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Solving the Lane-Emden and Emden-Fowler equations on Cantor Sets by the Local Fractional Homotopy Analysis Method

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Abstract: Within the definition of the local fractional derivative, we present a reliable and effective method for solving the Lane-Emden and Emden-Fowler models in the current article. Numerous fields of research and engineering can benefit from the use of these models. The proposed technique is named the local fractional homotopy analysis method. The method is applied for both models of k order and tested for several examples. This method proves to be effective in handling such models when compared to other similar models. The results indicate the robustness of the technique and the possibility of its application to other models in the future.

Keywords: Non-differentiable solutions on Cantor sets, homotopy analysis method, local fractional calculus, Lane-Emden equation, Emden-Fowler equation.

1 Introduction

The Lane-Emden and Emden-Fowler equations are two nonlinear differential equations of the second order, which are credited to Jonathan Lane who was the first to propose his first equation, in 1870, in a work presented as the first to study the internal structure of a star. Then, after the discovery of these models, researchers over the years were trying to find several solutions to such models to understand their behaviors. In addition, the discovery of fractional calculus along with its several definitions opens the door to better simulate such models and add more to their physical significance. Given the importance of this type of equation in mathematical physics, theoretical physics, and chemical physics, a lot of work has solved these two equations of integer order using several methods [1,2,3,4]. Also, Fazly et. al in [5] discovered some finite Morse index solutions for the fractional-order Lane-Emden equation. Also, the homotopy perturbation method was adapted for solving the fractional lane Emden equations by Wei et al. in [6]. The existence of positive weak solutions to the fractional-order Lane-Emden model was found in [7] by Ao et al. Saadah et al. [8] investigated the solution if the fractional Lane-Emden model through the Laplace transforms technique with the verification through the residual error approximation. The fractional Mayer neuro-swarm heuristic technique has been proposed for solving the fractional singular Lane-Emden model by Sabir et al. [9]. Many other works are introduced for solving both the integer-order and the fractional-order models. We only mention some of them and the rest can be found as [10,11,12,13,14]. As for the Emden-Fowler equation of the integer-order or the fractional-order, there are also many works that have been interested in solving it and discussing its solutions, we mention some of them [15,16,17,20].

The homotopy analysis approach, created in 1992 by Liao Shijun, is one of the best techniques for solving linear or nonlinear issues in engineering, chemistry, medicine, etc [21,22,23,24]. This method was also developed by Shehu Maitama and Weidong Zhao in [25] to the fractal case for solving non-differentiable problems in the sense of the local

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fractional operator. So, in this work, we are interested in applying the local homotopy analysis method (LFHAM) to solve the Lane-Emden and Emden-Fowler models on Cantor sets. The LFHAM method is considered an efficient technique for solving such a model due to its robustness and the ability to provide accurate solutions closed with the exact results.

The paper is organized as follows: Some basic definitions and properties of the local fractional derivative, local fractional integral, and some important results has been presented in section 2. Section 3 provides an analysis of the suggested approach. In section 4, the two proposed equations are solved by using the LFHAM. Finally, section 5 gives the conclusion of the study.

2 Notions of Local Fractional Calculus

The fundamental definitions and theories of local fractional calculus, such as the local fractional derivative and integral, as well as several significant findings, are presented in this section.

Definition 1. *$\varphi(\zeta) \in C_{\xi}[a,b]$ of order ξ at $\zeta = \zeta_0$ has the local fractional derivative as follows ([26]-[28]):*

$$\varphi^{(\xi)}(\zeta) = \left.\frac{d^{\xi}\varphi}{d\zeta^{\xi}}\right|_{\zeta=\zeta_0} = \lim_{\zeta \to \zeta_0} \frac{\Delta^{\xi}(\varphi(\zeta) - \varphi(\zeta_0))}{(\zeta - \zeta_0)^{\xi}}, \tag{1}$$

as well as

$$\Delta^{\xi}(\varphi(\zeta) - \varphi(\zeta_0)) \cong \Gamma(1+\xi)\left[(\varphi(\zeta) - \varphi(\zeta_0))\right]. \tag{2}$$

the class of functions known as local fractional continuous on the interval $[a,b]$ is denoted by $C_{\xi}(a,b)$.

