Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/100207

Solution of Conformable Fractional Heat Equation Using Fractional Bessel Functions

Almajeda Naamneh, Sharifa Alsharif and Edris Rawashdeh*

Department of Mathematics, Faculty of Sciences, Yarmouk Univeristy, Irbid, Jordan

Received: 2 Sep. 2022, Revised: 18 Oct. 2022, Accepted: 20 Jan. 2023
Published online: 1 Apr. 2024

Abstract: A second order linearly independent solution of the fractional Bessel equation is defined by utilizing the Wronskian matrix and the fractional Bessel function of the first kind of complex order. Additionally, as an application, a precise solution to a reformulated fractional type heat equation in one and two dimensions in a circular plate is produced.

Keywords: Conformable derivative, Bessel functions, conformable heat equation.

1 Introduction, Motivation and Preliminaries

Bessel functions and related integrals are constantly required in applied mathematics and physics. They arise in spherical symmetry problems. Moreover, they are considered to be the basic solutions of Bessel's differential equation

$$t^2y'' + ty' + \left(t^2 - p^2\right) = 0, \quad p \in \mathbb{R}.$$

Bessel functions are named after the great mathematician W. Bessel $(1784-1846)$, while D. Bernoulli (1732) is known to be the first one who introduced Bessel functions. These function can be obtained by solving the wave equations

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

in spherical coordinates [1]. Therefore, Bessel functions form an essential pillar in Fourier analysis.

Bessel equations has two fundamental solutions $J_n(x)$and $Y_n(x)$. $J_n(x)$ and $Y_n(x)$ are called the Bessel function of first and second order, respectively. For more details about Bessel functions, one can see the books by Luke [2] and Watson [3].

Fractional calculus became a very attractive to mathematician and many different forms of fractional differential operator were introduced; see [1,3,4,5]. Most of them used an integral form. All definitions appeared couldn't satisfy the usual properties of standard derivative except linearity property. In 2014, Khalil and et al. [6] give a new definition of fractional derivative called "conformable fractional derivative".

Definition 1.*[6] Let $f : [0,\infty) \to \mathbb{R}$ be a function. The α^{th} order "conformable fractional derivative" of f is defined by*

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0,a), a > 0$, and $\lim_{t\to 0^+} D^{\alpha}(f)(t)$ exists, then define $D^{\alpha}(f)(0) = \lim_{t\to 0^+} D^{\alpha}(f)(t)$.

* Corresponding author e-mail: sharifa@yu.edu.jo

The conformable fractional derivative satisfies the multiplication rule, the quotient rule, however does not satisfy the chain rule. In addition, the new definition has results that are a natural generalization of the corresponding Roll's Theorem and Mean Value Theorem [6,7]. Note that the function may be $\alpha-$differentiable but not differentiable.

With the use of conformable fractional derivative, the fractional Bessel second order differential equation was reformulated as follows

$$x^{2\alpha}D^{\alpha}D^{\alpha}y+\alpha x^{\alpha}D^{\alpha}y+(x^{2\alpha}-\alpha^{2}p^{2})y=0.$$

Further, the fractional Bessel function of the first kind of order αp was obtained as

$$J_{\alpha p}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k+p}k!\Gamma(k+p+1)}x^{2\alpha k+\alpha p},$$

where $\alpha p\in\mathbb{R}$ and $\alpha\in(0,1)$, [8].

In this paper for $v\in\mathbb{C}$, with the use of the fractional Bessel function of the first kind, we define a second order linearly independent solution of the fractional Bessel equation and verify some of its orthogonality features, using complex order αv and the Wronskian matrix. Additionally, a reformulated fractional type heat equation in one and two dimensions has been solved for a circular plate.

2 Fractional Bessel Functions of the Second Kind

For $v\in\mathbb{C}$, the complex order fractional Bessel function of the first kind αv is defined as

$$J_{\alpha v}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k+v}k!\ \Gamma(k+v+1)}x^{2\alpha k+\alpha v}.$$

To determine the second linearly independent solution, let us first discuss the behavior of $J_{\alpha v}(x)$ as $x\to 0$.

Theorem 1. *Let $J_{\alpha v}(x)$ be the fractional Bessel function of the first kind of complex order, then*

$$\lim_{x\to 0}J_{\alpha v}(x)=\begin{cases}0, & Re(v)>0\\ 1, & v=0\\ \pm\infty, & Re(v)<0,\ v\notin\mathbb{Z}\end{cases}$$

Proof. If $Re(v)>0$, then

$$\begin{aligned}\lim_{x\to 0}J_{\alpha v}(x)&=\lim_{x\to 0}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k+v}k!\Gamma(k+v+1)}x^{2\alpha k+\alpha v}\\ &=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k+v}k!\Gamma(k+v+1)}\lim_{x\to 0}x^{2\alpha k+\alpha v}\\ &=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k+v}k!\Gamma(k+v+1)}\lim_{x\to 0}(x^{\alpha})^{2k}(x^{\alpha})^{v}\\ &=\frac{1}{(2\alpha)^{v}\Gamma(v+1)}\lim_{x\to 0}(x^{\alpha})^{v}=0,\end{aligned}$$

since $Re(v)>0$.

