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Solutions of Fuzzy Fractional Differential Equations

Atimad Harir[1,*], *Said Melliani*[2] *and Lalla Saadia Chadli* [2]

[1] Laboratory of Mathematical Modeling and Economic Calculation Department of Economics, Faculty of Economics and Management, University Hassan 1er, Settat, Morocco
[2] Laboratory of Applied Mathematics and Scientific Computing Sultan Moulay Slimane University P.O. Box 523, Beni Mellal, 23000, Morocco

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Abstract: The concept of generalized conformable fractional derivative is used to explore fuzzy fractional differential equations in this research. The extension of the class of differentiable fuzzy mappings underpins this concept. The summations and differences of two functions are produced using this derivative, and we prove the existence and uniqueness of a solution to a fuzzy fractional differential equation.

Keywords: fuzzy fractional differential equation, conformable fractional derivative, fuzzy number.

1 Introduction

In this paper we will consider the fractional differential equation

$$\begin{aligned} y^{(q)}(t) &= F(t, y(t)) \\ y(0) &= y_0 \end{aligned} \tag{1}$$

where $t \in (0,a)$ and y_0 is a fuzzy number. $y^{(q)}$ is the conformable fractional derivative of y of order $q \in (0,1]$ [1,2,3]. There are many suggestions to define a fuzzy fractional derivative and in consequence, to study eq. (1.1). see for [4,5] and fractional order introduced by [6]. The genralized derivative of a set value function was made by [7,8,9] and studied by [10,11,12], for generalized conformable fractional derivative studied by [13]. However, in this study, we look into the possibility of "other solutions" (local existence of two solutions is possible under the generalized conformable fractional derivative concept). The summoning as well as the distinction between two functions are introduced and demonstrated. We show several examples of the complex behavior of fuzzy fractional differential equation solutions(1).

2 Preliminaries

Let us denote by $\mathbb{R}_{\mathscr{F}} = \{\upsilon : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties:

$(i)\upsilon$ is normal i.e, there exists an $x_0 \in \mathbb{R}$ such that $\upsilon(x_0) = 1$,
$(ii)\upsilon$ is fuzzy convex i.e for $x, y \in \mathbb{R}$ and $0 < \lambda \leq 1$,

$$\upsilon(\lambda x + (1-\lambda)y) \geq \min[\upsilon(x), \upsilon(y)]$$

$(iii)\upsilon$ is upper semicontinuous,
$(iv)[\upsilon]^0 = cl\{x \in \mathbb{R} \mid \upsilon(x) > 0\}$ is compact. Then $\mathbb{R}_{\mathscr{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathscr{F}}$.

* Corresponding author e-mail: atimad.harir@gmail.com

For $0 < \alpha \leq 1$ denote $[\upsilon]^{\alpha} = \{x \in \mathbb{R} \mid \upsilon(x) \geq \alpha\}$, then from (i) to (iv) it follows that the α-level sets $[\upsilon]^{\alpha} \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which is denoted by $[\upsilon]^{\alpha} = [\upsilon_1^{\alpha}, \upsilon_2^{\alpha}]$. By $P_K(\mathbb{R})$ we denote the family of all nonempty compact convex subsets of $\mathbb{R}$, and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual.

Theorem 1.[4] *If $\upsilon \in \mathbb{R}_{\mathscr{F}}$, then*

$(i)[\upsilon]^{\alpha} \in P_K(\mathbb{R})$ *for all* $0 \leq \alpha \leq 1$
$(ii)[\upsilon]^{\alpha_2} \subset [\upsilon]^{\alpha_1}$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$
$(iii)\{\alpha_k\} \subset [0,1]$ *is a nondecreasing sequence which converges to* α *then*

$$[\upsilon]^{\alpha} = \bigcap_{k \geq 1} [\upsilon]^{\alpha_k}$$

Conversely, if $A_{\alpha} = \left\{[\upsilon_1^{\alpha}, \upsilon_2^{\alpha}]; \alpha \in (0,1]\right\}$ is a family of closed real intervals verifying (i) and (ii), then $\{A_{\alpha}\}$ defined a fuzzy number $\upsilon \in \mathbb{R}_{\mathscr{F}}$ such that $[\upsilon]^{\alpha} = A_{\alpha}$ for $0 < \alpha \leq 1$ and $[\upsilon]^{0} = \overline{\cup_{0<\alpha<1} A_{\alpha}} \subset A_0$

Lemma 1.[14] *Let $\upsilon, v : X \to [0,1]$ be the fuzzy sets. Then $\upsilon = v$ if and only if $[\upsilon]^{\alpha} = [v]^{\alpha}$ for all $\alpha \in [0,1]$*