Definition 2. *In the interval $[a,b]$, the local fractional integral of $\varphi(\zeta)$ of order ξ is defined by ([26]-[28])*

$$\begin{aligned} {}_aI_b^{(\xi)}\varphi(\zeta) &= \frac{1}{\Gamma(1+\xi)}\int_a^b \varphi(\zeta)(d\zeta)^{\xi} \\ &= \frac{1}{\Gamma(1+\xi)} \lim_{\Delta\zeta \longrightarrow 0} \sum_{j=0}^{N-1} \varphi(\zeta_j)(\Delta\zeta_j)^{\xi}, \end{aligned} \tag{3}$$

with
$\Delta\zeta_j = \zeta_{j+1} - \zeta_j$, $\Delta\zeta = \max\{\Delta\zeta_0, \Delta\zeta_1, \Delta\zeta_2, \cdots\}$ *and* $\left[\zeta_j, \zeta_{j+1}\right]$, $\zeta_0 = a$, $\zeta_N = b$, *is a partition of* $[a,b]$.

2.1 Several Interesting Local Fractional Calculus Results

Definition 3. *In fractal space, the sine, cosine, and the Mittag-Leffler functions are defined as ([26]-[28])*

$$E_{\xi}(\zeta^{\xi}) = \sum_{\eta=0}^{+\infty} \frac{\zeta^{\eta\xi}}{\Gamma(1+\eta\xi)}, \quad 0 < \xi \leqslant 1, \tag{4}$$

$$\sin_{\xi}(\zeta^{\xi}) = \sum_{\eta=0}^{+\infty} (-1)^{\eta} \frac{\zeta^{(2\eta+1)\xi}}{\Gamma(1+(2\eta+1)\xi)}, \quad 0 < \xi \leqslant 1, \tag{5}$$

$$\cos_{\xi}(\zeta^{\xi}) = \sum_{\eta=0}^{+\infty} (-1)^{\eta} \frac{\zeta^{2\eta\xi}}{\Gamma(1+2\eta\xi)}, \quad 0 < \xi \leqslant 1, \tag{6}$$

The characteristics of certain functions' local fractional derivative and integral are provided by ([26],[27],[28])

$$\frac{d^{\xi}}{d\zeta^{\xi}} \frac{\zeta^{\eta\xi}}{\Gamma(1+\eta\xi)} = \frac{\zeta^{(\eta-1)\xi}}{\Gamma(1+(\eta-1)\xi)}, \tag{7}$$

$$\frac{d^{\xi}}{d\zeta^{\xi}} E_{\xi}(\zeta^{\xi}) = E_{\xi}(\zeta^{\xi}), \tag{8}$$

$$\frac{d^{\xi}}{d\zeta^{\xi}}\sin_{\xi}(\zeta^{\xi})=\cos_{\xi}(\zeta^{\xi}), \tag{9}$$

$$\frac{d^{\xi}}{d\zeta^{\xi}}\cos_{\xi}(\zeta^{\xi})=-\sin_{\xi}(\zeta^{\xi}), \tag{10}$$

$${}_0I_{\zeta}^{(\xi)}\frac{\zeta^{\eta\xi}}{\Gamma(1+\eta\xi)}=\frac{\zeta^{(\eta+1)\xi}}{\Gamma(1+(\eta+1)\xi)}. \tag{11}$$

3 Implementation of Homotopy Analysis Method

The fundamental concept of this approach was presented by Maitama and Zhao [25], wherein they took into consideration the subsequent nonlinear local fractional partial differential equation:

$$\theta\left[\omega(\mu,\zeta)\right]=0, \tag{12}$$

The nonlinear operator is θ; the independent variables are μ and ζ; the local fractional unknown function is $\omega(\mu,\zeta)$. Applying the principles of the usual homotopy analysis technique as suggested by Liao [21], we construct a convex non-differentiable homotopy called the zero order deformation equation:

$$(1-\rho)U[\psi(\mu,\zeta;\rho)-\omega_0(\mu,\zeta)]=\rho\hbar H(\mu,\zeta)\theta[\psi(\mu,\zeta;\rho)], \tag{13}$$

in when $\rho\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is the nonzero convergence-control parameter, $H(\mu,\zeta)\neq 0$ is the local fractional nonzero auxiliary function, $\psi(\mu,\zeta;\rho)$ is the local fractional unknown function, $\omega_0(\omega,\zeta)$ is an initial estimate of $\omega(\mu,\zeta)$, and $U_{\xi}=\frac{\partial^{\xi}}{\partial\zeta^{\xi}}$. The linear local fractional operator has the following property:

$$U_{\xi}\left[\psi(\mu,\zeta)\right]=0,\ \text{when}\ \psi(\mu,\zeta)=0. \tag{14}$$