Now if $v=0$, then

$$\lim_{x\to 0}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k}k!\Gamma(k+1)}x^{2\alpha k}=1.$$

Finally, if $Re(v)<0$, we have

$$\begin{aligned}\lim_{x\to 0}J_{\alpha v}(x)&=\lim_{x\to 0}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2\alpha)^{2k+v}k!\Gamma(k+v+1)}x^{2\alpha k+\alpha v}\\ &=\frac{1}{(2\alpha)^{v}\Gamma(v+1)}\lim_{x\to 0}(x^{\alpha})^{v}\\ &=\pm\infty,\end{aligned}$$

since $Re(v)<0$.

As a conclusion of Theorem 1, it is easy to see that the two solution $J_{\alpha v}(x)$ and $J_{-\alpha v}(x)$ are linearly independent and any linear combination is also a solution if $v \notin \mathbb{Z}$, else they are linearly dependent. Thus we define

$$Y_{\alpha v}(x) = \frac{\cos(v\pi)J_{\alpha v}(x) - J_{-\alpha v}(x)}{\sin(v\pi)},$$

the fractional Bessel function of the second kind of order αv.

For $p \in \mathbb{Z}$, we define

$$Y_{\alpha p}(x) = \lim_{v \to p} Y_{\alpha v}(x).$$

Definition 2.[9] *For two functions y_1 and y_2 satisfying the conformable fractional linear differential equation*

$$D^\alpha D^\alpha y + P(x)D^\alpha y + Q(x)y = 0,$$

where $0 < \alpha \leq 1$. The fractional Wronskian of the solutions is defined by

$$W_\alpha(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ D^\alpha(y_1) & D^\alpha(y_2) \end{vmatrix} = e^{-\int \frac{P(x)}{x^{1-\alpha}} dx}.$$

Theorem 2. *$J_{\alpha v}(x)$ and $Y_{\alpha v}(x)$ are two linearly independent solutions of the fractional Bessel equation for all $v \in \mathbb{C}$.*

Proof. This can be shown by computing the Wronskian determinant for conformable fractional differential equations [10].

First, rewrite the fractional Bessel second order differential equation in the form

$$D^\alpha D^\alpha y + \frac{\alpha}{x^\alpha} D^\alpha y + (1 - \frac{\alpha^2 v^2}{x^{2\alpha}})y = 0,$$

then the Wronskian determinant of $J_{\alpha v}(x)$ and $Y_{\alpha v}(x)$ turns out to be

$$\begin{aligned} W(J_{\alpha v}(x), Y_{\alpha v}(x)) &= \begin{vmatrix} J_{\alpha v}(x) & Y_{\alpha v}(x) \\ D^\alpha(J_{\alpha v}(x)) & D^\alpha(Y_{\alpha v}(x)) \end{vmatrix} \\ &= e^{-\int \frac{\alpha}{x^\alpha x^{1-\alpha}} dx} \\ &= e^{-\int \frac{\alpha}{x} dx} \\ &= e^{-\ln(x)^\alpha} = \frac{1}{(x)^\alpha}. \end{aligned}$$

Since $W(J_{\alpha v}(x), Y_{\alpha v}(x)) \neq 0$; then the result holds.

3 Orthogonality and Normalization of Fractional Bessel Function

Given a fixed nonnegative integer p, the function $J_{\alpha p}(x)$ has an infinite number of positive zeros; $z_{k,\alpha p}$. Figure 1.1 describes the graph of $J_{\alpha p}(x)$ with $\alpha = 0.75$ and $p = 2$, where it is clear that it has infinitely number of positive zeros.