The following arithmetic operations on fuzzy numbers are well known and frequently used below. If $\upsilon, v \in \mathbb{R}_{\mathscr{F}}$ then

$$[\upsilon + v]^{\alpha} = [\upsilon_1^{\alpha} + v_1^{\alpha}, \upsilon_2^{\alpha} + v_2^{\alpha}]$$
$$[\lambda \upsilon]^{\alpha} = \lambda [\upsilon]^{\alpha} = \begin{cases} [\lambda \upsilon_1^{\alpha}, \lambda \upsilon_2^{\alpha}] \text{ if } \lambda \geq 0 \\ [\lambda \upsilon_2^{\alpha}, \lambda \upsilon_1^{\alpha}] \text{ if } \lambda < 0 \end{cases}$$

Definition 1.[15, 16] *Let $\upsilon, v \in \mathbb{R}_{\mathscr{F}}$. If there exists $w \in \mathbb{R}_{\mathscr{F}}$ such as $\upsilon = v + w$ then w is called the H-difference of υ, v and it is denoted $\upsilon \ominus v$.*

Define $d : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{+} \cup \{0\}$ by the equation

$$d(\upsilon, v) = \sup_{\alpha \in [0,1]} d_H\left([\upsilon]^{\alpha}, [v]^{\alpha}\right), \quad \text{for all } \upsilon, v \in \mathbb{R}_{\mathscr{F}}$$

where d_H is the Hausdorff metric.

$$d_H\left([\upsilon]^{\alpha}, [v]^{\alpha}\right) = \max\left\{|\upsilon_1^{\alpha} - v_1^{\alpha}|, |\upsilon_2^{\alpha} - v_2^{\alpha}|\right\}$$

It is well known that $(\mathbb{R}_{\mathscr{F}}, d)$ is a complete metric space. We list the following properties of $d(\upsilon, v)$

$$d(\upsilon + w, v + w) = d(\upsilon, v) \quad \text{and} \quad d(\upsilon, v) = d(v, \upsilon)$$
$$d(k\upsilon, kv) = |k| d(\upsilon, v)$$
$$d(\upsilon, v) \leq d(\upsilon, w) + d(w, v)$$

for all $\upsilon, v, w \in \mathbb{R}_{\mathscr{F}}$ and $\lambda \in \mathbb{R}$.
Let (A_k) be a sequence in $P_K(\mathbb{R})$ converging to A. Then Theorem in [10] gives us an expression for the limit.

Theorem 2.[10] *If $d(A_k, A) \to 0$ as $k \to \infty$ then*

$$A = \bigcap_{k \geq 1} \overline{\bigcup_{m \geq k} A_m}.$$

Let $(0,a) \subset \mathbb{R}$ be an interval. We denote by $C((0,a), \mathbb{R}_{\mathscr{F}})$ the space of all continuous fuzzy functions on $(0,a)$ is a complete metric space with respect to the metric d.

3 Conformable Fractional Differentiability

Now we offer our new definition of fractional derivative of order $q \in (0,1]$, which is the simplest, most natural, and efficient definition available.

Definition 2.[13] *Let $F:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be a fuzzy function. q^{th} order fuzzy conformable fractional derivative of F is defined by*

$$T_q(F)(t)=\lim_{\varepsilon\to0^+}\frac{F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{F(t)\ominus F\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}.$$

for all $t>0,\ q\in(0,1)$. Let $F^{(q)}(t)$ stands for $T_q(F)(t)$. Hence

$$F^{(q)}(t)=\lim_{\varepsilon\to0^+}\frac{F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{F(t)\ominus F\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}$$

If F is q-differentiable in some $(0,a)$, and $\lim_{t\to0^+}F^{(q)}(t)$ exists, then

$$F^{(q)}(0)=\lim_{t\to0^+}F^{(q)}(t)$$

and the limits (in the metric d)

*Remark.*From the definition, it directly follows that if F is q-differentiable then the multi valued mapping F_α is q-differentiable for all $\alpha\in[0,1]$ and

$$T_qF_\alpha=\left[F^{(q)}(t)\right]^\alpha \tag{2}$$

Here T_qF_α is denoted the conformable fractional derivative of F_α of order q.

