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# An Efficient Model for Solving Fractional Order Partial Differential Equations

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**Abstract:** To solve fractional order differential equations, a hybrid technique named "the natural Adomian decomposition method (NADM) with Caputo fractional operator (CFO)" has been applied. The combination approach incorporates both the natural transform method (NTM) and the Adomian decomposition method (ADM). Two challenges are overcome to validate and demonstrate the efficacy of the current process. It is also shown that the results acquired using the suggested technique are extremely like those obtained using other strategies. For a range of science and engineering difficulties, the proposed solution has been shown to be efficient, dependable, and simple to implement.

**Keywords:** Fractional differential equations; Adomian decomposition method, Natural transform; Caputo fractional operators.

## 1 Introduction

In recent years, engineering and applied sciences have shown a great deal of fractional calculation. The principles of fractional calculus is located in [1,2]. One kind of differential equation are fractional differential equations, or FDEs, which are considered a broad kind of differential equations, involve derivatives of any complex or real order. Fractional partial differential equations can be used to solve a variety of problems in the real world, and they've been discovered to be a tool that's useful for interpreting and modeling all areas of science and mathematical applications concerns [5,6,8].

The precise and estimated results for PDEs with fractions has recently received a lot of attention (PDEs). For the solution of fractional PDEs, numerous motivated strategies have been used in this work such as HAM, expansion methods, HATM, FDM, operational method, VIM, HPM, direct approach, Lie symmetry analysis, DTM, reproducing kernel method, EDTM, mesh less methods, SVIM, SDM, LHPM, and LVIM [3,4,7,9,10,11,12,13].

The goal of this paper is to show how NT and ADM can be combined, the method we use is called the NADM, and use it to resolve the NFPDEs. The rest is separated as, in section 2, some FC definitions are giving. We go over some keys of natural transform definitions and characteristics in section 3. In section 4, the analysis of the NADM with CFO is achieved. Examples of NADM are shown in the fifth section . Chapter 6 is where this paper's conclusion is found.

## 2 Preliminaries

In this section, we'll go over some of the most important fractional calculus definitions and formulas [1,2,7].

**Definition 1.***Consider* $Ł(p), p>0$, *is a real function.* $Ł(p)$ *is* $C_i, i \in R$, *if* $\exists \imath > i, s.t. Ł(p) = p^{\imath} g(p)$ *where* $g(p) \in C[0,\infty)$ *and* $\imath$ *is a real number, and it is* $C_i^m$ *if* $Ł^{(m)} \in C_i, m \in N$.

**Definition 2.***Consider* $Ł \in C_i$. *For any* $\alpha \geq 0$, *the Riemann-Liouville fractional operator of* $\alpha$, *is given as*

$$J^{\alpha}Ł(p) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_0^p (p-q)^{\alpha-1} Ł(q)dq, & \alpha > 0, p > 0 \\ J^{\alpha}Ł(p) = f(p), & \alpha = 0 \end{cases} \tag{1}$$

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*where $\Gamma(\cdot)$ is the well-known Gamma function.*

**Definition 3.** *The fractional derivative of $Ł \in C^m_{-1}$ is defined as*

$$D^\alpha Ł(p) = J^{m-\alpha} D^m Ł(p) = \frac{1}{\Gamma(m-\alpha)} \int_0^p (p-q)^{m-\alpha-1} Ł^{(m)}(q) dq, \tag{2}$$

*for $m-1 < \alpha \leq m, m \in N, p > 0$. Which is in the Caputo concept.*

**Definition 4.** *The series expansion defines a Mittag-Leffler type one-parameter function:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in C \tag{3}$$

**Lemma 1.**

$$D^\alpha J^\alpha Ł(p) = Ł(p) \qquad if \qquad p > 0$$

$$J^\alpha D^\alpha Ł(p) = Ł(p) - \sum_{k=0}^{m-1} Ł^k(0^+) \frac{p^k}{k!} \quad if \quad m-1 < \alpha < m \tag{4}$$

*For $m-1 < \alpha \leq m, m \in N, Ł \in C^m_i, i \geq -1$.*

## 3 Natural Transform definitions and properties

Here, there is some background of the natural transform approach [4,9].