A lot of choice is available when selecting the auxiliary linear operator and the first guess when using the homotopy analysis method. It follows naturally if $\rho=0$ and $\rho=1$:

$$\psi(\mu,\zeta;0)=\omega_0(\mu,\zeta)\ \text{and}\ \psi(\mu,\zeta;1)=\omega(\mu,\zeta), \tag{15}$$

Consequently, the solution $\psi(\mu,\zeta;\rho)$ differs from the initial guess $w_0(\mu,\zeta)$ to the solution $w(\mu,\zeta)$ as ρ increases from 0 to 1. Applying the local fractional Taylor series to expand $\psi(\mu,\zeta;\rho)$ with respect to ρ, we arrive at the following deduction:

$$\psi(\mu,\zeta;\rho)=\omega_0(\mu,\zeta)+\sum_{\eta=1}^{+\infty}\omega_{\eta}(\mu,\zeta)\rho^{\eta}, \tag{16}$$

as well as

$$\omega_{\eta}(\mu,\zeta)=\left[\frac{1}{\eta!}\frac{\partial^{\eta}\psi(\mu,\zeta;\rho)}{\partial\rho^{\eta}}\right]_{\rho=0}. \tag{17}$$

In the event that the convergence-control parameter, auxiliary function, auxiliary linear operator, and starting guess are all appropriately selected, (16) will converge at $\rho=1$

$$\omega(\mu,\zeta)=\omega_0(\mu,\zeta)+\sum_{\eta=1}^{+\infty}w_{\eta}(\mu,\zeta), \tag{18}$$

is a solution to the initial problem (12). The zero deformation (13) can be used to infer the governing equation, as mentioned in (16).

A local fractional vector is defined as follows:

$$\mathbf{w}_{\eta}=\{\omega_0(\mu,\zeta),\omega_1(\mu,\zeta),\omega_1(\mu,\zeta),\ldots,\omega_{\eta}(\mu,\zeta)\}. \tag{19}$$

By dividing by $\eta!$ and differentiating (13) η-times in relation to the embedding parameter ρ, setting $\rho = 0$, and so on, we derive the ηth-order deformation

$$U_\xi[\omega_\eta(\mu,\zeta) - \chi_\eta \omega_{\eta-1}(\mu,\zeta)] = \hbar H(\mu,\zeta)\Re_\eta(\mathbf{w}_\eta,\mu,\zeta), \tag{20}$$

in which

$$\Re_\eta(\mathbf{w}_{\eta-1},\mu,\zeta) = \left[\frac{1}{(\eta-1)!}\frac{\partial^{\eta-1}\theta[\psi(\mu,\zeta;\rho)]}{\partial\rho^{\eta-1}}\right]_{\rho=0}, \tag{21}$$

and

$$\chi_\eta = \begin{cases} 0, & \eta \leqslant 1, \\ 1, & \eta > 1. \end{cases} \tag{22}$$

We determine the following by using the local fractional integral operator on both sides of (20):

$$\begin{aligned}\omega_\eta(\mu,\zeta) &= \chi_\eta \omega_{\eta-1}(\mu,\zeta) - \chi_\eta \sum_{k=0}^{\eta-1} \omega_{\eta-1}^{(k)}(\mu,0^+)\frac{\zeta^{k\xi}}{\Gamma(k\xi+1)} \\ &+\hbar_0 I_\zeta^{(\xi)}[H(\mu,\zeta)\Re_\eta(\mathbf{w}_{\eta-1},\mu,\zeta)].\end{aligned} \tag{23}$$

The following are the simple series solutions of $\omega_\eta(\mu,\zeta)$ for $\eta \geq 1$ at ηth-order deformation equation that may be obtained by using Mathematica:

$$\omega(\mu,\zeta) = \sum_{\eta=0}^{+\infty} \omega_\eta(\mu,\zeta). \tag{24}$$

You can see the paper [25] for the convergence of the serie (24).

4 Applications to Some Local Fractional Models

In this part, we will solve the Lane-Emden equation of index k with local fractional derivative and the Emden-Fowler equation of index k with local fractional derivative using the approach indicated above [25], which combines the HAM method with the local fractional derivative to find his non-differentiable solutions on Cantor sets.

