eq. (22), we get:

$$\begin{aligned} \Psi(w,q)=g\left[q^{-m}+w^{-n}\right]^{-1}&\left(\sum_{j=0}^{m-1}q^{2-m+j}G_j(w)+\sum_{k=0}^{n-1}w^{-n+k}H_k(q)+F(w,q)\right.\\ &\left.-S_u^{\vartheta_1}E_v^{\vartheta_2}\left[N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)\right]\right). \end{aligned} \tag{23}$$

Taking $\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}$ of Eq. (23), we get the solution $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ of Eq. (1)

$$\begin{aligned} \psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}&\left[\frac{1}{q^{-m}+w^{-n}}\left(\sum_{j=0}^{m-1}q^{2-m+j}G_j(w)+\sum_{k=0}^{n-1}w^{-n+k}H_k(q)+F(w,q)\right.\right.\\ &\left.\left.-S_u^{\vartheta_1}E_v^{\vartheta_2}\left[N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)\right]\right)\right]. \end{aligned} \tag{24}$$

Use the decomposition strategy now by assuming,

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\sum_{i=0}^{\infty}\psi_i\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right). \tag{25}$$

Decomposition of the nonlinear term yields:

$$N\left(\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)\right)=\sum_{i=0}^{\infty}A_i, \tag{26}$$

regarding a few Adomian polynomials

$$A_i(\psi_0,\psi_1,\psi_2,...,\psi_n)=\frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\psi_i\right)\right]_{\lambda=0}, i=0,1,2,\cdots.$$

By replacing Eqs. (26) and (25) in Eq. (24), we obtain:

$$\begin{aligned} \sum_{i=0}^{\infty}\psi_i\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}&\left[\frac{1}{q^{-m}+w^{-n}}\left(\sum_{j=0}^{m-1}q^{2-m+j}G_j(w)+\sum_{k=0}^{n-1}w^{-n+k}H_k(q)+F(w,q)\right.\right.\\ &\left.\left.-S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\sum_{i=0}^{\infty}A_i\right]\right)\right]. \end{aligned} \tag{27}$$

After that, we have the following recurrence relationships:

$$\psi_0\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[\frac{1}{q^{-m}+w^{-n}}\left(\sum_{j=0}^{m-1}q^{2-m+j}G_j(w)+\sum_{k=0}^{n-1}w^{-n+k}H_k(q)+F(w,q)\right)\right],\tag{28}$$

$$\psi_{r+1}\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=-\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[\frac{1}{q^{-m}+w^{-n}}S_u^{\vartheta_1}E_v^{\vartheta_2}[A_r]\right],\quad r\geq 0.\tag{29}$$

Thus, we have the following as the solution to Eq. (1)

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\psi_0\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)+\psi_1\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)+\psi_2\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)+\cdots.\tag{30}$$

5 Elucidative Examples

This section provides some examples on nonlinear conformable fractional partial derivatives are solved to demonstrate the performance and the efficiency of the CDSE-Decomposition method.

Example 2. Consider the following nonlinear conformable fractional partial differential equation

$$\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}}+\psi\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}=0,\quad 0<\vartheta_1,\vartheta_2\leq 1,\tag{31}$$

with The IC

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},0\right)=\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right),\tag{32}$$

and the BC

$$\psi\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)=0.\tag{33}$$

Solution. Operating the CDSE transformation to (31), conformable Sumudu transform on Eq. (32), and the conformable Elzaki transform to Eq. (33), we get,

$$\Psi(w,q)=\frac{wq^2}{(1+w^2)}-qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right].\tag{34}$$

Taking $\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}[\Psi(w,q)]$ of (34), we get

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)-\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right]\right].\tag{35}$$

Now, applying the decomposition method, we substitute Eq. (25) in Eq. (35) and with the results in Equations. (28), and (29), we get the following solution components

$$\psi_0\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right),\tag{36}$$

$$\psi_1\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=-\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi_0\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}\right]\right]=-\frac{v^{\vartheta_2}}{2\vartheta_2}\sin\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right),\tag{37}$$

$$\begin{aligned}\psi_2\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)&=-\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi_0\frac{\partial^{\vartheta_1}\psi_1}{\partial u^{\vartheta_1}}+\psi_1\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}\right]\right]\\&=\frac{v^{2\vartheta_2}}{2\vartheta_2^2}\left(\sin\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right)\cos\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)+\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\cos\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right)\right),\end{aligned}\tag{38}$$

$$\psi_3\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right) = -\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi_0\frac{\partial^{\vartheta_1}\psi_2}{\partial u^{\vartheta_1}}+\psi_2\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}+\psi_1\frac{\partial^{\vartheta_1}\psi_1}{\partial u^{\vartheta_1}}\right]\right]$$
$$= -\frac{v^{3\vartheta_2}}{12\vartheta_2^3}\left(\sin\left(4\frac{u^{\vartheta_1}}{\vartheta_1}\right)-5\sin^2\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\sin\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right)\right). \tag{39}$$