**Definition 5.** *The function $Ł(p)$ for $q \in R$ has a natural transform defined by*

$$\mathbb{N}[Ł(q)] = R(\omega, \mu) = \int_{-\infty}^{\infty} e^{-\omega q} Ł(\mu q) dq, \quad \omega, \mu \in (-\infty, \infty) \tag{5}$$

*The $\mathbb{NT}$ of $Ł(q)$ is $\mathbb{N}[Ł(q)]$, and $\omega$ and $\mu$ are the $\mathbb{NT}$ variables. Define $Ł(q)H(q)$, $H(.)$ is Heaviside function, $q \in (0, \infty)$, and*

$$\mathscr{A} = \{Ł(q) : \exists M, \zeta_1, \zeta_2 > 0, with\ |Ł(q)| \leq M e^{\frac{|q|}{\zeta_j}}, for\ q \in (-1)^j \times [0, \infty), j \in Z^+\}$$

*The natural transform, often known as the $\mathbb{NT}$, is defined as follows:*

$$\mathbb{N}[Ł(q)\mathscr{H}(q)] = \mathbb{N}^+[Ł(q)] = R^+(\omega, \mu) = \int_0^{\infty} e^{-\omega q} Ł(\mu q) dq, \quad \omega, \mu \in (-\infty, \infty) \tag{6}$$

*Note that if $\mu = 1$, (6), it's possible to reduce it to the Laplace transform, and if $\omega = 1$, (6) can be reduced to the Sumuda transform.*

## 4 Natural Adomian Decomposition Method ($\mathbb{N}$ADM) Analysis

Suppose that the general fractional nonlinear PDEs with Caputo fractional operator

$$D_q^\alpha Ł_i(p,q) + \mathscr{R} Ł_i(p,q) + \mathscr{N} Ł_i(p,q) = \Im_i(p,q), \quad 0 < \alpha \leq 1 \tag{7}$$

depending on

$$Ł_i(p,0) = \Im_i(p), \tag{8}$$

where $D_q^{(\alpha)} Ł_i(p,q)$ which is the CFD derivative of $Ł_i(p,q)$.The LDO is represented by $\mathscr{R}$, whereas the NLDO is $\mathscr{N}$, and the source terms are given by $\Im_i(p,q)$.

By taking $\mathbb{NT}$ to (7), we get

$$\mathbb{V}_i(q,\omega,\mu)=\frac{\mu^\alpha}{\omega^\alpha}\sum\frac{\omega^{\alpha-(k+1)}}{\mu^{\alpha-k}}[D^k Ł_i(p,q)]_{q=0}+\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\Im_i(p,q)]-\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}Ł_i(p,q)+\mathscr{N}Ł_i(p,q)] \tag{9}$$

Apply the inverse $\mathbb{NT}$ to (9), we obtain

$$Ł_i=\phi_i(p,q)-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}Ł_i(p,q)+\mathscr{N}Ł_i(p,q)]\right] \tag{10}$$

From the nonhomogeneous term to the essential initial condition, $\phi_i(p,q)$ is an increasing function. Now, consider $Ł_i(p,q)$ is an infinite series solution of the type

$$\mathscr{N}Ł(p,q)=\sum_{n=0}^{\infty}A_n(p,q) \tag{11}$$

Then, by using (11), we may formulate (10) as follows:

$$\sum_{n=0}^{\infty}L_n(p,q)=\phi_i(p,q)-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}\sum_{n=0}^{\infty}Ł_n(p,q)]+\sum_{n=0}^{\infty}A_n\right] \tag{12}$$

$A_n(p,q)$ is an $\mathbb{AD}$ polynomial and is defined as follows:

$$A_n(p,q)=\frac{1}{n!}\frac{d^n}{d\xi^n}\mathscr{N}\left[\sum_{i=0}^{n}\xi^i Ł_i\right]_{\xi=0} \tag{13}$$

We can infer by comparing the two sides of (12)

$$Ł_0=\phi_i(p,q)$$

$$Ł_1=-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}Ł_0(p,q)+A_0]\right]$$

$$\vdots$$

We continue in this direction until we achieve the broad relation given by

$$Ł_{n+1}(p,q)=-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}p_n(p,q)+A_n]\right],\quad n\geq 1$$