Definition 3.[13] *Let $F:I\to\mathbb{R}_{\mathscr{F}}$ be a fuzzy function and $q\in(0,1]$. One says, F is $q_{(1)}$-differentiable at point $t>0$ if there exists an element $F^{(q)}(t)\in\mathbb{R}_{\mathscr{F}}$ such that for all $\varepsilon>0$ sufficiently near to 0, there exist $F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)$, $F(t)\ominus F\left(t-\varepsilon t^{1-q}\right)$ and the limits (in the metric d)*

$$\lim_{\varepsilon\to0^+}\frac{F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{F(t)\ominus F\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}=F^{(q)}(t) \tag{3}$$

F is $q_{(2)}$-differentiable at $t>0$ if for all $\varepsilon<0$ sufficiently near to 0, there exist $F\left(t+\varepsilon t^{1-q}\right)\ominus F(t),F(t)\ominus F\left(t-\varepsilon t^{1-q}\right)$

$$\lim_{\varepsilon\to0^-}\frac{F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)}{\varepsilon}=\lim_{\varepsilon\to0^-}\frac{F(t)\ominus F\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}=F^{(q)}(t) \tag{4}$$

If F *is $q_{(n)}$-differentiable at $t>0$, we denote its q-derivatives $(q\in(0,1])$ by $F_n^{(q)}(t)$, for $n=1,2$*

Theorem 3.*Let $F:I\to\mathbb{R}_{\mathscr{F}}$ be fuzzy function, where $F_\alpha(t)=[f_1^\alpha(t),f_2^\alpha(t)]$, $\alpha\in[0,1]$*

(i)If F is $q_{(1)}$-differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q-differentiable and

$$\left[F^{\left(q_{(1)}\right)}(t)\right]^\alpha=\left[(f_1^\alpha)^{(q)}(t),(f_2^\alpha)^{(q)}(t)\right].$$

(ii)If F is $q_{(2)}$-differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q-differentiable and

$$\left[F^{\left(q_{(2)}\right)}(t)\right]^\alpha=\left[(f_2^\alpha)^{(q)}(t),(f_1^\alpha)^{(q)}(t)\right].$$

Theorem 4.*Let $F,G:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be generalized conformable differentiability such that if both F and G are $q_{(1)}$-differentiable or $q_{(2)}$-differentiable then $F+G$ and $F\ominus G$ generalized q -differentiable and*

$(i)T_q(F+G)(t)=T_qF(t)+T_qG(t)$
$(ii)T_q(F\ominus G)(t)=T_qF(t)\ominus T_qG(t)$

*Proof.*We present the details only for case (i), since the other case is anlogous. Since F is $q_{(1)}$-differentiable it follows that $F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)$ exists i.e there exists $\upsilon_1\left(t,\varepsilon t^{1-q}\right)$ such that

$$F\left(t+\varepsilon t^{1-q}\right)=F(t)+\upsilon_1\left(t,\varepsilon t^{1-q}\right) \tag{5}$$

Analogously since G is $q_{(1)}$-differentiable there exists $v_1\left(t,\varepsilon t^{1-q}\right)$ such that

$$G\left(t+\varepsilon t^{1-q}\right)=G(t)+v_1\left(t,\varepsilon t^{1-q}\right)$$

and we get

$$F\left(t+\varepsilon t^{1-q}\right)+G\left(t+\varepsilon t^{1-q}\right)=F(t)+G(t)+\upsilon_1\left(t,\varepsilon t^{1-q}\right)+v_1\left(t,\varepsilon t^{1-q}\right) \tag{6}$$

that is the H-difference

$$\left(F\left(t+\varepsilon t^{1-q}\right)+G\left(t+\varepsilon t^{1-q}\right)\right)\ominus(F(t)+G(t))=\upsilon_1\left(t,\varepsilon t^{1-q}\right)+v_1\left(t,\varepsilon t^{1-q}\right) \tag{7}$$

By similar reasoning we get that there exist $\upsilon_2\left(t,\varepsilon t^{1-q}\right)$ and $v_2\left(t,\varepsilon t^{1-q}\right)$ such that

$$F(t)=F\left(t-\varepsilon t^{1-q}\right)+\upsilon_2\left(t,\varepsilon t^{1-q}\right)$$
$$G(t)=G\left(t-\varepsilon t^{1-q}\right)+v_2\left(t,\varepsilon t^{1-q}\right)$$

and so

$$(F(t)+G(t))=\left(F\left(t-\varepsilon t^{1-q}\right)+G\left(t-\varepsilon t^{1-q}\right)\right)+\upsilon_2\left(t,\varepsilon t^{1-q}\right)+v_2\left(t,\varepsilon t^{1-q}\right)$$

that is the H-difference

$$(F(t)+G(t))\ominus\left(F\left(t-\varepsilon t^{1-q}\right)+G\left(t-\varepsilon t^{1-q}\right)\right)=\upsilon_2\left(t,\varepsilon t^{1-q}\right)+v_2\left(t,\varepsilon t^{1-q}\right) \tag{8}$$

We observe that

$$\lim_{\varepsilon\to0^+}\frac{\upsilon_1\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{\upsilon_2\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=F^{(q)}(t)\quad\text{and}$$
$$\lim_{\varepsilon\to0^+}\frac{v_1\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{v_2\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=G^{(q)}(t).$$

Finally, by multiplying (5) and (2) with $\frac{1}{\varepsilon}$ and passing to limit with $\lim_{\varepsilon\to0^+}$ we get that $F+G$ is $q_{(1)}$-differentiable and $T_q(F+G)(t)=T_qF(t)+T_qG(t)$ The case when F and G are $q_{(2)}$-differentiable is similar to the previous one.