4.1 The Local Fractional Lane-Emden Equation of Index k

We take into account the following equation:

$$\frac{\partial^{2\xi}T(\zeta)}{\partial\zeta^{2\xi}} + \frac{2\Gamma(1+\xi)}{\zeta^\xi}\frac{\partial^\xi T(\zeta)}{\partial\zeta^\xi} + U^k(\zeta) = 0,\ 0 < \xi \leqslant 1, \tag{25}$$

under the following conditions:

$$T(0) = 1,\quad \frac{\partial^\xi T(0)}{\partial\zeta^\xi} = 0. \tag{26}$$

To facilitate the solution of (27), we use the following transformation:

$$V(\zeta) = \frac{\zeta^\xi}{\Gamma(1+\xi)}T(\zeta), \tag{27}$$

so that

$$\begin{aligned}\frac{\partial^\xi V(\zeta)}{\partial\zeta^\xi} &= \frac{\zeta^\xi}{\Gamma(1+\xi)}\frac{\partial^\xi T(\zeta)}{\partial\zeta^\xi} + T(\zeta), \\ \frac{\partial^{2\xi} V(\zeta)}{\partial\zeta^{2\xi}} &= \frac{\zeta^\xi}{\Gamma(1+\xi)}\frac{\partial^{2\xi} T(\zeta)}{\partial\zeta^{2\xi}} + 2\frac{\partial^\xi T(\zeta)}{\partial\zeta^\xi}.\end{aligned} \tag{28}$$

After replacing (27) and (28) in (25), the new local fractional differential equation that results is as follows:

$$\frac{\partial^{2\xi}V(\zeta)}{\partial\zeta^{2\xi}}+\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}V^k(\zeta)=0,\quad k=0,1,2,..., \tag{29}$$

with the initial conditions

$$T(0)=1,\ \frac{\partial^{\xi}T(0)}{\partial\zeta^{\xi}}=0. \tag{30}$$

It makes sense to use $V_0(\zeta)=\frac{\zeta^{\xi}}{\Gamma(1+\xi)}$ as the first guess in accordance with (29) and the steps of this procedure. The linear operator that we can use is:

$$U_{\xi}[\psi(\zeta;\rho)]=\frac{\partial^{2\xi}}{\partial\zeta^{2\xi}}[\psi(\zeta;\rho)], \tag{31}$$

with $U_{\xi}[C]=0$ as its property, and C being an integral constant:
The nonlinear operator is defined as follows:

$$\theta[\psi(\zeta;\rho)]=\frac{\partial^{2\xi}\psi(\zeta;\rho)}{\partial\zeta^{2\xi}}+\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}\psi^k(\zeta;\rho).$$

The zero-order deformation equation is as follows, based on the preceding LFHAM steps:

$$(1-\rho)U[\psi(\zeta;\rho)-V_0(\zeta)]=\rho\hbar H(\zeta)\theta[\psi(\zeta;\rho)]. \tag{32}$$

This leads to: when $\rho=0$ and $\rho=1$, we get

$$\psi(\zeta;0)=V_0(\zeta),\ \text{and}\ \ \psi(\zeta;1)=V(\zeta). \tag{33}$$

Next, the following is the definition of the ηth-order deformation equation:

$$U_{\xi}[V_{\eta}(\zeta)-\chi_{\eta}V_{\eta-1}(\zeta)]=\hbar H(\zeta)\Re_{\eta}(\mathbf{V}_{\eta-1},\zeta), \tag{34}$$

as well as

$$\Re_{\eta}(\mathbf{V}_{\eta-1},\zeta)=\frac{\partial^{2\xi}V_{\eta-1}(\zeta)}{\partial\zeta^{2\xi}}+\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}V^k_{\eta-1}(\zeta), \tag{35}$$

and

$$\chi_{\eta}=\begin{cases}0, & \eta\leqslant 1,\\ 1, & \eta>1.\end{cases} \tag{36}$$

Applying the local fractional integral on the ηth-order deformation of (34) while setting $H(\zeta)=1$ yields the following result:

$$\begin{aligned}V_{\eta}(\zeta)&=\chi_{\eta}V_{\eta-1}(\zeta)-\chi_{\eta}V(0)\\&\quad+\hbar_0 I_{\zeta}^{(2\xi)}\left[\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}V^k_{\eta-1}(\zeta)\right].\end{aligned} \tag{37}$$