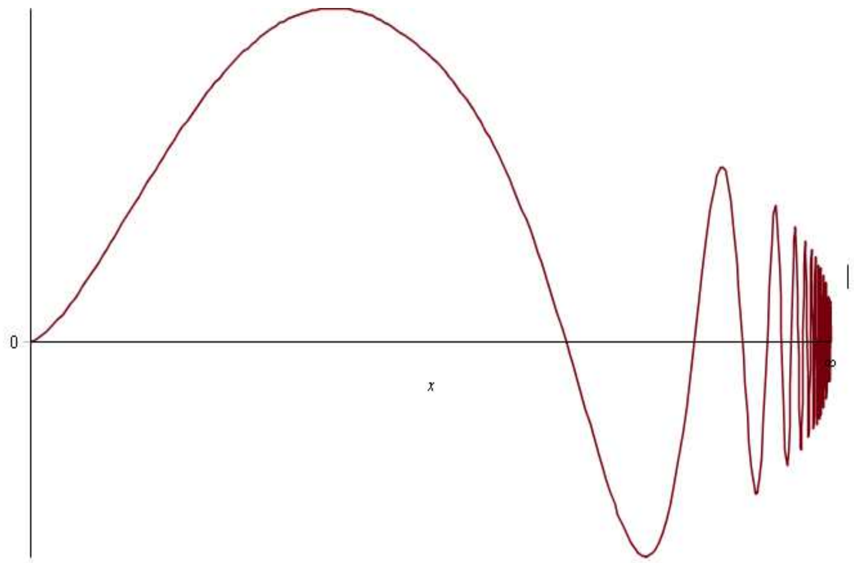

Figure 1.1: $J_{\alpha p}(x)$ with $\alpha = 0.75$ and $p = 2$.

To determine the coefficients A_k in a given series of the form

$$f(x) = \sum_{k=1}^{\infty} A_k J_{\alpha p}(z_{k,\alpha p} x),$$

we need to determine the orthogonality relation as in the following theorem.

Theorem 3. *For a fixed integer $p \geq 0$ and a, b are distinct positive zeros of $J_{\alpha p}(x)$, we have*

$$\int_0^1 x^\alpha J_{\alpha p}(ax) J_{\alpha p}(bx) d^\alpha x = 0$$

and

$$\int_0^1 x^\alpha J_{\alpha p}^2(ax) d^\alpha x = \frac{1}{2\alpha}(J_{\alpha p+\alpha}(a))^2.$$

Proof. If we write the fractional Bessel equation

$$x^{2\alpha} D^\alpha D^\alpha y + \alpha x^\alpha D^\alpha y + (x^{2\alpha} - \alpha^2 p^2) y = 0$$

in the form

$$x^\alpha D^\alpha (x^\alpha D^\alpha y) + (x^{2\alpha} - \alpha^2 p^2) y = 0$$

and replace x by ax, we get

$$x^\alpha D^\alpha (x^\alpha D^\alpha y) + ((ax)^{2\alpha} - \alpha^2 p^2) y = 0. \tag{3.1}$$

Thus, the equation has a solution

$$y = J_{\alpha p}(ax).$$

Similarly, replace x by bx, we get

$$x^\alpha D^\alpha (x^\alpha D^\alpha y) + ((bx)^{2\alpha} - \alpha^2 p^2) y = 0. \tag{3.2}$$

Then

$$y = J_{\alpha p}(bx)$$

is also a solution. Multiply the differential equation 3.1 by $J_{\alpha p}(bx)$ and the differential equation 3.2 by $J_{\alpha p}(ax)$, subtract the resulting equations and divide by x^α then add and subtract the term $x^\alpha D^\alpha (J_{\alpha p}(ax)) D^\alpha (J_{\alpha p}(bx))$ to get

$$D^\alpha [x^\alpha J_{\alpha p}(bx) \cdot D^\alpha (J_{\alpha p}(ax)) - x^\alpha J_{\alpha p}(ax) \cdot D^\alpha (J_{\alpha p}(bx))] + (a^{2\alpha} - b^{2\alpha}) x^\alpha J_{\alpha p}(ax) J_{\alpha p}(bx) = 0.$$

Integrate the resulting equation from 0 to 1 to obtain

$$\begin{matrix}[x^{\alpha}J_{\alpha p}(bx)\cdot D^{\alpha}(J_{\alpha p}(ax)) \\ -x^{\alpha}J_{\alpha p}(ax)\cdot D^{\alpha}(J_{\alpha p}(bx))]\end{matrix}\Big|_0^1+\int_0^1(a^{2\alpha}-b^{2\alpha})x^{\alpha}J_{\alpha p}(ax)J_{\alpha p}(bx)d^{\alpha}x=0.$$

If a and b are distinct positive zeros of $J_{\alpha p}(x)$, then

$$(a^{2\alpha}-b^{2\alpha})\int_0^1 x^{\alpha}J_{\alpha p}(ax)J_{\alpha p}(bx)d^{\alpha}x=0.$$

Thus

$$\int_0^1 x^{\alpha}J_{\alpha p}(ax)J_{\alpha p}(bx)d^{\alpha}x=0.$$

Back to the fractional Bessel equation, replace x by ax to get

$$x^{2\alpha}D^{\alpha}D^{\alpha}y+\alpha x^{\alpha}D^{\alpha}y+((ax)^{2\alpha}-\alpha^2p^2)y=0.$$

