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Fractional Spectral Approaches for Some Fractional Partial Differential Equations

Saad Zagloul Rida[1], *Hussein Shafei Hussein*[1], *Alla Hassan Noureldeen*[2], *and Mona Mahmoud Farag*[2,*]

[1] Mathematics Department, Faculty of Science, South Valley University, Postal code: 83523, Qena, Egypt
[2] Mathematics Department, Faculty of Science, Aswan University, Postal code: 81528, Aswan, Egypt

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Abstract: The multi-order linear and nonlinear partial differential equations of fractional order have been solved numerically. A computational strategy based on fractional spectral operational matrices (**OM**) and generalized fractional Lageurre (**GFL**), generalized fractional shifted Legendre (**GFSL**), generalized fractional modified Bernstein (**GFMB**) are provided. Our fractional spectral methods have been used to solve nonlinear fractional partial differential equations and fractional Korteweg-de Vries-Burgers (**KdVB**) equation. Making use of a variety of test and application examples, the effectiveness of the numerical solution have been satisfied.

Keywords: fractional Caputo, fractional Korteweg- de Vries and Burger equations, operational matrices, special functions.

1 Introduction

Fractional differential equations described a variety of natural phenomena, engineering theories, economic and business models, etc. it have been crucial in helping to solve many physical issues during the past century [1], [2], [3] and [4]. Some of these occurrences are complicated and extremely challenging to comprehend and address. However, if they are modelled by common fractional differential equations, they are simple to analyse and solve. The majority of fractional differential equations have been treated numerically over the past twenty years, as there is no precise solution to such problems.

Fractional order integrals and derivatives operators are involved in the problem, it is difficult or sometimes impossible to identify the analytical solution to the majority of fractional partial differential equations. As a result, there is considerable impetus for the development of reliable and effective numerical approaches to tackle fractional problems that are challenging to answer analytically numerically.

operational matrices technique paired with polynomials is one of the effective and reliable computing approaches utilised to discover the approximative solutions to fractional ordinary and partial differential equations see [5], [6] and [7]. Therefore, we developed a polynomials into fractional functions with operational matrices to solve fractional problems using **MATLAB**.

In this study, we will use techniques GFLOM, GFSLOM and GFMBOM to deal with various linear test examples of fractional partial differential equations. On the other hand, we illustrate the development of Korteweg–de Vries–Burgers equation that occurs in many physical environments.

The breakdown of the paper's structure is as follows: We present the fractional derivative and a few approximation functions based on specific polynomials' fractional versions in section 2. In section 3, we present the problem statment. In section 4, we describe how the operational matrices method based on fractional functions was developed to solve multi-order linear and nonlinear fractional partial differential equations. We present how to state th solution of problem in section 5. We demonstrate numerical findings for a few test issues and applications that claim to be efficient and effective in section 6. In section 7, a conclusion is reached.

* Corresponding author e-mail: monamahmoud.7744@aswu.edu.eg

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2 Basic Concepts

In this section, we present important definitions utilised in this paper to find the solution of fractional partial differential equations.

2.1 Fractional Caputo

Definition 1. *Fractional derivatives Caputo* ${}^{C}D^{\mu}$ *of order* $\mu > 0$*, is obtained by [8]:*

$$ {}^{C}D^{\mu}u(t) = \begin{cases} \frac{1}{\Gamma(\lceil\mu\rceil-\mu)} \int_0^t (t-\tau)^{\lceil\mu\rceil-\mu-1} f^{(\lceil\mu\rceil)}(\tau)d\tau, & \lceil\mu\rceil - 1 < \mu < \lceil\mu\rceil, t \geq 0 \\ \frac{d^{(\lceil\mu\rceil)}u(t)}{dt^{\lceil\mu\rceil}}, & \mu = \lceil\mu\rceil, \end{cases} \tag{1} $$

where $\lceil\mu\rceil$ denotes the lowest integer bigger than or equal to μ. If the function $u(t) = t^n$ then the Caputo fractional derivative is:

$$ {}^{C}D^{\mu}t^n = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\mu)} t^{n-\mu}, & n > \mu, \\ n\, t^{n-1} & \mu = n. \end{cases} \tag{2} $$

For constants $\lambda_k, \;\; k = 1, 2, ..., n$, we have [9]:

$$ {}^{C}D^{\mu} \sum_{k=1}^{n} \lambda_k u_k(t) = \sum_{k=1}^{n} \lambda_k \; {}^{C}D^{\mu} u_k(t), $$

then

$$ {}^{C}D^{\mu}\left[\lambda_1 u_1(t) + \lambda_2 u_2(t) + ... + \lambda_n u_n(t)\right] = \left[\lambda_1 \, {}^{C}D^{\mu}u_1(t) + \lambda_2 \, {}^{C}D^{\mu}u_2(t) + ... + \lambda_n \, {}^{C}D^{\mu}u_n(t)\right]. $$

Differentiation operator of Caputo will be identical of ordinary differential operator when μ is integer number.