Finally, we have an approximate solution

$$Ł(p,q)=\sum_{n=0}^{\infty}Ł_n(p,q)$$

## 5 Applications

### *5.1 Example*

Consider

$$D_q^\alpha Ł(p,q)=Ł_{pp}(p,q)+Ł(p,q)\qquad 0<\alpha\leq 1 \tag{14}$$

subject to initial conditions

$$Ł(p,0)=e^{-p}+p \tag{15}$$

Apply Natural Transform to (14), we get

$$\frac{\omega^\alpha}{\mu^\alpha}\mathbb{N}[Ł(p,q)]-\sum_{k=0}^{n-1}\frac{\omega^{\alpha-(k+1)}}{\mu^{\alpha-k}}[D^kŁ]_{q=0}=\mathbb{N}[Ł_{pp}(p,q)-Ł(p,q)] \tag{16}$$

Thus, from (15) and (16) and by taking the inverse, we have

$$Ł = (e^{-p} + p) + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[Ł_{pp}(p,q) - Ł(p,q)]\right] \tag{17}$$

Consider infinite series solution for the function $Ł(p,q)$ as follows:

$$\sum_{n=0}^{\infty} Ł_n = (e^{-p} + p) + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\left[\sum_{n=0}^{\infty}(Ł_n)_{pp}(p,q) - \sum_{n=0}^{\infty}(Ł_n)(p,q)\right]\right], \quad n \geq 0 \tag{18}$$

Now, comparing both sides of (18)

$$Ł_0 = e^{-p} + p$$

$$Ł_1 = \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\left[(Ł_0)_{pp} - Ł_0\right]\right] = -p\frac{q^\alpha}{\Gamma(\alpha+1)}$$

$$Ł_2(p,q) = p\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\left(\frac{\Gamma(\alpha+1)\mu^\alpha}{\Gamma(\alpha+1)\omega^{\alpha+1}}\right)\right] = p\frac{q^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\vdots$$

Similarly, we continue to arrive at the following approximation

$$Ł(p,q) = \sum_{n=0}^{\infty} Ł_n(p,q) = Ł_0(p,q) + Ł_1(p,q) + Ł_2(p,q) + \ldots$$

$$= e^{-p} + p - p\frac{q^\alpha}{\Gamma(\alpha+1)} + p\frac{q^{2\alpha}}{\Gamma(2\alpha+1)} - \ldots \tag{19}$$

If $\alpha$ approaches 1, the exact solution of (14) is

$$Ł(p,q) = e^{-p} + p + p\left(-q + \frac{q^2}{2!} - \ldots\right)$$

As a result, if $\alpha = 1$, the approximate solution approaches the exact solution rapidly.

### *5.2 Example*

Consider

$$D_q^\alpha Ł(p,q) = Ł_{pp}(p,q) - Ł_p(p,q) + Ł(p,q)Ł_{pp}(p,q) + Ł^2(p,q) + Ł(p,q) \qquad 0 < \alpha \leq 1 \tag{20}$$

subject to initial conditions

$$Ł(p,0) = e^p \tag{21}$$

Apply Natural Transform to (20), we get

$$\mathbb{N}[D_q^\alpha Ł] = \mathbb{N}\left[Ł_{pp} - Ł_p + ŁŁ_{pp} + Ł^2 + Ł\right] \tag{22}$$

Thus, from (21) and (22) and by taking the inverse NT to (22), we get

$$Ł = e^p + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\left[Ł_{pp} - Ł_p + ŁŁ_{p,q} + Ł^2 + Ł\right]\right] \tag{23}$$

Assume infinite series solutions for the unknown function $Ł, Ł_{pp}$, and $Ł^2$ as follows:

$$Ł(p,q)Ł_{pp}(p,q) = \sum_{n=0}^{\infty} A_n(p,q)$$

$$Ł^2(p,q) = \sum_{n=0}^{\infty} B_n(p,q) \tag{24}$$

where the nonlinear terms are represented by the Adomian polynomials $A_n$ and $B_n$. We may rewrite (23) in the form using (24)

$$\sum_{n=0}^{\infty} Ł_n(p,q) = e^p + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\Big[\frac{\partial^2}{\partial p^2}\sum_{n=0}^{\infty} Ł_n - \frac{\partial}{\partial p}\sum_{n=0}^{\infty} Ł_n + \sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} Ł_n\Big]\right],\ n \geq 0 \tag{25}$$