As a result, we have the solution to the Eq. (31) as:

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right) = \sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right) - \frac{v^{\vartheta_2}}{2\vartheta_2}\sin\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right) + \frac{v^{2\vartheta_2}}{2\vartheta_2^2}\left(\sin\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right)\cos\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)+\sin\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\cos\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right)\right)$$
$$-\frac{v^{3\vartheta_2}}{12\vartheta_2^3}\left(\sin\left(4\frac{u^{\vartheta_1}}{\vartheta_1}\right)-5\sin^2\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)\sin\left(2\frac{u^{\vartheta_1}}{\vartheta_1}\right)\right). \tag{40}$$

the exact solution if $\vartheta_1 = \vartheta_2 = 1$ is

$$\psi(u,v) = \sin(u) - \frac{v}{2}\sin(2u) + \frac{v^2}{2}\left(\sin(2u)\cos(u)+\sin(u)\cos(2u)\right)$$
$$-\frac{v^3}{12}\left(\sin(4u)-5\sin^2(u)\sin(2u)\right). \tag{41}$$

In the following figure, Figure 1 we sketch the approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (31) at different fractional orders $\vartheta_1 = \vartheta_2 = 1, 0.9, 0.8$.

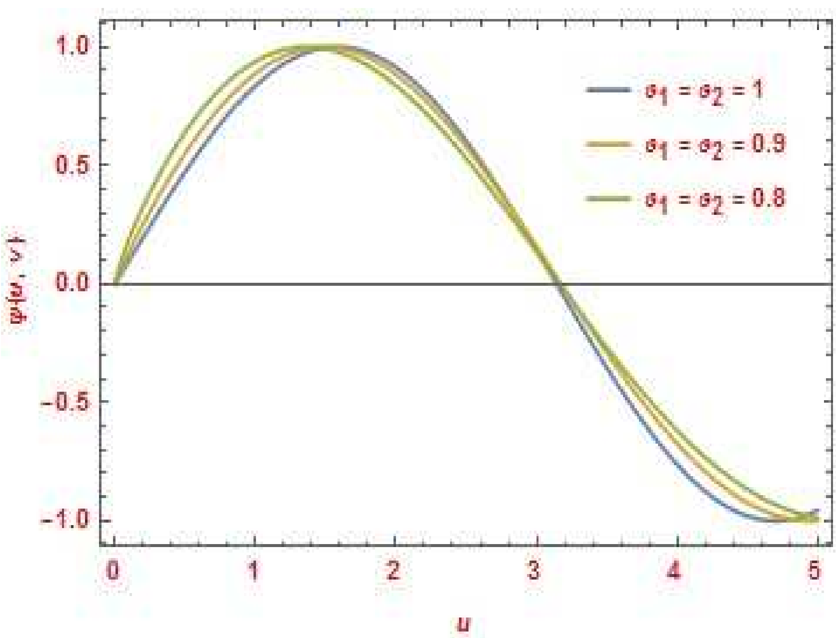

Fig. 1: The approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (40) at $\vartheta_1 = \vartheta_2 = 1, 0.9, 0.8$.

Example 3. Consider the fourth order KdV problem of nonlinear conformable fractional partial differential equation

$$\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}} + \frac{\partial^{4\vartheta_1}\psi}{\partial u^{4\vartheta_1}} + \psi\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}} - \psi\frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}} = 0, \quad 0 < \vartheta_1, \vartheta_2 \le 1, \tag{42}$$

with the IC

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},0\right) = e^{\frac{u^{\vartheta_1}}{\vartheta_1}}, \tag{43}$$

and the BCs

$$\psi\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\psi_u\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\psi_{uu}\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\psi_{uuu}\left(0,\frac{v^{\vartheta_2}}{\vartheta_2}\right)=e^{-\frac{v^{\vartheta_2}}{\vartheta_2}}. \tag{44}$$

Solution. By implementing CDSE transformation to (42), conformable Sumudu transform to (43), and the conformable Elzaki transform to (44), we get,

$$\Psi(w,q)=\frac{q^2}{(1-w)(1+q)}+\frac{qw^4}{q+w^4}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi\frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}}-\psi\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right]. \tag{45}$$