2.2 Generalized Fractional Laguerre

Now, we presente fractional Laguerre functions $L_{\alpha}^{\gamma,\beta}(x)$ as [10]:

$$ L_{\alpha}^{\beta,\gamma}(x) = \sum_{k=0}^{\lceil\alpha\rceil+1} (-\gamma)^k \frac{\Gamma(\alpha+\beta+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)\Gamma(\beta+k+1)} x^k. \tag{3} $$

Lemma 1. *If* $L_{\alpha}^{\beta,\gamma}(x)$ *is a generalized fractional Laguerre function, Then fractional-order derivative of it gives as:*

$$ {}^{C}D^{\mu}L_{\alpha}^{\beta,\gamma}(x) = \sum_{k=1}^{\lceil\alpha\rceil+1} (-\gamma)^k \frac{\Gamma(\alpha+\beta+1)}{\Gamma(1-\mu+k)\Gamma(1-k+\alpha)\Gamma(\beta+k+1)} x^{k-\mu}, \quad x \in \mathbb{R}, \;\; \mu > 0, \quad \alpha, \; \beta, \; \gamma > 0. \tag{4} $$

2.3 Generalied Fractional Shifted Legendre

The fractional version of shifted Legendre function $P_{\alpha}(x)$ can be get from [11]:

$$ P_{\alpha}(x) = \sum_{k=0}^{1+\lceil\alpha\rceil} \sum_{m=0}^{k} \frac{(-1)^{m+k} \left(\Gamma(\alpha+1)\right)^2 \; x^{m+\alpha-k}}{\Gamma(m+1)\Gamma(k-m+1)\Gamma(k+1)\left(\Gamma(\alpha-k+1)\right)^2}. \tag{5} $$

Lemma 2. *If* $P_{\alpha}(x)$ *be a generalized fractional shifted Legendre function, consequently the fractional-order derivative is obtained by:*

$$ {}^{C}D^{\mu}P_{\alpha}(x) = \sum_{k=0}^{1+\lceil\alpha\rceil} \sum_{m=0}^{k} \frac{(-1)^{k+m} \left(\Gamma(1+\alpha)\right)^2 \Gamma(m+\alpha-k+1)}{\Gamma(m+1)\Gamma(k-m+1)\Gamma(k+1)\left(\Gamma(1+\alpha-k)\right)^2 \Gamma(m+\alpha-k-\mu+1)} \; x^{m+\alpha-k-\mu}. \tag{6} $$

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2.4 Generalied Fractional Modified Bernstein

Fractional modified Berntein function in order α^{th} on the interval [0,1] can be introduced in [11] by:

$$B_\alpha(x)=\sum_{k=0}^{\lceil\alpha\rceil+1}\sum_{m=0}^{k}\frac{(-1)^{\lceil\alpha\rceil+m+k}\Big(\Gamma(1+\alpha)\Big)^3\Gamma(m-k+\alpha+1)}{\Gamma(m+1)\Gamma(k-m+1)\Gamma(k+1)\Big(\Gamma(\alpha-k+1)\Big)^2\Gamma(m+2\alpha-k+1)}x^{m+\alpha-k}. \tag{7}$$

Lemma 3. *If $B_\alpha(x)$ be a fractional modified Bernstein function, Then the fractional derivative of it is defined by:*

$$^CD^\mu B_\alpha(x)=\sum_{k=0}^{\lceil\alpha\rceil+1}\sum_{m=0}^{k}\frac{(-1)^{m+k+\lceil\alpha\rceil}\Big(\Gamma(\alpha+1)\Big)^3}{\Gamma(m+1)\Gamma(k-m+1)\Gamma(k+1)\Big(\Gamma(\alpha-k+1)\Big)^2}*$$
$$\frac{\Big(\Gamma(m+\alpha-k+1)\Big)^2}{\Gamma(m+2\alpha-k+1)\Gamma(m+\alpha-k-\mu+1)}x^{m+\alpha-k-\mu}. \tag{8}$$