Theorem 5.*Let $F,G:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be generalized conformable differentiability such that F is $q_{(1)}$-differentiable and G is $q_{(2)}$-differentiable or F is $q_{(2)}$-differentiable and G is $q_{(1)}$-differentiable on an interval (β,σ). If the H-difference $F(t)\ominus G(t)$ exists for $t\in(\beta,\sigma)$ then $F+G$ and $F\ominus G$ generalized q-differentiable and*

$(i)T_q(F+G)(t)=T_qF(t)\ominus(-T_qG(t))$ for all $t\in(\beta,\sigma)$
$(ii)T_q(F\ominus G)(t)=T_qF(t)+(-1)T_qG(t)$ for all $t\in(\beta,\sigma)$

*Proof.*We present the details only for case (ii), since the other case is anlogous. Since F is $q_{(1)}$-differentiable it follows that $F\left(t+\varepsilon t^{1-q}\right)\ominus F(t)$ exists i.e there exists $\upsilon_1\left(t,\varepsilon t^{1-q}\right)$ such that

$$F\left(t+\varepsilon t^{1-q}\right)=F(t)+\upsilon_1\left(t,\varepsilon t^{1-q}\right) \tag{9}$$

Analogously since G is $q_{(2)}$-differentiable there exists $v_1\left(t,\varepsilon t^{1-q}\right)$ such that

$$G(t)=G\left(t+\varepsilon t^{1-q}\right)+v_1\left(t,\varepsilon t^{1-q}\right) \tag{10}$$

and we get

$$F\left(t+\varepsilon t^{1-q}\right)+G(t)=F(t)+G\left(t+\varepsilon t^{1-q}\right)+\upsilon_1\left(t,\varepsilon t^{1-q}\right)+v_1\left(t,\varepsilon t^{1-q}\right).$$

Since the H-differences $F(t)\ominus G(t)$ and $F\left(t+\varepsilon t^{1-q}\right)\ominus G\left(t+\varepsilon t^{1-q}\right)$ exist for $\varepsilon>0$ such that $t+\varepsilon t^{1-q}\in(\beta,\sigma)$, we get

$$F\left(t+\varepsilon t^{1-q}\right)\ominus G\left(t+\varepsilon t^{1-q}\right)=F(t)\ominus G(t)+\upsilon_1\left(t,\varepsilon t^{1-q}\right)+v_1\left(t,\varepsilon t^{1-q}\right)$$

that is the H-difference

$$\left(F\left(t+\varepsilon t^{1-q}\right)\ominus G\left(t+\varepsilon t^{1-q}\right)\right)\ominus(F(t)\ominus G(t))=\upsilon_1\left(t,\varepsilon t^{1-q}\right)+v_1\left(t,\varepsilon t^{1-q}\right) \tag{11}$$

By similar reasoning we get that there exist $\upsilon_2\left(t,\varepsilon t^{1-q}\right)$ and $v_2\left(t,\varepsilon t^{1-q}\right)$ such that

$$F(t)=F\left(t-\varepsilon t^{1-q}\right)+\upsilon_2\left(t,\varepsilon t^{1-q}\right)$$
$$G\left(t-\varepsilon t^{1-q}\right)=G(t)+v_2\left(t,\varepsilon t^{1-q}\right)$$

and so

$$(F(t)\ominus G(t))\ominus\left(F\left(t-\varepsilon t^{1-q}\right)\ominus G\left(t-\varepsilon t^{1-q}\right)\right)=\upsilon_2\left(t,\varepsilon t^{1-q}\right)+v_2\left(t,\varepsilon t^{1-q}\right) \tag{12}$$

We observe that

$$\lim_{\varepsilon\to 0^+}\frac{u_1\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=\lim_{\varepsilon\to 0^+}\frac{\upsilon_2\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=F^{(q)}(t)\quad\text{and}$$
$$\lim_{\varepsilon\to 0^+}\frac{v_1\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=\lim_{\varepsilon\to 0^+}\frac{v_2\left(t,\varepsilon t^{1-q}\right)}{\varepsilon}=(-1)G^{(q)}(t)$$

Finally, by multiplying (4) and (7) with $\frac{1}{\varepsilon}$ and passing to limit with $\lim_{\varepsilon\to 0^+}$ we get that $F\ominus G$ is $q_{(1)}$-differentiable and $T_q(F\ominus G)(t)=T_qF(t)+$ (-1)$T_qG(t)$. The case when F is $q_{(2)}$-differentiable and G is $q_{(1)}$-differentiable is similar to the previous one.