According to (16), (17) and (37), we obtain:

$$\begin{aligned}V_1&=\hbar_0 I_{\zeta}^{(2\xi)}\left[\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}V_0^k\right],\\V_2&=(1+\hbar)V_1+\hbar_0 I_{\zeta}^{(2\xi)}\left[\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}kV_0^{k-1}V_1\right],\\V_3&=(1+\hbar)V_2+\hbar_0 I_{\zeta}^{(2\xi)}\left[\frac{\zeta^{(1-k)\xi}}{(\Gamma(1+\xi))^{1-k}}\left(\frac{k(k-1)}{2}V_0^{k-2}V_1^2+kV_0^{k-1}V_2\right)\right],\\&\ \ \vdots\end{aligned} \tag{38}$$

The first terms of the local fractional homotopy analysis technique of (29) are obtained by the formulas (38) and the first guess $V_0(\zeta)$ are as follows

$$\begin{array}{c} V_0(\zeta)=\frac{\zeta^{\xi}}{\Gamma(1+\xi)}, \\ V_1(\zeta)=\hbar\frac{\zeta^{3\xi}}{\Gamma(1+3\xi)}, \\ V_2(\zeta)=\left(\hbar+\hbar^2\right)\frac{\zeta^{3\xi}}{\Gamma(1+3\xi)}+k\hbar^2\frac{\zeta^{5\xi}}{\Gamma(1+5\xi)}, \\ V_3(\zeta)=\hbar(1+\hbar)^2\frac{\zeta^{3\xi}}{\Gamma(1+3\xi)}+2k\left(\hbar^2+\hbar^3\right)\frac{\zeta^{5\xi}}{\Gamma(1+5\xi)} \\ +\hbar^3\left[k^2+\frac{k(k-1)}{2}\frac{\Gamma(1+\xi)\Gamma(1+5\xi)}{\Gamma(1+3\xi)^2}\right]\frac{\zeta^{7\xi}}{\Gamma(1+7\xi)}, \\ \vdots \end{array} \tag{39}$$

and so forth. The similar method can be used to compute the other LFHAM terms.
The use of formula (27), gives the first terms of the approximate solution of equation (25), as follows:

$$\begin{array}{c} T_0(\zeta)=1, \\ T_1(\zeta)=\hbar\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}, \\ T_2(\zeta)=\left(\hbar+\hbar^2\right)\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}+k\hbar^2\frac{\Gamma(1+\xi)}{\Gamma(1+5\xi)}\zeta^{4\xi}, \\ T_3(\zeta)=\hbar(1+\hbar)^2\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}+2k\left(\hbar^2+\hbar^3\right)\frac{\Gamma(1+\xi)}{\Gamma(1+5\xi)}\zeta^{4\xi} \\ +\hbar^3\left[k^2+\frac{k(k-1)}{2}\frac{\Gamma(1+\xi)\Gamma(1+5\xi)}{\Gamma(1+3\xi)^2}\right]\frac{\Gamma(1+\xi)}{\Gamma(1+7\xi)}\zeta^{6\xi}, \\ \vdots \end{array} \tag{40}$$

Then, the approximate series of the solution function of equation (25), becomes:

$$\begin{array}{c} T(\zeta)=1+\hbar\left(3+3\hbar+\hbar^2\right)\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}+k\hbar^2(3+2\hbar)\frac{\Gamma(1+\xi)}{\Gamma(1+5\xi)}\zeta^{4\xi} \\ +\hbar^3\left[k^2+\frac{k(k-1)}{2}\frac{\Gamma(1+\xi)\Gamma(1+5\xi)}{\Gamma(1+3\xi)^2}\right]\frac{\Gamma(1+\xi)}{\Gamma(1+7\xi)}\zeta^{6\xi}+\cdots \end{array}. \tag{41}$$

Substiting $h=-1$ in (41) and according to the formula (24), we get:

$$\begin{aligned} T(\zeta)=1&-\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}+k\frac{\Gamma(1+\xi)}{\Gamma(1+5\xi)}\zeta^{4\xi} \\ &-\left[k^2+\frac{k(k-1)}{2}\frac{\Gamma(1+\xi)\Gamma(1+5\xi)}{\Gamma(1+3\xi)^2}\right]\frac{\Gamma(1+\xi)}{\Gamma(1+7\xi)}\zeta^{6\xi}+\cdots \end{aligned} \tag{42}$$

We'll go over the solutions in the three previously stated scenarios, even for the equation containing the local fractional derivative.