3 Problem Statement

We consider the generic nonlinear fractional-order partial differential equations:

$$H\big(t,x,U(x,t),U^{(\mu_i)}_{x^j}(x,t),U^{(\mu_i)}_{t^k}(x,t),U^{(\mu_i)}_{x^jt^k}(x,t)\big)=0,\quad i=1,2,...,m,\quad j,k=0,1,2,...,m, \tag{9}$$

$$U(x,t)=\Big[u_1(x,t)\ \ u_2(x,t)\ \ ...\ \ u_n(x,t)\Big]^T,\quad \eta_i>0,\ x,t\in[0,w],\ w,\ m\in\mathbb{N},$$

subject to the initial conditions:

$$I^{(l)}\Big(U(x_l,0)\Big)=\sigma_l,\ \ l=0,1,...,\lceil\mu\rceil, \tag{10}$$

and boundary condition

$$B^{(l)}\Big(U(x_l,t_{l+1})\Big)=\rho_{l,l+1},\ \ l=0,1,...,\lceil\mu\rceil, \tag{11}$$

the right hand side function H, is nonlinear in general and the constants $\{\sigma_l,\rho_{l,m-1}\}_{l,m=0}^{\lceil\mu\rceil}$ are given. Operational matrices are fundamental building component in the development of approximation methods. The objective of operational matrices is to substitute the matrix notation of a specified derivative term. We use a different strategy in this study. We apply the GFL,FSL and FMB functions discussed in section 2 to linear and nonlinear partial differential equaions of fractional differential equations.

4 Operational Matrices of Fractional Differential

Let $\varphi(t)\in\Big\{L_\alpha^{\beta,\gamma}(t),\ P_\alpha(t),\ B_\alpha(t)\Big\}$ and the function $u_n(t)$ can be approximated as:

$$u_n(t)\simeq\sum_{l=0}^{n}b_l\varphi_l(t), \tag{12}$$

where b_l denotes the coefficient:

$$b_l=\sum_{l=0}^{n}u_l\theta_{ml} \tag{13}$$

for some numbers $\{\theta_{ml}\}_{m,l=0}^{n}$ depends on any functions that used in this paper. Equations (12),(13) can be written in matrix form as:

$$U=B^T\Phi_n(t), \tag{14}$$

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$$B = U^T \Theta \tag{15}$$

where $\Phi_n(t) \equiv \left[[\varphi_l]_{l=0}^n\right]^T$ and the unknown coefficients $B^T = [b_l]_{l=0}^n$. Combining Eq.(15) and Eq.(14), we conclude that:

$$\begin{aligned} U &= U\Theta^T \Phi_n(t), \\ I &= \Theta^T \Phi_n(t), \\ \Theta^T &= \Phi_n^{-1}(t). \end{aligned} \tag{16}$$

Once Θ is defined by Eq.(16), then the coefficients representation Eq.(15) is also defined. For an approximation $u_n(t)$, the fractional order derivative is defined by [16]:

$${}^C D^\mu u_n(t) = UM^\mu(t), \tag{17}$$

where

$$M^\mu(t) = \Theta^T \left[{}^C D^\mu \Phi_n(t)\right] \tag{18}$$

This approximation of fractional derivative operational matrices of order μ is depended on the error analysis and results of the approximate function.

5 Solution of Problem

Let the solution of problem (9) is:

$$u_n(x,t) = \psi_n(x)\omega_n(t) \Rightarrow U = \Psi\varpi, \tag{19}$$

where $\psi(x)$ and $\varpi(t)$ are defined by:

$$\omega_n(t) \simeq \sum_{l=0}^{n} b_l \varphi_l(t), \quad \psi_n(x) \simeq \sum_{k=0}^{n} a_k \varphi_k(x), \tag{20}$$

and its derivatives are:

$${}^C D^\mu \omega_n(t) = \varpi M^\mu(t), \quad {}^C D^\mu \psi_n(x) = \Psi M^\mu(x) \tag{21}$$

that it is solved by Eq.(16). Substituting in Eq.(9), we get:

$$\begin{aligned} H\big(t,x,\Psi^T \varphi_k(x)\varpi^T \varphi_l(t), \Psi^T M_{x^j}^{(\mu_i)}(x)\varpi^T \varphi_l(t), \Psi^T \varphi_k(x)\varpi^T M_{t^k}^{(\mu_i)}(t), \\ \Psi^T M_{x^j}^{(\mu_i)}(x)\varpi^T M_{t^k}^{(\mu_i)}(t)\big) = 0, \end{aligned} \tag{22}$$

with initial conditions:

$$I^{(k,l)}\left(\Psi^T \varphi_k(x)\varpi^T \varphi_l(0)\right) = \sigma_{k,l}, \quad k,l = 0,1,...,\lceil\mu\rceil, \tag{23}$$

and boundary condition

$$B^{(k,l)}\left(\Psi^T \varphi_k(x)\varpi^T \varphi_{l+1}(t)\right) = \rho_{k,l+1}, \quad k,l = 0,1,...,\lceil\mu\rceil, \tag{24}$$