Lemma 2.*If $F:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be generalized conformable differentiability and $\lambda\in\mathbb{R}$ then $T_q(\lambda F)(t)=\lambda T_qF(t)$*

*Proof.*By Definition 4 the statement of the Lemma follows easily.

Theorem 6.*Theorem [13] Let $q\in(0,1]$*

(i)If F is (1)-differentiable and F is $q_{(1)}$-differentiable then

$$T_{q_{(1)}}F(t)=t^{1-q}D_1^1F(t)$$

(ii)If F is (2)-differentiable and F is $q_{(2)}$-differentiable then

$$T_{q_{(2)}}F(t)=t^{1-q}D_2^1F(t)$$

Lemma 3.*Let $q\in(0,1]$. If both F and G*

–are $q_{(1)}$-differentiable and are (1)-differentiable or
–are $q_{(2)}$-differentiable and are (2)-differentiable

then

(i) $T_q(F+G)(t)=t^{1-q}DF(t)+t^{1-q}DG(t)$
(ii) $T_q(F\ominus G)(t)=t^{1-q}DF(t)\ominus t^{1-q}DG(t)$

*Proof.*By a similar reasoning as in the proof of Theorem 6, pose that $h=\varepsilon t^{1-q}$ in proof then $\varepsilon=ht^{q-1}$, we obtain (i) and (ii)

Lemma 4.*Let $q\in(0,1]$. If*

–F is $q_{(1)}$-differentiable and is (1)-differentiable and G is $q_{(2)}$-differentiable and is (2)-differentiable and or
–F is $q_{(2)}$-differentiable and is (2)-differentiable and G is $q_{(1)}$-differentiable and is (1)-differentiable

on an interval (β,σ). If the H-difference $F(t)\ominus G(t)$ exists for $t\in(\beta,\sigma)$ then

(i) $T_q(F+G)(t)=t^{1-q}DF(t)\ominus\left(-t^{1-q}DG(t)\right)$
(ii) $T_q(F\ominus G)(t)=t^{1-q}DF(t)+(-1)t^{1-q}DG(t)$

for all $t\in(\beta,\sigma)$

*Proof.*By a similar reasoning as in the proof of Theorem 5, pose that $h=\varepsilon t^{1-q}$ in proof then $\varepsilon=ht^{q-1}$, we obtain (i) and (ii)

Theorem 7.*If a function $F:(0,a)\to\mathbb{R}_{\mathscr{F}}$ is q-differentiable at $t_0>0, q\in(0,1]$, Denote $F_\alpha(t)=[f_1^\alpha(t),f_2^\alpha(t)], \alpha\in[0,1]$. Then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are continuous at t_0, so F is continuous at t_0*

Proof. If $\varepsilon > 0$ and $\alpha \in [0,1]$, we have:

$$\left[F\left(t_0+\varepsilon t_0^{1-q}\right) \ominus F(t_0)\right]^\alpha = \left[f_1^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_1^\alpha(t_0), f_2^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_2^\alpha(t_0)\right]$$

Dividing and multiplying by ε, we have:

$$\left[F\left(t_0+\varepsilon t_0^{1-q}\right) \ominus F(t_0)\right]^\alpha = \left[\frac{f_1^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_1^\alpha(t_0)}{\varepsilon}\cdot\varepsilon, \frac{f_2^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_2^\alpha(t_0)}{\varepsilon}\cdot\varepsilon\right]$$

Similarly, we obtain:

$$\left[F(t_0) \ominus F\left(t_0-\varepsilon t_0^{1-q}\right)\right]^\alpha = \left[\frac{f_1^\alpha(t_0) - f_1^\alpha\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\varepsilon, \frac{f_2^\alpha(t_0) - f_2^\alpha\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\varepsilon\right]$$

Then

$$\lim_{\varepsilon\to 0^+}\left[F\left(t_0+\varepsilon t_0^{1-q}\right) \ominus F(t_0)\right]^\alpha = \left[\lim_{\varepsilon\to 0^+}\frac{f_1^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_1^\alpha(t_0)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\right.$$
$$\left., \lim_{\varepsilon\to 0^+}\frac{f_2^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_2^\alpha(t_0)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\right]$$