Then

$$y=J_{\alpha p}(ax)$$

is a solution. Multiply the differential equation by $2D^{\alpha}y$, we have

$$2x^{2\alpha}D^{\alpha}yD^{\alpha}D^{\alpha}y+2\alpha x^{\alpha}D^{\alpha}yD^{\alpha}y+2((ax)^{2\alpha}-\alpha^2p^2)yD^{\alpha}y=0,$$

which is equivalent to

$$D^{\alpha}[x^{2\alpha}(D^{\alpha}y)^2]+D^{\alpha}[a^{2\alpha}x^{2\alpha}y^2]-2\alpha a^{2\alpha}x^{\alpha}y^2-D^{\alpha}[\alpha^2p^2y^2]=0.$$

Therefore

$$D^{\alpha}[x^{2\alpha}(D^{\alpha}J_{\alpha p}(ax))^2]+D^{\alpha}[a^{2\alpha}x^{2\alpha}J^2_{\alpha p}(ax)]-2\alpha a^{2\alpha}x^{\alpha}J^2_{\alpha p}(ax)-D^{\alpha}[\alpha^2p^2J^2_{\alpha p}(ax)]=0.$$

Integrate from 0 to 1 for both side to obtain

$$2\alpha a^{2\alpha}\int_0^1 x^{\alpha}J^2_{\alpha p}(ax)d^{\alpha}x=[x^{2\alpha}(D^{\alpha}J_{\alpha p}(ax))^2+(a^{2\alpha}x^{2\alpha}-\alpha^2p^2)J^2_{\alpha p}(ax)]_0^1.$$

At $x=0$, $J_{\alpha p}(0)=0$. At $x=1$,

$$D^{\alpha}J_{\alpha p}(ax)=a^{\alpha}D^{\alpha}J_{\alpha p}(a)=a^{\alpha}D^{\alpha}[J_{\alpha p}(ax)]_{x=1}.$$

So we have

$$\int_0^1 x^{\alpha}J^2_{\alpha p}(ax)d^{\alpha}x=\frac{1}{2\alpha a^{2\alpha}}[(a^{\alpha}D^{\alpha}J_{\alpha p}(a))^2+(a^{2\alpha}-\alpha^2p^2)J^2_{\alpha p}(a)].$$

Since a is a positive zeros of $J_{\alpha p}(x)$ then $J_{\alpha p}(a)=0$ and

$$\int_0^1 x^{\alpha}J^2_{\alpha p}(ax)d^{\alpha}x=\frac{1}{2\alpha}(D^{\alpha}J_{\alpha p}(a))^2=\frac{1}{2\alpha}(J_{\alpha p+\alpha}(a))^2.$$

Corollary 1. *Let f be a function defined on the interval $0\le x\le 1$, and that it has a Fourier-fractional Bessel series expansion given by*

$$f(x)=\sum_{k=1}^{\infty}A_kJ_{\alpha p}(z_{k,\alpha p}x).$$

where $z_{k,\alpha p}$ are the zero's of $J_{\alpha p}$. Then the coefficients A_k *are*

$$A_k=\frac{2\alpha}{(J_{\alpha p+\alpha}(z_{k,\alpha p}))^2}\int_0^1 f(x)J_{\alpha p}(z_{k,\alpha p}x)x^{\alpha}d^{\alpha}x.$$

*Proof.*Multiply both side of the series by $x^{\alpha}J_{\alpha p}(z_{k,\alpha p}x)$ and integrate from 0 to 1 to get

$$\int_0^1 f(x)J_{\alpha p}(z_{k,\alpha p}x)x^{\alpha}d^{\alpha}x = \sum_{k=1}^{\infty} A_k \int_0^1 x^{\alpha}J^2_{\alpha p}(z_{k,\alpha p}x)d^{\alpha}x$$

$$= \frac{1}{2\alpha}\sum_{k=1}^{\infty} A_k(J_{\alpha p+\alpha}(z_{k,\alpha p}))^2.$$

$$= \frac{1}{2\alpha}A_r(J_{\alpha p+\alpha}(z_{r,\alpha p}))^2.$$

Therefore

$$A_r = \frac{2\alpha}{(J_{\alpha p+\alpha}(z_{r,\alpha p}))^2}\int_0^1 f(x)J_{\alpha p}(z_{r,\alpha p}x)x^{\alpha}d^{\alpha}x.$$