So, to obtain the unknown vector U, we constract the unconstrained optimization problem with the objective function:

$$\begin{aligned} S(U_k) = \left|\left|\left(H\big(t,x,\Psi^T \varphi_k(x)\varpi^T \varphi_l(t), \Psi^T M_{x^j}^{(\mu_i)}(x)\varpi^T \varphi_l(t), \Psi^T \varphi_k(x)\varpi^T M_{t^k}^{(\mu_i)}(t), \Psi^T M_{x^j}^{(\mu_i)}(x)\varpi^T M_{t^k}^{(\mu_i)}(t)\big)\right)\right|\right|_2^2 + \\ \left|\left|I^{(k,l)}\left(A^T \Phi_l(x) B^T \Phi_l(0)\right) - \sigma_{k,l}\right|\right|_2^2 + \left|\left|B^{(k,l)}\left(A^T \Phi_l(x) B^T \Phi_{l+1}(t)\right) - \rho_{k,l+1}\right|\right|_2^2. \end{aligned} \tag{25}$$

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6 Results and Discussion

6.1 Testing Examples

Example 1. Consider the following linear FPDE [18]:

$$D_t^{\mu}u(x,t) - \frac{1}{2}x^2 D_x^{\eta}u(x,t) = g(x,t), \ \ 0 \le x,t \le 1,$$

where $0 < \mu \le 1 < \eta \le 2$ and the exact solution is $u(x,t) = te^x$. Initial and boundary conditions are $u(x,0) = 0,\ u(0,t) = t,\ u(1,t) = te^1$, and $g(x,t)$ is depended on the exact solution $u(x,t)$.

In figure 1, we comparison between exact and approximate solution by using GFMBOM on $D_t^{\mu}u(x,t)$, GFLOM on $D_x^{\eta}u(x,t)$ at $\mu = 0.5, \eta = 2.0$. Through table 1, we present maximum absolute error at $t = 1, \mu = 0.5, \eta = 2.0$ and comparison it by results in [18] and also show the norm of each case of matrix used. Comparison of absolute errors for our method at different value of $\mu, \alpha = 2.5$ in figure 2.

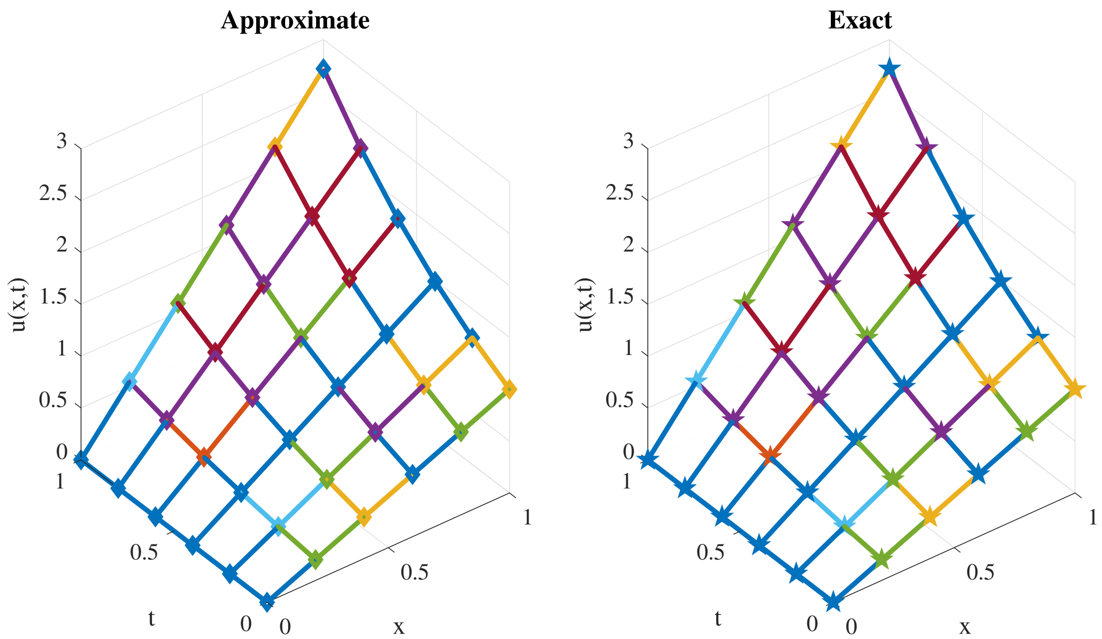

Fig. 1: Approximate and exact solution by GFLOM and GFMBOM of example (1).