Similarly, we obtain:

$$\lim_{\varepsilon\to 0^+}\left[F(t_0) \ominus F\left(t_0-\varepsilon t_0^{1-q}\right)\right]^\alpha = \left[\lim_{\varepsilon\to 0^+}\frac{f_1^\alpha(t_0) - f_1^\alpha\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\right.$$
$$\left., \lim_{\varepsilon\to 0^+}\frac{f_2^\alpha(t_0) - f_2^\alpha\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\right]$$

Let $h = \varepsilon t_0^{1-q}$. Then

$$\lim_{h\to 0^+}\left[F(t_0+h) \ominus F(t_0)\right]^\alpha = \left[(f_1^\alpha)^{(q)}(t_0)\cdot 0, (f_2^\alpha)^{(q)}(t_0)\cdot 0\right]$$

Similarly, we obtain:

$$\lim_{h\to 0^+}\left[F(t_0-h)\right]^\alpha = \left[F(t_0)\right]^\alpha$$

which implies that

$$\lim_{h\longrightarrow 0^+}\left[F(t_0+h)\right]^\alpha = \left[F(t_0)\right]^\alpha$$

Similary, we obtain:

$$\lim_{h\longrightarrow 0^+}\left[F(t_0-h)\right]^\alpha = \left[F(t_0)\right]^\alpha$$

Hence, F is continuous at t_0.

Remark. If $F:(0,a)\to\mathbb{R}_{\mathscr{F}}$ is q -differentiable for $q\in(0,1]$ and $F^{(q)}(t)$ is continuous, then we denote $F\in C((0,a),\mathbb{R}_{\mathscr{F}})$.

Let $q\in(0,1]$ and $F:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be such that $[F(t)]^\alpha = [f_1^\alpha(t), f_2^\alpha(t)]$ for all $t\in(0,a)$ and $\alpha\in[0,1]$. Suppose that $f_1^\alpha, f_2^\alpha \in C((0,a),\mathbb{R})\cap L^1((0,a),\mathbb{R})$ for all $\alpha\in[0,1]$ and let

$$A_\alpha =: \left[\int_0^t \frac{f_1^\alpha}{x^{1-q}}(x)dx, \int_0^t \frac{f_2^\alpha}{x^{1-q}}(x)dx\right], \quad t\in(0,a) \tag{13}$$

Lemma 5. *The family $\{A_\alpha;\alpha\in[0,1]\}$, given by eq (5), defined a fuzzy number $F\in\mathbb{R}_{\mathscr{F}}$ such that $[F]^\alpha = A_\alpha$*

*Proof.*For $\alpha < \beta$ we have that $f_1^\alpha(x) \leq f_1^\beta(x)$ and $f_2^\alpha(x) \geq f_2^\beta(x)$. It follows that $A_\alpha \supseteq A_\beta$. Since $f_1^0(x) \leq f_1^{\alpha_n}(x) \leq f_1^1(x)$ we have

$$\left|x^{q-1} f_i^{\alpha_n}(x)\right| \leq \max\left\{a^{q-1}\left|f_i^0(x)\right|, a^{q-1}\left|f_i^1(x)\right|\right\} =: g_i(x)$$

for $\alpha_n \in [0,1]$ and $i = 1,2$. Obviously, g_i is integrable on $(0,a)$. Therefore, if $\alpha_n \uparrow \alpha$ then by the Lebesque's Dominated convergence Theorem, we have

$$\lim_{n\to\infty} \int_0^t \frac{f_i^{\alpha_n}}{x^{1-q}}(x)dx = \int_0^t \frac{f_i^\alpha}{x^{1-q}}(x)dx, i = 1,2$$

From Theorem 1, the proof is complete.

Definition 4.*Let $F \in C\left((0,a), \mathbb{R}_{\mathscr{F}}\right) \cap L^1\left((0,a), \mathbb{R}_{\mathscr{F}}\right)$. Define the fuzzy fractional integral for $q \in (0,1]$*

$$I_q(F)(t) = I_1\left(t^{q-1}F\right)(t) = \int_0^t \frac{F}{x^{1-q}}(x)dx$$

by

$$\begin{aligned}
\left[I_q(F)(t)\right]^\alpha &= \left[I\left(t^{q-1}F\right)(t)\right]^\alpha = \left[\int_0^t \frac{F}{x^{1-q}}(x)dx\right]^\alpha \\
&= \left[\int_0^t \frac{f_1^\alpha}{x^{1-q}}(x)dx, \int_0^t \frac{f_2^\alpha}{x^{1-q}}(x)dx\right]
\end{aligned}$$

where the integral $\int_0^t \frac{f_i^\alpha}{x^{1-q}}(x)dx$, for $i = 1,2$ is the usual Riemann improper integral.