Case 1 : For $k=0$, the following provides the exact solution to the equation (25):

$$T(\zeta)=1-\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}.$$

Case 2 : For $k=1$, the solution $T(\zeta)$ in the form of a series, is given by:

$$T(\zeta)=\Gamma(1+\xi)\left(1-\frac{\zeta^{2\xi}}{\Gamma(1+3\xi)}+\frac{\zeta^{4\xi}}{\Gamma(1+5\xi)}-\frac{\zeta^{6\xi}}{\Gamma(1+7\xi)}+\cdots\right). \tag{43}$$

As a result, we obtain the exact solution that follows:

$$T(\zeta)=\frac{\sin_{\xi}\left(\zeta^{\xi}\right)}{\frac{\zeta^{\xi}}{\Gamma(1+\xi)}}. \tag{44}$$

Case 3 : The series form solution $T(\zeta)$ for $k=5$ is as follows:

$$T(\zeta)=\Gamma(1+\xi)\left(\begin{array}{c}1-\frac{\zeta^{2\xi}}{\Gamma(1+3\xi)}+\frac{5\zeta^{4\xi}}{\Gamma(1+5\xi)}-\left[25+\frac{10\Gamma(1+5\xi)\Gamma(1+\xi)}{(\Gamma(1+3\xi))^2}\right]\\ \times\frac{\zeta^{6\xi}}{\Gamma(1+7\xi)}+\cdots\end{array}\right). \tag{45}$$

If $\xi=1$, in each of the above three cases, we obtain the same results as those reported in the papers [29] and [30].

4.2 The Local Fractional Emden-Fowler Equation of Index k

Let the equation be

$$\frac{\partial^{2\xi}T(\zeta)}{\partial\zeta^{2\xi}}+\frac{2\Gamma(1+\xi)}{\zeta^{\xi}}\frac{\partial^{\xi}T(\zeta)}{\partial\zeta^{\xi}}+a\zeta^{r\xi}T^{k}(\zeta)=0,\ 0<\xi\leqslant 1, \tag{46}$$

with

$$T(0)=1,\ T_{\zeta}^{(\xi)}(0)=0. \tag{47}$$

Note that the exact solutions for the case $\xi=1$, are only available for $k=0;1$ and 5.

The equation can only be solved if we eliminate the value $\zeta=0$, which is the sole value that gives us trouble.

Using the transformations from (27) and (28) into (46), we are able to solve this problem and obtain the new local fractional differential equation that follows:

$$\frac{\partial^{2\xi}V(\zeta)}{\partial\zeta^{2\xi}}+\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}}V^{k}(\zeta)=0,\quad r=0,1,2,..., \tag{48}$$

considering the initial condition

$$V(0)=0,\ V_{\zeta}^{(\xi)}(0)=1. \tag{49}$$

It makes sense to start with $V_0(\zeta)=\frac{\zeta^{\xi}}{\Gamma(1+\xi)}$, as suggested by (48) and the LFHAM's steps.

The linear operators that we use are:

$$U_{\xi}[\psi(\zeta;\rho)]=\frac{\partial^{\xi}}{\partial\zeta^{\xi}}[\psi(\zeta;\rho)], \tag{50}$$

with the integral constant C having the property $U_{\xi}[C]=0$.

The nonlinear operator has the following definition:

$$\theta[\psi(\zeta;\rho)]=\frac{\partial^{2\xi}\psi(\zeta;\rho)}{\partial\zeta^{2\xi}}+\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}}\psi^{k}(\zeta;\rho). \tag{51}$$

The zero-order deformation equation is as follows, based on the preceding LFHAM steps:

$$(1-\rho)U[\psi(\zeta;\rho)-V_0(\zeta)]=\rho\hbar H(\zeta)\theta[\psi(\zeta;\rho)]. \tag{52}$$