Table 1: Maximum absolute error and norm error by GFLOM and GFMBOM of example 1.

$n \times n$	[18]	$Error$	L_2
3×3	–	$5.36E-06$	$2.87E-06$
4×4	$1.02E-02$	$2.06E-04$	$1.62E-05$
5×5	–	$6.63E-07$	$2.68E-08$

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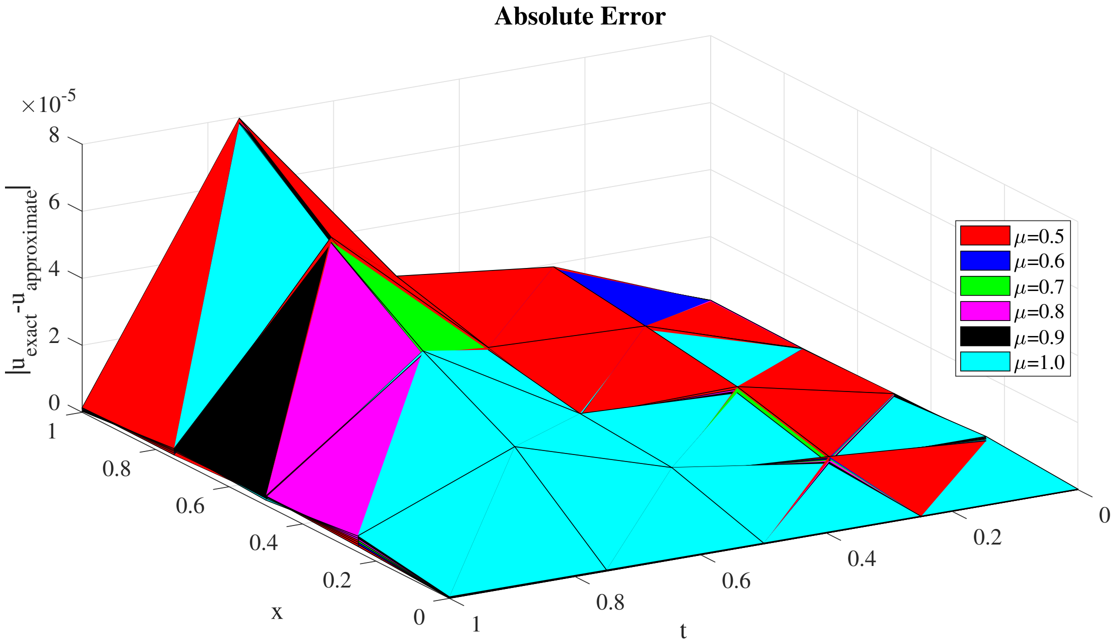

Fig. 2: Absolute error of example 1 at $\eta = 2.0$ by GFLOM and GFMBOM.

Example 2. Consider the following nonhomogeneous FPDE [18]:

$$D_t^{\mu} u(x,t) + D_t^{\eta} u(x,t) - D_x^{\xi} u(x,t) = g(x,t); \quad 0 \leq x,t \leq 1,$$

where $\mu, \eta \in [0,1], \xi \in [0,2]$ initial condition $u(x,0) = x - x^2$, boundary conditions $u(0,t) = u(1,t) = 0$, exact solution $u(x,t) = (x - x^2)(1 + t^2)$ and $g(x,t)$ is depended on the exact solution.

In table 2, we illustrate the numerical results at fixed values of the variable $x = 1$, maximum absolute errors at $t = 1$ and matrix 4×4 of [18] equal $3.0126e - 3$ while using our method equal $2.12e - 4$. Comparison between exact and approximate solution by using GFMBOM on $D_x^{\xi} u(x,t)$, GFSLOM on $D_t^{\mu} u(x,t), D_t^{\eta} u(x,t)$ at $\mu = 0.5, \eta = 2.0$ in figure 3. Figure 4, contaite absolute error at $\mu = 0.9, \eta = 0.6, \xi = 2.0$ by our methods.