For $q = 1$, we obtain $IF(t) = \int_0^t F(x)dx$, that is the integral operator. Also, the following properties are obvious.

$(i) I_q cF(t) = cI_qF(t)$ for each $c \in \mathbb{R}$
$(ii) I_q(F+G)(t) = I_qF(t) + I_qG(t)$.

Theorem 8.*Let $F : (0,a) \to \mathbb{R}_{\mathscr{F}}$ be continuous fuzzy function. Then $I_q(F)(t)$ is $q_{(1)}$-differentiable and we have $T_{q_{(1)}} I_q(F)(t) = F(t)$*

*Proof.*Since F is continuous, then $I_q(F)(t)$ is clearly $q_{(1)}$-differentiable. Hence,

$$\begin{aligned}
\left[T_{q_{(1)}} I_q(F)(t)\right]^\alpha &= \left[t^{1-q} D_1^1 I_q(F)(t)\right]^\alpha \\
&= \left[t^{1-q}\frac{d}{dt}\int_0^t \frac{f_1^\alpha(x)}{x^{1-q}}dx, t^{1-q}\frac{d}{dt}\int_0^t \frac{f_2^\alpha(x)}{x^{1-q}}dx\right] \\
&= \left[t^{1-q}\frac{f_1^\alpha(t)}{t^{1-q}}, t^{1-q}\frac{f_2^\alpha(t)}{t^{1-q}}\right] \\
&= [F(t)]^\alpha
\end{aligned}$$

4 Solving Fuzzy Fractional Differential Equations

We study the fuzzy initial value problem

$$\begin{aligned}
y^{(q)}(t) &= F(t, y(t)), \quad q \in (0,1] \\
y(0) &= y_0
\end{aligned} \tag{14}$$

where $F : (0,a) \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$ is a continuous fuzzy mapping and y_0 is fuzzy number.

Theorem 9.*Let $F : (0,a) \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$ be continuous and assume that there exists a $k > 0$ such that*

$$d(F(t,x), F(t,y)) \leq kd(x,y)$$

for all $t \in (0,a), x, y \in \mathbb{R}_{\mathscr{F}}$. Then the problem (9) has two unique solution on $(0,a)$

Proof. If in the problem (9) we consider the conformable derivative $y^{(q)}$ for all $q \in (0,1]$ in the first form (i) theorem 5, then from Theorem 6.1 in [10] and using Definition 4 and Lemma 5, there exists an unique solution on $(0,a)$. In the same way, if we consider the conformable derivative $y^{(q)}$ for all $q \in (0,1]$ in the second form (ii) theorem 5 then, analogously to the proof of Theorem 6.1 in [10] and using Definition 4 and Lemma 5, we can prove that there exists an unique solution on $(0,a)$. This proves there exists an unique solution for each lateral direction, and the proof is now complete.

Following [10], we observe that the relations (i) and (ii) in Theorem 5 give us an useful procedure to solve eq (9). Denote $[y(t)]^\alpha = [y_1^\alpha(t), y_2^\alpha(t)]$, $\quad [y_0]^\alpha = \left[y_{01}^\alpha, y_{02}^\alpha\right]$ and

$$[F(t,y(t))]^\alpha = [f_1^\alpha(t, y_1^\alpha(t), y_2^\alpha(t)), f_2^\alpha(t, y_1^\alpha(t), y_2^\alpha(t))]$$

Then, we have the following alternatives for solving eq (9):

CaseI. If we consider $y^{(q)}(t)$ by using coformable derivative in the first form (i), then we have

$$\left[y^{(q_{(1)})}(t)\right]^\alpha = \left[(y_1^\alpha)^{(q)}(t), (y_2^\alpha)^{(q)}(t)\right]$$

Now, we proceed as follows:

1.Solve the differential system

$$\begin{cases} (y_1^\alpha)^{(q)}(t) = f_1^\alpha(t, y_1^\alpha(t), y_2^\alpha(t)), & y(0) = y_{01}^\alpha \\ (y_2^\alpha)^{(q)}(t) = f_2^\alpha(t, y_1^\alpha(t), y_2^\alpha(t)), & y(0) = y_{02}^\alpha \end{cases} \tag{15}$$

for y_1^α and y_2^α.

2.Ensure that $[y_1^\alpha(t), y_2^\alpha(t)]$ and $\left[(y_1^\alpha)^{(q)}(t), (y_2^\alpha)^{(q)}(t)\right]$ are valid α-cuts.