4 Fractional Heat Equation in One Circular Plate

Using conformable fractional derivative, many authors tried to reformulate the general form of heat equation to fractional form, see [10,11,12]. One of these forms is the homogeneous conformable heat equation defined on a radial symmetric plate [12],

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(r,t) = \beta[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}] \tag{4.1}$$

$$u(r,0) = T_0$$

$$u(0,t) = 0$$

$$|u(1,t)| < \infty \quad \text{as} \quad r \to 0^+$$

for $0 < r < 1$, $t > 0$ and $\alpha \in (0,1)$. Using separation of variable method, the general solution is

$$u(r,t) = 2T_0\sum_{k=1}^{\infty}\frac{1}{\lambda_k J_1(\lambda_k)}e^{-\lambda_k^2\beta t}J_0(\lambda_k r), \tag{4.2}$$

where λ_k is the $k'th$ positive zero of J_0.

With the use of the fractional Bessel functions, we obtain an exact solution for the fractional heat conduction equation in circular disk of radius 1 as follows:

Theorem 4. *For $0 < r < 1$, $t > 0$ and $\alpha \in (0,1)$, the exact solution of the fractional heat conduction equation*

$$\frac{\partial}{\partial t}u(r,t) = c^2(\frac{\partial^{\alpha}}{\partial r^{\alpha}}(\frac{\partial^{\alpha}}{\partial r^{\alpha}}u(r,t)) + \frac{\alpha}{r^{\alpha}}\frac{\partial^{\alpha}}{\partial r^{\alpha}}u(r,t)) \tag{4.3}$$

$$u(1,t) = 0$$

$$|u(r,t)| < \infty \; as \; r \to 0$$

$$u(r,0) = f(r) = T_0$$

is given by

$$u(r,t) = 2\alpha T_0\sum_{k=1}^{\infty}\frac{1}{\lambda_k J_{\alpha}(\lambda_k)}\,e^{-\lambda_k^{2\alpha}c^2 t}J_0(\lambda_k r).$$

Proof. By separation of variable method, let

$$u(r,t) = R(r)T(t).$$

Then

$$R(r)\frac{\partial}{\partial t}T(t) = c^2T(t)(\frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial^\alpha}{\partial r^\alpha}R(r)) + \frac{\alpha}{r^\alpha}\frac{\partial^\alpha}{\partial r^\alpha}R(r)).$$

So

$$\frac{1}{c^2}\frac{T'(t)}{T(t)} = \frac{1}{R(r)}(D^\alpha D^\alpha R(r) + \frac{\alpha}{r^\alpha}D^\alpha R(r)) = k = -\lambda^2.$$

Thus, we obtain two ordinary differential equations

$$T'(t) + \lambda^2c^2T(t) = 0 \quad (4.4)$$

and

$$D^\alpha D^\alpha R(r) + \frac{\alpha}{r^\alpha}D^\alpha R(r) + \lambda^2 R(r) = 0. \quad (4.5)$$

Note that we choose the separation constant k to be negative since else the factor $T(t)$ doesn't go to zero as $t \to \infty$.

The solution of the ordinary differential equation (4.4) is

$$T(t) = e^{-\lambda^2c^2t}.$$

Multiply the differential equation (4.5) by $r^{2\alpha}$, we have

$$r^{2\alpha}D^\alpha D^\alpha R(r) + \alpha r^\alpha D^\alpha R(r) + \lambda^2 r^{2\alpha}R(r) = 0,$$

which is a fractional Bessel differential equation of order 0. Therefore, the solution is

$$R(r) = c_1J_0(\lambda^{\frac{1}{\alpha}}r) + c_2Y_0(\lambda^{\frac{1}{\alpha}}r).$$

Since Y_0 is not bounded as $r \to 0$, then we must have $c_2 = 0$. So the solution is

$$R(r) = c_1J_0(\lambda^{\frac{1}{\alpha}}r).$$

Applying the boundary condition $R(1) = 0$, gives us

$$c_1J_0(\lambda^{\frac{1}{\alpha}}) = 0.$$

In order to get a nontrivial solution, we must have

$$J_0(\lambda^{\frac{1}{\alpha}}) = 0$$

and $\lambda^{\frac{1}{\alpha}} = \lambda_1, \lambda_2, \lambda_3, \ldots$ ($\lambda^{\frac{1}{\alpha}} = \lambda_k$, $k = 1,2,3,\ldots$) are the $k'th$ positive root of J_0. Thus

$$R_k(r) = J_0(\lambda_k r)$$

and the fundamental solution given by

$$u_k(r,t) = e^{-\lambda_k^{2\alpha}c^2t}J_0(\lambda_k r)$$

satisfies the differential equation (4.3) and the boundary conditions for each positive integer k. The general solution

$$u_k(r,t) = \sum_{k=1}^{\infty} A_k\, e^{-\lambda_k^{2\alpha}c^2t}J_0(\lambda_k r)$$

also satisfies the initial condition. To determine the coefficient A_k, we must have

$$u(r,0) = f(r) = \sum_{k=1}^{\infty} A_k\, J_0(\lambda_k r),$$

where

$$A_k = \frac{2\alpha}{J_\alpha^2(\lambda_k)} \int_0^1 r^\alpha f(r)\, J_0(\lambda_k r) d^\alpha r$$