Table 2: Absolute error of example 2 with $\xi = 2.0, x = 1$ by using GFSLOM and GFMBOM.

t	$\mu = 0.6, \eta = 0.3$	$\mu = 0.8, \eta = 0.5$	$\mu = 1.0, \eta = 0.7$
0	$1.97E-04$	$1.40E-04$	$6.32E-05$
0.2	$2.15E-04$	$1.44E-04$	$6.06E-05$
0.4	$2.33E-04$	$1.59E-04$	$6.65E-05$
0.6	$2.66E-04$	$1.82E-04$	$7.75E-05$
0.8	$3.20E-04$	$2.20E-04$	$9.42E-05$
1.0	$3.90E-04$	$2.70E-04$	$1.16E-04$

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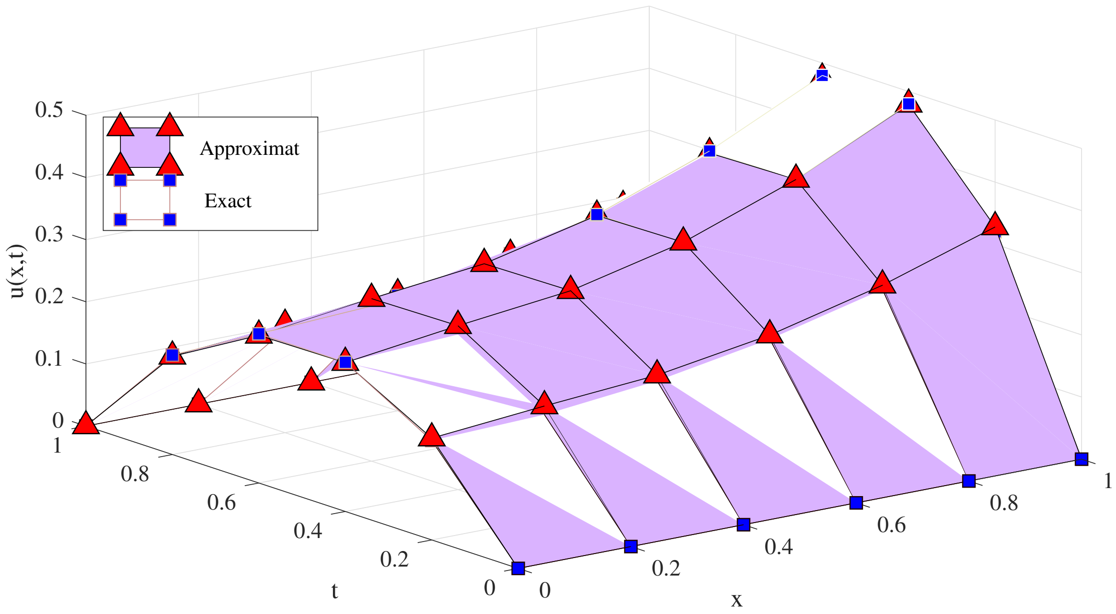

Fig. 3: Approximate and exact solution by GFSLOM and GFMBOM $\mu = 0.5, \eta = 0.2, \xi = 2.0$ of example 2.