3.By using Theorem 1, we construct a fuzzy solution $y(t)$ such that

$$[y(t)]^\alpha = [y_1^\alpha(t), y_2^\alpha(t)] \text{ for all } \alpha \in [0,1]$$

CaseII. If we consider $y^{(q)}(t)$ by using coformable derivative in the second form (ii), then we have

$$\left[y^{(q_{(2)})}(t)\right]^\alpha = \left[(y_2^\alpha)^{(q)}(t), (y_1^\alpha)^{(q)}(t)\right]$$

Now, we proceed as follows:

1.Solve the differential system

$$\begin{cases} (y_1^\alpha)^{(q)}(t) = f_2^\alpha(t, y_1^\alpha(t), y_2^\alpha(t)), y(0) = y_{01}^\alpha \\ (y_2^\alpha)^{(q)}(t) = f_1^\alpha(t, y_1^\alpha(t), y_2^\alpha(t)), y(0) = y_{02}^\alpha \end{cases} \tag{16}$$

for y_1^α and y_2^α

2.Ensure that $[y_1^\alpha(t), y_2^\alpha(t)]$ and $\left[(y_2^\alpha)^{(q)}(t), (y_1^\alpha)^{(q)}(t)\right]$ are valid α -cuts.

3.By using Theorem (1), we obtain a new fuzzy solution $y(t)$ such that $[y(t)]^\alpha = [y_1^\alpha(t), y_2^\alpha(t)]$ for all $\alpha \in [0,1]$

Example 1. Let us consider the fuzzy problem

$$\begin{aligned} y^{(q)}(t) &= -y(t), \quad \text{for all } q \in (0,1] \\ y(0) &= y_0 \end{aligned} \tag{17}$$

where y_0 is a triangular fuzzy number. If we consider $y^{(q)}(t)$ in the first form (i), we have solve the following differential system:

$$\begin{cases} (y_1^\alpha)^{(q)}(t) = -y_2^\alpha(t), & y(0) = y_{01}^\alpha \\ (y_2^\alpha)^{(q)}(t) = -y_1^\alpha(t), & y(0) = y_{02}^\alpha \end{cases} \tag{18}$$

The solutions of this system are

$$y_1^\alpha(t) = \frac{y_{01}^\alpha - y_{02}^\alpha}{2} e^{\frac{tq}{q}} + \frac{y_{01}^\alpha + y_{02}^\alpha}{2} e^{-\frac{t^q}{q}}$$
$$y_2^\alpha(t) = \frac{y_{02}^\alpha - y_{01}^\alpha}{2} e^{\frac{iq}{q}} + \frac{y_{01}^\alpha + y_{02}^\alpha}{2} e^{-\frac{t^q}{q}}$$

and we see that $y_1^\alpha(t) \le y_2^\alpha(t)$ for all $t \in (0,a)$. Therefore, the fuzzy fuction $y(t)$ solving eq (17) has α-cuts for all $\alpha \in [0,1]$

$$[y(t)]^\alpha = \left[\frac{y_{01}^\alpha - y_{02}^\alpha}{2} e^{\frac{t^q}{q}} + \frac{y_{01}^\alpha + y_{02}^\alpha}{2} e^{-\frac{t^q}{q}}, \frac{y_{02}^\alpha - y_{01}^\alpha}{2} e^{\frac{t^q}{q}} + \frac{y_{01}^\alpha + y_{02}^\alpha}{2} e^{-\frac{t^q}{q}}\right]$$

Now, if we consider $y^{(q)}(t)$ in the second form (ii), we solve the following differential system:

$$\begin{cases} (y_1^\alpha)^{(q)}(t) = -y_1^\alpha(t), & y(0) = y_{01}^\alpha \\ (y_2^\alpha)^{(q)}(t) = -y_2^\alpha(t), & y(0) = y_{02}^\alpha \end{cases} \tag{19}$$

and the solution of this system are

$$y_1^\alpha(t) = y_{01}^\alpha e^{-\frac{t^q}{q}} \quad \text{and} \quad y_2^\alpha(t) = y_{02}^\alpha e^{-\frac{t^q}{q}}$$

and we see that $y_1^\alpha(t) \le y_2^\alpha(t)$ for all $t \ge 0$. Therefore, the fuzzy fuction $y(t)$ solving eq (17) in this case has α-cuts for all $\alpha \in [0,1]$

$$[y(t)]^\alpha = \left[y_{01}^\alpha e^{-\frac{t^q}{q}}, y_{02}^\alpha e^{-\frac{t^q}{q}}\right]$$

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