Through integration by substitution, let $\lambda_k r = x$, and if $f(r) = T_0$, we have

$$\begin{aligned} A_k =& \frac{2\alpha}{J_\alpha^2(\lambda_k)} \frac{T_0}{(\lambda_k)^{\alpha+1}} \int_0^{\lambda_k} x^\alpha\, J_0(x) d^\alpha x \\ =& \frac{2\alpha}{J_\alpha^2(\lambda_k)} \frac{T_0}{(\lambda_k)^{\alpha+1}} [x^\alpha\, J_\alpha(x)]_0^{\lambda_k} \\ =& \frac{2\alpha T_0}{\lambda_k J_\alpha(\lambda_k)}. \end{aligned}$$

Thus, the general solution is

$$u(r,t) = 2\alpha T_0 \sum_{k=1}^{\infty} \frac{1}{\lambda_k J_\alpha(\lambda_k)}\, e^{-\lambda_k^{2\alpha} c^2 t} J_0(\lambda_k r).$$

As an application of the fractional Bessel functions of the first and second order, an exact solution of the fractional heat equation in two dimensional circular plate is given in the following Theorem.

Theorem 5.*For* $0 < r < 1$, $0 < \theta < (2\alpha\pi)^{\frac{1}{\alpha}}$, $t > 0$ *and* $\alpha \in (0,1)$, *the exact solution of the fractional heat conduction equation in two dimension circular disk,*

$$\frac{\partial}{\partial t} u(r,\theta,t) = c^2 (\frac{\partial^\alpha}{\partial r^\alpha} (\frac{\partial^\alpha}{\partial r^\alpha} u(r,\theta,t)) + \frac{\alpha}{r^\alpha} \frac{\partial^\alpha}{\partial r^\alpha} u(r,\theta,t) + \frac{1}{r^{2\alpha}} \frac{\partial^\alpha}{\partial \theta^\alpha} (\frac{\partial^\alpha}{\partial \theta^\alpha} u(r,\theta,t))),$$

$$\begin{aligned} u(1,\theta,t) =& 0, \\ |u(r,\theta,t)| <& \infty \;\; as \;\; r \to 0, \\ u(r,0,t) =& u(r,(2\alpha\pi)^{\frac{1}{\alpha}},t), \\ \frac{\partial^\alpha}{\partial \theta^\alpha} u(r,0,t) =& \frac{\partial^\alpha}{\partial \theta^\alpha} u(r,(2\alpha\pi)^{\frac{1}{\alpha}},t), \\ u(r,\theta,0) =& f(r,\theta) = T_0, \end{aligned}$$

is given by

$$u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{n,m} r) e^{-\lambda_{n,m}^{2\alpha} c^2 t}\, [a_{n,m} \cos(\frac{m}{\alpha}\theta^\alpha) + b_{n,m} \sin(\frac{m}{\alpha}\theta^\alpha)].$$

*Proof.*By separation of variable method, let

$$u(r,\theta,t) = R(r) \cdot \Phi(\theta) \cdot T(t).$$

Then

$$R(r)\Phi(\theta) \frac{\partial}{\partial t} T(t) = c^2 T(t) [\Phi(\theta) [\frac{\partial^\alpha}{\partial r^\alpha} (\frac{\partial^\alpha}{\partial r^\alpha} R(r)) + \frac{\alpha}{r^\alpha} \frac{\partial^\alpha}{\partial r^\alpha} R(r)] + \frac{R(r)}{r^{2\alpha}} \frac{\partial^\alpha}{\partial \theta^\alpha} (\frac{\partial^\alpha}{\partial \theta^\alpha} \Phi(\theta))]$$

and

$$\frac{1}{c^2} \frac{\acute{T}(t)}{T(t)} = \frac{1}{R(r)} (D^\alpha D^\alpha R(r) + \frac{\alpha}{r^\alpha} D^\alpha R(r)) + \frac{1}{r^{2\alpha}} \frac{1}{\Phi(\theta)} D^\alpha D^\alpha \Phi(\theta) = k = -\lambda^2.$$