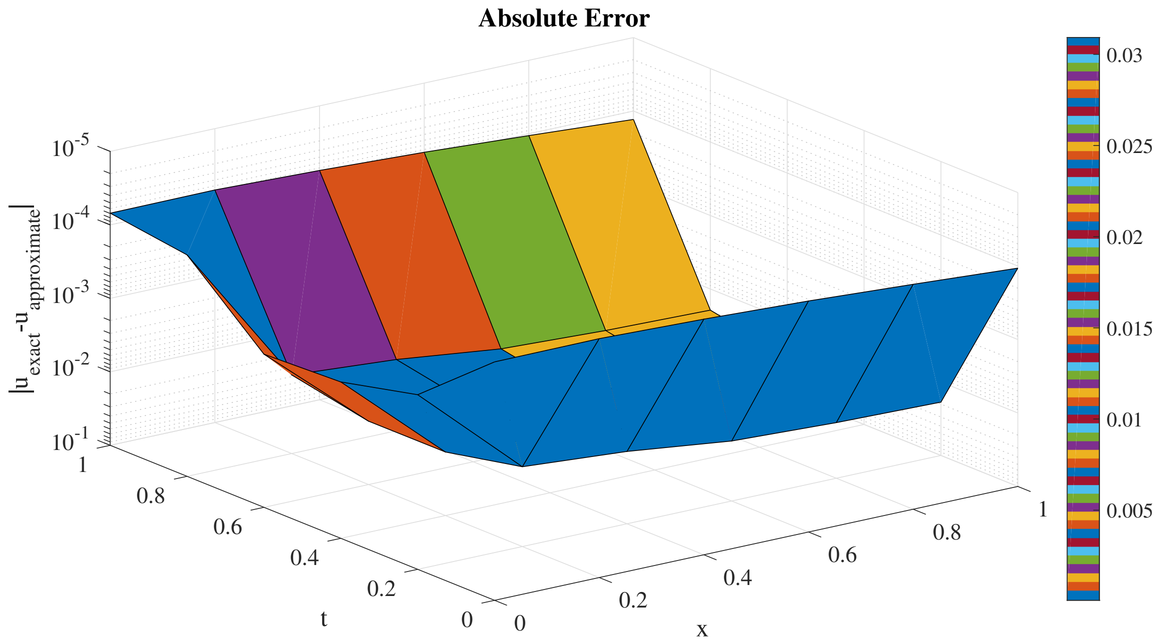

Fig. 4: Error by GFSLOM and GFMBOM of example 2.

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6.2 Application Example

Example 3. Consider the general nonlinear time fractional partial differential equation [17], [18], [19]:

$$D_t^{\mu}u(x,t)+\lambda_1 uD_x^{\eta}u(x,t)-\lambda_2 D_x^{\delta}u(x,t)+\lambda_3 D_x^{\xi}u(x,t)=g(x,t),$$

where $0\leq x,t\leq 1,\mu,\eta\in(0,1],\ \delta\in(1,2],\ \xi\in(2,3]$,conditions $u(0,t)=f_1,u(x,0)=f_2,u(1,t)=f_3,u(x,1)=f_4$, and $g(x,t),f_i,i=1,2,3,4$ are depended on the exact solution.

Case 1: If we take $\lambda_1=1,\eta=1,\lambda_2=0,\lambda_3=1$ and $\xi=3$, we get KdV equation which exact solution is $u(x,t)=t^2 sin(x)$.
Case 2: When we put $\lambda_1=1,\eta=1,\lambda_2=1,\lambda_3=0$ and $\delta=2$, we obtained Burgers equation where $u(x,t)=te^x$.
Case 3: To compensate $\lambda_1=1,\lambda_2=0.125,\lambda_3=\frac{1}{2},\xi=3$, and $\delta=2$ for obtaining KdV-Burgers equation at exact solution is $u(x,t)=\frac{12}{25}\lambda_2^2\left(1-tanh(\theta)-\frac{1}{2}sech^2(\theta)\right),\ \ \theta=\frac{1}{5}\lambda_2 x-\frac{12}{125}\lambda_2^3 t$.

Results:

Case 1: table 3 present numerical results via cpu-time by GFSLOM and GFLOM where the properties of laptob that we use is (processing: intel core i5, RAM: 6 GB). Figure 5 illustrate a comparison between numerical solutions at different μ with exact solution by GFLOM and GFMBOM. We state the absolute error in figure 6 by GFLOM and GFMBOM.
Case 2: Comparison between exact solution and approximate solutions in figure 7. Numerical results at different μ by GFSLOM in figure 8.
Case 3: figure 9 show the equivalence of the approximate solution via the exact solution at $\mu=1,\eta=1$. A numerical results at $\alpha=2.5$ by GFLOM in table 4.

Table 3: Norm error by GFSLOM and GFLOM at $\mu=.05,\eta=1.0$ and $\xi=3.0$ of example 3 **case 1**.

$n\times n$	$CPU-Time$	L_2
3×3	5.92 s	$2.69E-05$
4×4	14.09 s	$1.77E-05$
5×5	72.81 s	$3.80E-06$

Table 4: Norm error via GFLOM at $\mu=1.0$ of example 3 **case 3**.

η	$CPU-Time$	L_2
1.0	7.73 m	$4.29E-09$
0.9	7.69 m	$1.28E-07$
0.8	7.74 m	$2.39E-07$
0.7	7.31 m	$3.33E-07$