As a result, if $\rho=0$ and $\rho=1$, we obtain:

$$\psi(\zeta;0)=V_0(\zeta),\ \text{and}\ \ \psi(\zeta;1)=V(\zeta). \tag{53}$$

The definition of the ηth-order deformation equation is thus:

$$U_{\xi}[V_{\eta}(\zeta)-\chi_{\eta}V_{\eta-1}(\zeta)]=\hbar H(\zeta)\Re_{\eta}(\mathbf{V}_{\eta-1},\zeta), \tag{54}$$

as well as

$$\Re_{\eta}(\mathbf{V}_{\eta-1},\zeta)=\frac{\partial^{2\xi}V_{\eta-1}(\zeta)}{\partial\zeta^{2\xi}}+\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}}V_{\eta-1}^{k}(\zeta), \tag{55}$$

and

$$\chi_\eta = \begin{cases} 0, & \eta \leqslant 1, \\ 1, & \eta > 1. \end{cases} \tag{56}$$

When we use the local fractional integral on the ηth-order deformation of (54), setting $H(\zeta) = 1$, we obtain:

$$\begin{aligned} V_\eta(\zeta) &= \chi_\eta V_{\eta-1}(\zeta) - \chi_\eta V(0) \\ &+ \hbar_0 I_\zeta^{(2\xi)} \left[\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}} V_{\eta-1}^k(\zeta) \right]. \end{aligned} \tag{57}$$

According to (16), (17) and (57), we obtain:

$$\begin{aligned} V_1 &= \hbar_0 I_\zeta^{(2\xi)} \left[\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}} V_0^k \right], \\ V_2 &= (1+\hbar)V_1 + \hbar_0 I_\zeta^{(2\xi)} \left[\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}} k V_0^{k-1} V_1 \right], \\ V_3 &= (1+\hbar)V_2 + \hbar_0 I_\zeta^{(2\xi)} \left[\frac{a\zeta^{(1+r-k)\xi}}{(\Gamma(1+\xi))^{1-k}} \left(\frac{k(k-1)}{2} V_0^{k-2} V_1^2 + k V_0^{k-1} V_2 \right) \right], \\ &\vdots \end{aligned} \tag{58}$$

The local fractional homotopy analysis approach of (48) yields the following initial terms, which lead to the formulas (58):

$$\begin{aligned} V_0(\zeta) &= \frac{\zeta^\xi}{\Gamma(1+\xi)}, \\ V_1(\zeta) &= a\hbar \frac{\Gamma[1+(r+1)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi]} \zeta^{(r+3)\xi}, \\ V_2(\zeta) &= a\left(\hbar+\hbar^2\right) \frac{\Gamma[1+(r+1)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi]} \zeta^{(r+3)\xi} + a^2 k \hbar^2 \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi] \times \Gamma[1+(2r+5)\xi]} \zeta^{(2r+5)\xi}, \\ V_3(\zeta) &= a\hbar(1+\hbar)^2 \frac{\Gamma[1+(r+1)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi]} \zeta^{(r+3)\xi} + 2a^2 k h^2 (1+\hbar) \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi] \times \Gamma[1+(2r+5)\xi]} \zeta^{(2r+5)\xi} \\ &+ k a^3 h^3 \left[k \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi]} + \frac{k-1}{2} \frac{\Gamma[1+(r+1)\xi]^2 \times \Gamma[1+(3r+5)\xi]}{\Gamma(1+\xi)\Gamma[1+(r+3)\xi]^2} \right] \frac{1}{\Gamma[1+(3r+7)\xi]} \zeta^{3r+7}, \\ &\vdots \end{aligned} \tag{59}$$

and onward. This method can also be used to compute the other terms by the LFHAM.
The preliminary terms of the approximate solution of (25) are given by using the formula (46), as follows:

$$\begin{aligned} T_0(\zeta) &= 1, \\ T_1(\zeta) &= a\hbar \frac{\Gamma[1+(r+1)\xi]}{\Gamma[1+(r+3)\xi]} \zeta^{(r+2)\xi}, \\ T_2(\zeta) &= a\left(\hbar+\hbar^2\right) \frac{\Gamma[1+(r+1)\xi]}{\Gamma[1+(r+3)\xi]} \zeta^{(r+2)\xi} + a^2 k \hbar^2 \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma[1+(r+3)\xi] \times \Gamma[1+(2r+5)\xi]} \zeta^{(2r+4)\xi}, \\ T_3(\zeta) &= a\hbar(1+\hbar)^2 \frac{\Gamma[1+(r+1)\xi]}{\Gamma[1+(r+3)\xi]} \zeta^{(r+2)\xi} + 2a^2 k h^2 (1+\hbar) \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma[1+(r+3)\xi] \times \Gamma[1+(2r+5)\xi]} \zeta^{(2r+4)\xi} \\ &+ k a^3 h^3 \left[k \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma[1+(r+3)\xi] \times \Gamma[1+(3r+5)\xi]} + \frac{k-1}{2} \frac{\Gamma[1+(r+1)\xi]^2}{\Gamma[1+(r+3)\xi]^2} \right] \frac{\Gamma[1+(3r+5)\xi]}{\Gamma[1+(3r+7)\xi]} \zeta^{3r+6}, \\ &\vdots \end{aligned} \tag{60}$$