Thus, we obtain the ordinary differential equation

$$T'(t) + \lambda^2 c^2 T(t) = 0$$

and the solution is

$$T(t) = e^{-\lambda^2 c^2 t}.$$

Also,

$$\frac{r^{2\alpha}D^{\alpha}D^{\alpha}R(r)+\alpha r^{\alpha}D^{\alpha}R(r)+\lambda^{2}r^{2\alpha}R(r)}{R(r)}=-\frac{D^{\alpha}D^{\alpha}\Phi(\theta)}{\Phi(\theta)}=\mu.$$

Thus

$$r^{2\alpha}D^{\alpha}D^{\alpha}R(r)+\alpha r^{\alpha}D^{\alpha}R(r)+(\lambda^{2}r^{2\alpha}-\mu)R(r)=0 \tag{4.6}$$

and

$$D^{\alpha}D^{\alpha}\Phi(\theta)+\mu\Phi(\theta)=0.$$

Case 1: If $\mu<0$; $\mu=-n^2$, then we have

$$\Phi(\theta)=C_1e^{\frac{n}{\alpha}\theta^{\alpha}}+C_2e^{\frac{-n}{\alpha}\theta^{\alpha}}.$$

But

$$\Phi(0)=\Phi((2\alpha\pi)^{\frac{1}{\alpha}})$$

and

$$D^{\alpha}\Phi(0)=D^{\alpha}\Phi((2\alpha\pi)^{\frac{1}{\alpha}}).$$

Therefore $C_1=0=C_2$. We conclude that μ cannot be negative.

Case 2: If $\mu\geq 0$; $\mu=m^2$ with $m\geq 0$ then we have

$$\Phi(\theta)=C_1\cos(\frac{m}{\alpha}\theta^{\alpha})+C_2\sin(\frac{m}{\alpha}\theta^{\alpha}).$$

But what is meant by boundary conditions on θ is that $\Phi(\theta)$ and $D^{\alpha}\Phi(\theta)$ is $\alpha-$periodic with period $P=(2\alpha\pi)^{\frac{1}{\alpha}}$. So if we apply the condition $\Phi(0)=\Phi((2\alpha\pi)^{\frac{1}{\alpha}})$ to the solution, we get $C_1=C_1\cos(2\pi m)+C_2\sin(2\pi m)$. This happens when $m=n\in\mathbb{Z}$, and since we can take it to be nonnegative due to the constants C_1 and C_2,

$$\Phi(\theta)=C_1\cos(m\theta)+C_2\sin(m\theta),\qquad C_1,C_2\in\mathbb{R},\ m\in\mathbb{N}\cup\{0\}$$

Rewrite the differential equation (4.6) as follows

$$r^{2\alpha}D^{\alpha}D^{\alpha}R(r)+\alpha r^{\alpha}D^{\alpha}R(r)+(\lambda^{2}r^{2\alpha}-\alpha^{2}\frac{m^2}{\alpha^2})R(r)=0,$$

which is fractional Bessel differential equation ($p=\frac{m}{\alpha}$), hence

$$R(r)=AJ_m(\lambda^{\frac{1}{\alpha}}r)+BY_m(\lambda^{\frac{1}{\alpha}}r).$$

Since Y_m is not bounded as $r\to 0$, then we must have $B=0$. So the solution is

$$R(r)=AJ_m(\lambda^{\frac{1}{\alpha}}r).$$

Applying the boundary condition $R(1)=0$, gives us

$$AJ_m(\lambda^{\frac{1}{\alpha}})=0.$$

In order to get a nontrivial solution, we must have

$$J_m(\lambda^{\frac{1}{\alpha}})=0$$

and $\lambda^{\frac{1}{\alpha}}=\lambda_{n,m}$ are the $n'th$ positive roots of J_m. So that

$$R_{n,m}(r)=J_m(\lambda_{n,m}r),$$

for $n\geq 1$ and $m\geq 0$. Also,

$$T_{n,m}(t)=e^{-\lambda_{n,m}^{2\alpha}c^2t}.$$

The fundamental solution is

$$u_{n,m}(r,\theta,t) = R_{n,m}(r)\cdot\Phi_m(\theta)\cdot T_{n,m}(t).$$

The general solution is

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{n,m}r)e^{-\lambda_{n,m}^{2\alpha}c^2t}\,[a_{n,m}\cos(\frac{m}{\alpha}\theta^\alpha)+b_{n,m}\sin(\frac{m}{\alpha}\theta^\alpha)].$$