Now, we get the equation through comparison both sides of (25)

$$Ł_0(p,q) = e^p$$

$$Ł_1(p,q) = \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\Big[(Ł_0)_{pp} - (Ł_0)_p + A_0 - B_0 + Ł_0\Big]\right]$$

$$= e^p \frac{q^\alpha}{\Gamma(\alpha+1)}$$

$$Ł_2(p,q) = \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\Big[(Ł_1)_{pp} - (Ł_1)_\ell + A_1 - B_1 + Ł_1\Big]\right]$$

$$= e^p \frac{q^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\vdots$$

Similarly, we continue to arrive at the following approximation

$$Ł(p,q) = \sum_{n=0}^{\infty} Ł_n(p,q) = Ł_0(p,q) + Ł_1(p,q) + Ł_2(p,q) + \ldots$$

$$= e^p\left(1 + \frac{q^\alpha}{\Gamma(\alpha+1)} + \frac{q^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right)$$

If $\alpha$ approaches 1, the exact solution of (20) is

$$= e^{p+q} \tag{26}$$

As a result, if $\alpha = 1$, the approximate solution approaches the exact solution rapidly.

## 6 Conclusions

In the idea of the Caputo fractional operator, the Adomian decomposition technique (ADM) and the natural transform method (NT) were both shown to be extremely successful in solving FPDEs. The solution is provided in a series form by the suggested algorithm, if there is an exact solution, it converges quickly. It is obvious from the findings that the NADM produces solutions that are extremely precise with only a few iterates. Because of the efficacy and versatility shown in the examples given, NADM can be operational to more higher order FPDEs, according to the findings of this study.

## References

[1] I. Podlubny, *FDEs, Mathematics in Science and Engineering*, Academic Press, New York (1999).
[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of FDEs*, Vol. 204, Amsterdam, London, and New York, 2006.
[3] D. Baleanu, H. K. Jassim, *Approximate Analytical Solutions of Goursat Problem within Local Fractional Operators*, Journal of Nonlinear Science and Applications, 9, 4829-4837 (2016).
[4] Loonker, D, Banerji, PK, *Solution of FODES by NT*, Int. J.Math. Eng. Sci., 12(2), 1-7 (2017).

[5] H. Ahmad, T. A. Khan, *VI algorithm-I with an auxiliary parameter for wave-like vibration equations*, Journal of Low Frequency Noise, Vibration and Active Control 38 (3-4) , 1113–1124 (2019).
[6] L. Mahto, S. Abbas, M. Hafayed, H. M. Srivastava,*Approximate controllability of sub DE with impulsive condition*, Mathematics 7 (2), 190 (2019).
[7] D. Baleanu, H. K. Jassim, *Exact Solution of 2-dimensional FPDEs* , Fractal Fractional, 4(21), 1-9 (2020) .
[8] H. M. Srivastava, R. M. Jena, S. Chakraverty, S. K. Jena,*Dynamic response analysis of fractionally-damped generalized Bagley–Torvik equation subject to external loads*, Russian Journal of Mathematical Physics, 27, 254-268 (2020) .
[9] L. K. Alzaki, H. K. Jassim,*The approximate analytical solutions of nonlinear FODEs* , IJNAA, 12(2), 527-535 (2021) .
[10] L. K. Alzaki, H. K. Jassim,*Time-FDEs with an Approximate Solution*, J. Niger. Soc. Phys. Sci., 4, 818 (2022).
[11] A. R. Saeid and L. K. Alzaki, *Analytical solutions for the nonlinear homogeneous FBE using a LFO*, J. College Edu. Pure Sci., 13(3), 1-17 (2023).
[12] A. R. Saeid and L. K. Alzaki, *FDEs with an approximate solution using the NVIM*, RNA 6(3), 107–120 (2023).
[13] L. K. Alzaki, H. K. Jassim,*Analytical Approximations for a System of FPDEs* , Progr. Fract. Differ. Appl., 10(1), 81-89 (2024).