Taking $\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}[\Psi(w,q)]$ of (45), we get

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=e^{\frac{u^{\vartheta_1}}{\vartheta_1}}.e^{-\frac{v^{\vartheta_2}}{\vartheta_2}}+\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[\frac{qw^4}{q+w^4}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi\frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}}-\psi\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right]\right]. \tag{46}$$

Now we use the decomposition method (25) in (46) and get the following solution components through (28) and (29).

$$\psi_0\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=e^{\frac{u^{\vartheta_1}}{\vartheta_1}}.e^{-\frac{v^{\vartheta_2}}{\vartheta_2}}, \tag{47}$$

$$\psi_1\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[\frac{qw^4}{q+w^4}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\psi_0\frac{\partial^{2\vartheta_1}\psi_0}{\partial u^{2\vartheta_1}}-\psi_0\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}\right]\right]=0, \tag{48}$$

$$\begin{aligned}\psi_2\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\Bigg[\frac{qw^4}{q+w^4}S_u^{\vartheta_1}E_v^{\vartheta_2}\Bigg[\left(\psi_1\frac{\partial^{2\vartheta_1}\psi_0}{\partial u^{2\vartheta_1}}+\psi_0\frac{\partial^{2\vartheta_1}\psi_1}{\partial u^{2\vartheta_1}}\right)\\ -\left(\psi_1\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}+\psi_0\frac{\partial^{\vartheta_1}\psi_1}{\partial u^{\vartheta_1}}\right)\Bigg]\Bigg]=0.\end{aligned} \tag{49}$$

As a result, we have the solution to the Eq. (42) as:

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=e^{\frac{u^{\vartheta_1}}{\vartheta_1}}.e^{-\frac{v^{\vartheta_2}}{\vartheta_2}}, \tag{50}$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\psi(u,v)=e^u.e^{-v}=e^{u-v}. \tag{51}$$

In the following figure, Figure 2 we sketch the approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (42) at different fractional orders $\vartheta_1=\vartheta_2=1,0.9,0.8$.

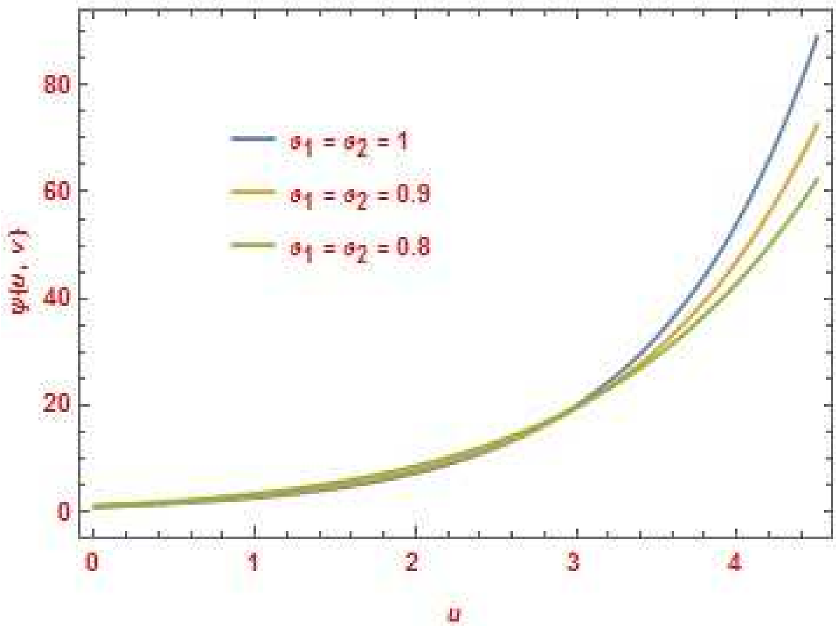

Fig. 2: The approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1}, \frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (50) at $\vartheta_1 = \vartheta_2 = 1, 0.9, 0.8$.