Setting $t=0$, we have

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{n,m}r)\,[a_{n,m}\cos(\frac{m}{\alpha}\theta^\alpha)+b_{n,m}\sin(\frac{m}{\alpha}\theta^\alpha)],$$

which is a Fourier series for $f(r,\theta)$ on the interval $[0,P]$ with $P=(2\alpha\pi)^{\frac{1}{\alpha}}$ holding r is fixed [13]. Therefore

$$\sum_{n=1}^{\infty} J_0(\lambda_{n,0}r)\,a_{n,0} = \frac{1}{2\pi}\int_0^P f(r,\theta)\frac{d\theta}{\theta^{1-\alpha}}, \quad for\ m=0.$$

$$\sum_{n=1}^{\infty} J_m(\lambda_{n,m}r)\,a_{n,m} = \frac{1}{\pi}\int_0^P f(r,\theta)\cos(\frac{m}{\alpha}\theta^\alpha)\frac{d\theta}{\theta^{1-\alpha}}, \quad for\ m\geq 1.$$

$$\sum_{n=1}^{\infty} J_m(\lambda_{n,m}r)\,b_{n,m} = \frac{1}{\pi}\int_0^P f(r,\theta)\sin(\frac{m}{\alpha}\theta^\alpha)\frac{d\theta}{\theta^{1-\alpha}}, \quad for\ m\geq 1.$$

These Fourier series coefficients are actually Fourier-fractional Bessel series expansion, so that

$$\begin{aligned} a_{n,0} =& \frac{2\alpha}{J_\alpha^2(\lambda_{n,0})}\int_0^1\left[\frac{1}{2\pi}\int_0^P f(r,\theta)d\theta\right]J_0(\lambda_{n,0}r)r^\alpha d^\alpha r \\ =& \frac{\alpha}{\pi J_\alpha^2(\lambda_{n,0})}\int_0^P\int_0^1 f(r,\theta)J_0(\lambda_{n,0}r)r^\alpha d^\alpha r\frac{d\theta}{\theta^{1-\alpha}}, \quad for\ m=0,\ n\geq 1. \end{aligned}$$

$$a_{n,m} = \frac{2\alpha}{\pi J_{m+\alpha}^2(\lambda_{n,m})}\int_0^P\int_0^1 f(r,\theta)\cos(\frac{m}{\alpha}\theta^\alpha)J_m(\lambda_{n,m}r)r^\alpha d^\alpha r\frac{d\theta}{\theta^{1-\alpha}}, \quad for\ m\geq 1,\ n\geq 1.$$

$$b_{n,m} = \frac{2\alpha}{\pi J_{m+\alpha}^2(\lambda_{n,m})}\int_0^P\int_0^1 f(r,\theta)\sin(\frac{m}{\alpha}\theta^\alpha)J_m(\lambda_{n,m}r)r^\alpha d^\alpha r\frac{d\theta}{\theta^{1-\alpha}}, \quad for\ m\geq 1,\ n\geq 1.$$

References

[1] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego GA,1999.
[2] Y. Luke, *Integral of Bessel functions*, McGraw-Hill, New York, 1962.
[3] G. N. Watson, *A Treatise on the theory of Bessel functions*, New York, Ny, Cambridge Mathematical Library, 1995.
[4] S. Lacroix, *Traite du calcul differentiel et du calcul integral*, Tome Troisiéme, Mme. VeCourcier, Paris, second edition, 1819.
[5] G. Samko, A. Kilbas and A. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach, Yverdon, 1993.
[6] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comp. Appl. Math.* **264**, 65-70 (2014).
[7] Sh. Al-Sharif and A. Malkawi, Modification of conformable fractional derivative with classical properties. *Italian J. P. Appl. Math.* **44**, 30-39 (2020).
[8] M. Abu Hammad, Sh. Al-Sharif, R. Khalil and A. Shmasneh, Fractional Bessel differential equation and fractional Bessel functions, *Italian J. P. Appl. Math.* **47**, 521–531(2022).
[9] M. Abu Hammad and R. Khalil, Abel's formula and Wronskian for conformable fractional differential equation, *Int. J. Diff. Equ. App.* **13**(3), 177-183 (2014).
[10] A.Kurt and Y. Çenesiz, The solutions of time and space conformable fractional heat equations with conformable Fourier transform, *Acta Univ. Sapientiae, Mathematica* **7**(2), 130–140 (2015).
[11] M. Abu Hammad, R. Khalil, Conformable fractional heat differential equation, *Int. J. Pure Appl. Math.* **94**(2), 215-221(2014).
[12] D. Avci, B. Iskender Eroglu and N. Ozdemir, Conformable heat equation on a radial symmetric plate, *Thermal Sci.* **21**(2), 819-826(2017).
[13] M. Abu Hammad and R. Khalil, Fractional Fourier series with applications, *J. Comput. Appl. Math.* **4**(6), 187-191 (2014).