By replacing $h = -1$ in (60) and using the formula (24), we arrive at the following:

$$T(\zeta) = 1 - A\zeta^{(r+2)\xi} + B\zeta^{(2r+4)\xi} - C\zeta^{3r+6} + \cdots \tag{61}$$

where

$$A = a\frac{\Gamma[1+(r+1)\xi]}{\Gamma[1+(r+3)\xi]} \quad B = a^2 k \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma[1+(r+3)\xi] \times \Gamma[1+(2r+5)\xi]},$$

$$C = a^3 k \left[k \frac{\Gamma[1+(r+1)\xi] \times \Gamma[1+(2r+3)\xi]}{\Gamma[1+(r+3)\xi] \times \Gamma[1+(3r+5)\xi]} + \frac{k-1}{2} \frac{\Gamma[1+(r+1)\xi]^2}{\Gamma[1+(r+3)\xi]^2} \right] \frac{\Gamma[1+(3r+5)\xi]}{\Gamma[1+(3r+7)\xi]}.$$

We'll go over the solutions in the three previously stated cases, even for the equation containing the local fractional derivative.

Case 1 : If $r = 0$ also $k = 0$, the exact solution to (46) can be found as follows:

$$T(\zeta) = 1 - a\frac{\Gamma(1+\xi)}{\Gamma(1+3\xi)}\zeta^{2\xi}. \tag{62}$$

Case 2 : If $r = 0$ also $k = 1$, the following is the exact solution to (46):

$$T(\zeta) = \Gamma(1+\xi)\left(1 - \frac{\left(\sqrt{a}\zeta^{\xi}\right)^2}{\Gamma(1+3\xi)} + \frac{\left(\sqrt{a}\zeta^{\xi}\right)^4}{\Gamma(1+5\xi)} - \frac{\left(\sqrt{a}\zeta^{\xi}\right)^6}{\Gamma(1+7\xi)} + \cdots\right), \tag{63}$$

Consequently, the following is the exact solution in the form of:

$$T(\zeta) = \frac{\sin_{\xi}\left(\sqrt{a}\zeta^{\xi}\right)}{\frac{\sqrt{a}\zeta^{\xi}}{\Gamma(1+\xi)}}. \tag{64}$$

Case 3 : The series form solution $T(\zeta)$ for $r = 0$ and $k = 5$ is as follows:

$$T(\zeta) = \Gamma(1+\xi)\left(\begin{array}{c} 1 - a\frac{\zeta^{2\xi}}{\Gamma(1+3\xi)} + a^2\frac{5\zeta^{4\xi}}{\Gamma(1+5\xi)} - a^3\left[25 + \frac{10\Gamma(1+5\xi)\Gamma(1+\xi)}{(\Gamma(1+3\xi))^2}\right] \\ \times\frac{\zeta^{6\xi}}{\Gamma(1+7\xi)} + \cdots \end{array}\right). \tag{65}$$

The results achieved in the study described in the publications [29] and [30] are identical in all three of the preceding situations if $\xi = 1$,.

5 Conclusion

The local fractional derivative was utilized in this research to solve the local fractional nonlinear Lane-Emden equation of index k and the local fractional Emden-Fowler equation of index k using the local fractional homotopy analysis method (LFHAM). We were able to solve these two nonlinear differential equations with local fractional derivative, which demonstrated the method's strength and efficacy. Additionally, if an exact solution exists, this method yields it in the form of a series that converges quickly to it. Due to the ease of using this method, its power, and its accuracy in solving the examples presented in this study, we can conclude that this method (LFHAM) can solve other nonlinear differential equations with a local fractional derivative and can be applied to other problems involving Cantor Sets.

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