Example 4. Consider the diffusion problem of nonlinear conformable fractional partial differential equation

$$\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}} - \frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}} = \psi^2 - \left(\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right)^2, \quad 0 < \vartheta_1, \vartheta_2 \le 1, \tag{52}$$

with the IC

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1}, 0\right) = e^{\frac{u^{\vartheta_1}}{\vartheta_1}}, \tag{53}$$

and the BCs

$$\psi\left(0, \frac{v^{\vartheta_2}}{\vartheta_2}\right) = \psi_u\left(0, \frac{v^{\vartheta_2}}{\vartheta_2}\right) = e^{\frac{v^{\vartheta_2}}{\vartheta_2}}. \tag{54}$$

Solution. According to the previous examples, we obtain the following solution components

$$\psi_0\left(\frac{u^{\vartheta_1}}{\vartheta_1}, \frac{v^{\vartheta_2}}{\vartheta_2}\right) = e^{\frac{u^{\vartheta_1}}{\vartheta_1}}.e^{\frac{v^{\vartheta_2}}{\vartheta_2}}, \tag{55}$$

$$\psi_1\left(\frac{u^{\vartheta_1}}{\vartheta_1}, \frac{v^{\vartheta_2}}{\vartheta_2}\right) = \left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[\frac{qw^2}{w^2-q}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(\psi_0{}^2\right) - \left(\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}\right)^2\right]\right] = 0, \tag{56}$$

$$\psi_2\left(\frac{u^{\vartheta_1}}{\vartheta_1}, \frac{v^{\vartheta_2}}{\vartheta_2}\right) = \left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[\frac{qw^2}{w^2-q}S_u^{\vartheta_1}E_v^{\vartheta_2}\left[(2\psi_0\psi_1) - (2\psi_{0_u}\psi_{1_u})\right]\right] = 0. \tag{57}$$

As a result, we have the solution to the Eq. (52) as:

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1}, \frac{v^{\vartheta_2}}{\vartheta_2}\right) = e^{\frac{u^{\vartheta_1}}{\vartheta_1}}.e^{\frac{v^{\vartheta_2}}{\vartheta_2}}, \tag{58}$$

putting $\vartheta_1 = \vartheta_2 = 1$; we get the exact solution,

$$\psi(u, v) = e^u.e^v = e^{u+v}. \tag{59}$$

In the following figure, Figure 3 we sketch the approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (52) at different fractional orders $\vartheta_1=\vartheta_2=1,0.9,0.8$.

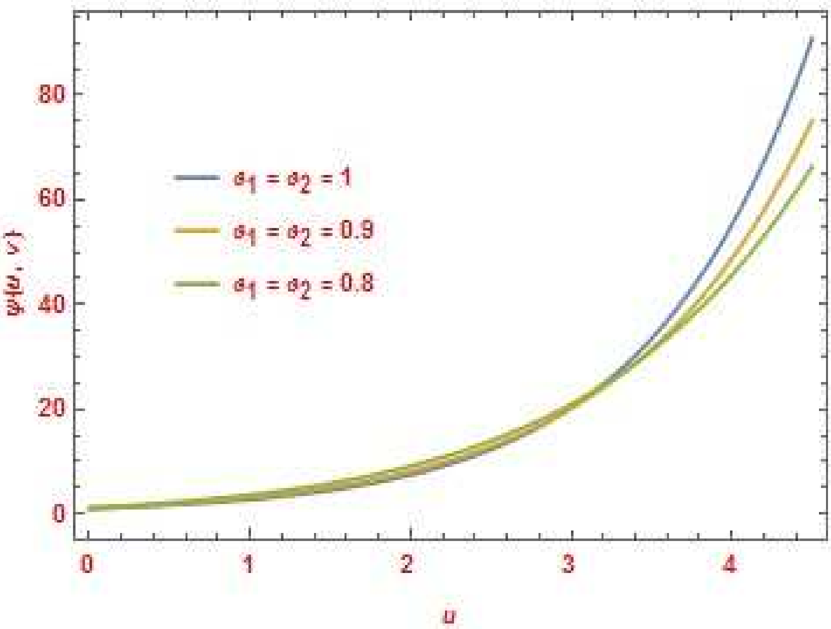

Fig. 3: The approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (58) at $\vartheta_1=\vartheta_2=1,0.9,0.8$.

Example 5. Consider the following nonlinear conformable fractional partial differential equation

$$\frac{\partial^{\vartheta_2}\psi}{\partial v^{\vartheta_2}}-\left(\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right)^2-\psi\frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}}=0,\quad 0<\vartheta_1,\vartheta_2\le 1, \tag{60}$$

with the IC

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},0\right)=\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2. \tag{61}$$

Solution. Applying CDSE transformation, and conformable Sumudu transform to Eq. (61), the result is,

$$\Psi(w,q)=2w^2q^2+qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right)^2+\psi\frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}}\right]. \tag{62}$$

Taking $\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}[\Psi(w,q)]$ of (62), we get

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2+\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(\frac{\partial^{\vartheta_1}\psi}{\partial u^{\vartheta_1}}\right)^2+\psi\frac{\partial^{2\vartheta_1}\psi}{\partial u^{2\vartheta_1}}\right]\right]. \tag{63}$$

Now, applying the decomposition method, we substitute Eq. (25) in Eq. (63) and with the results in Eqs. (28), and (29), we get the following solution components

$$\psi_0\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2, \tag{64}$$

$$\psi_1\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[\left(\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}\right)^2+\psi_0\frac{\partial^{2\vartheta_1}\psi_0}{\partial u^{2\vartheta_1}}\right]\right]=6\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right), \quad (65)$$

$$\psi_2\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(S_u^{\vartheta_1}\right)^{-1}\left(E_v^{\vartheta_2}\right)^{-1}\left[qS_u^{\vartheta_1}E_v^{\vartheta_2}\left[2\left(\frac{\partial^{\vartheta_1}\psi_0}{\partial u^{\vartheta_1}}\right)\left(\frac{\partial^{\vartheta_1}\psi_1}{\partial u^{\vartheta_1}}\right)+\psi_0\frac{\partial^{2\vartheta_1}\psi_1}{\partial u^{2\vartheta_1}}+\psi_1\frac{\partial^{2\vartheta_1}\psi_0}{\partial u^{2\vartheta_1}}\right]\right]$$
$$=36\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)^2. \quad (66)$$

As a result, we have the solution to the Eq. (60) as:

$$\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)=\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2+6\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)+36\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)^2+....=\frac{\left(\frac{u^{\vartheta_1}}{\vartheta_1}\right)^2}{1-6\left(\frac{v^{\vartheta_2}}{\vartheta_2}\right)}, \quad (67)$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\psi(u,v)=\frac{u^2}{1-6v}. \quad (68)$$

In the following figure, Figure 4 we sketch the approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (60) at different fractional orders $\vartheta_1=\vartheta_2=1,0.9,0.8$.

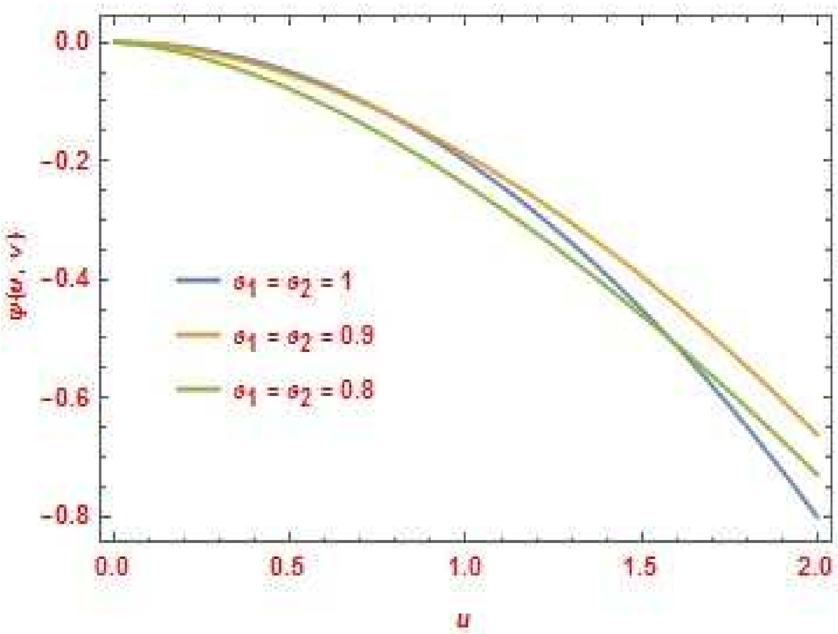

Fig. 4: The approximate solution of $\psi\left(\frac{u^{\vartheta_1}}{\vartheta_1},\frac{v^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (67) at $\vartheta_1=\vartheta_2=1,0.9,0.8$.

6 Conclusion

In this study, we defined the CDSE transformation. We began by putting the definition of CDSE transformation and computing to some basic functions. A few CDSE transformation related theorems and properties are presented and proved in the next section. To show the applicability and efficiency of the suggested transform, we have employed the CDSE transformation and the decomposition method to accomplish the precise solution of a broad class of nonlinear conformable fractional partial differential equations, including conformable fractional derivatives. Based on the obtained findings, we conclude that the provided technique is efficient, suitable, reliable, and adequate to acquire the accurate solutions of nonlinear conformable fractional partial differential equations according to the considered initial and boundary conditions. Additionally, as compared to other approaches, the necessary computation arisen in using the CDSE decomposition method is less than other numerical method. Therefore, we may state that a broad class of nonlinear fractional partial differential equation schemes can be solved using these approaches.

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