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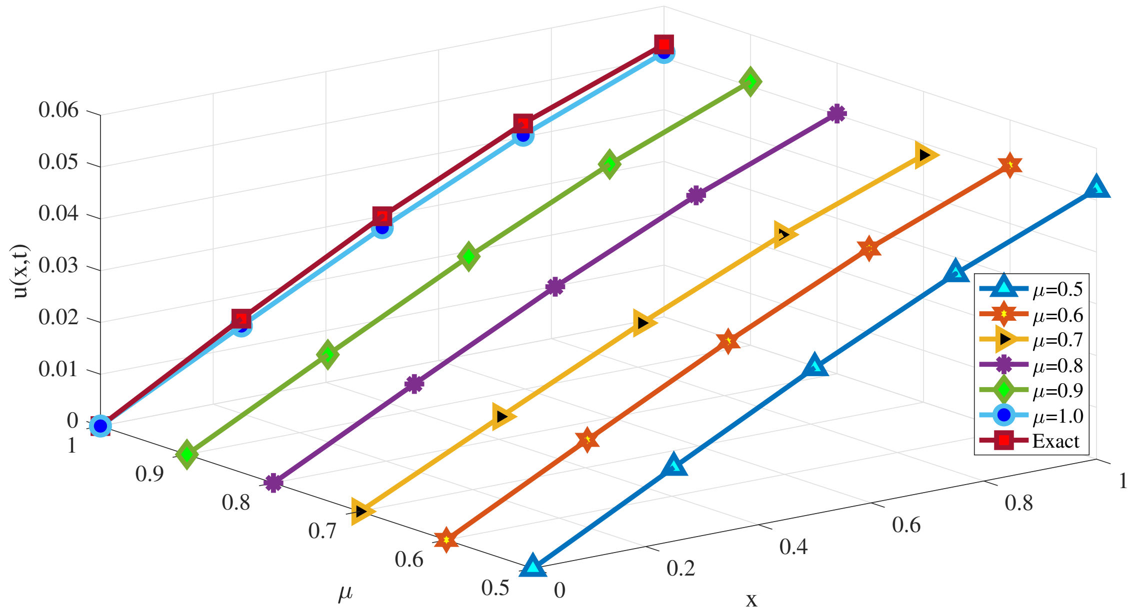

Fig. 5: Approximate and exact solution by GFLOM and GFMBOM of example (3) at $t = 0.25$ of **case 1**.

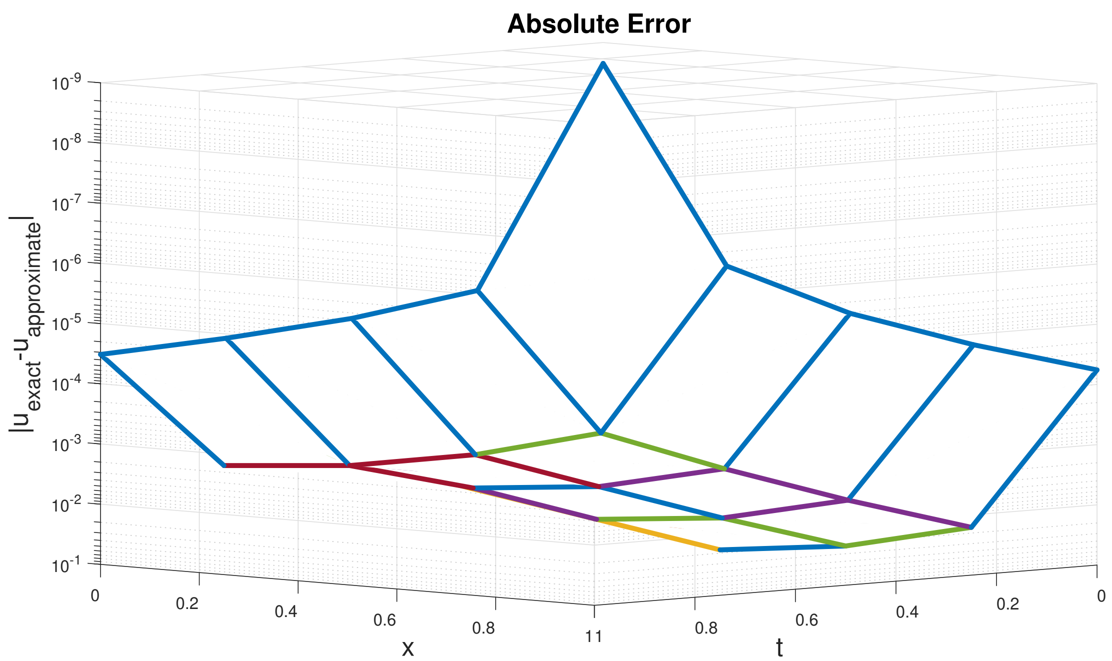

Fig. 6: Absolute error by GFLOM and GFMBOM of example (3) at $\mu = 1.0$ of **case 1**.

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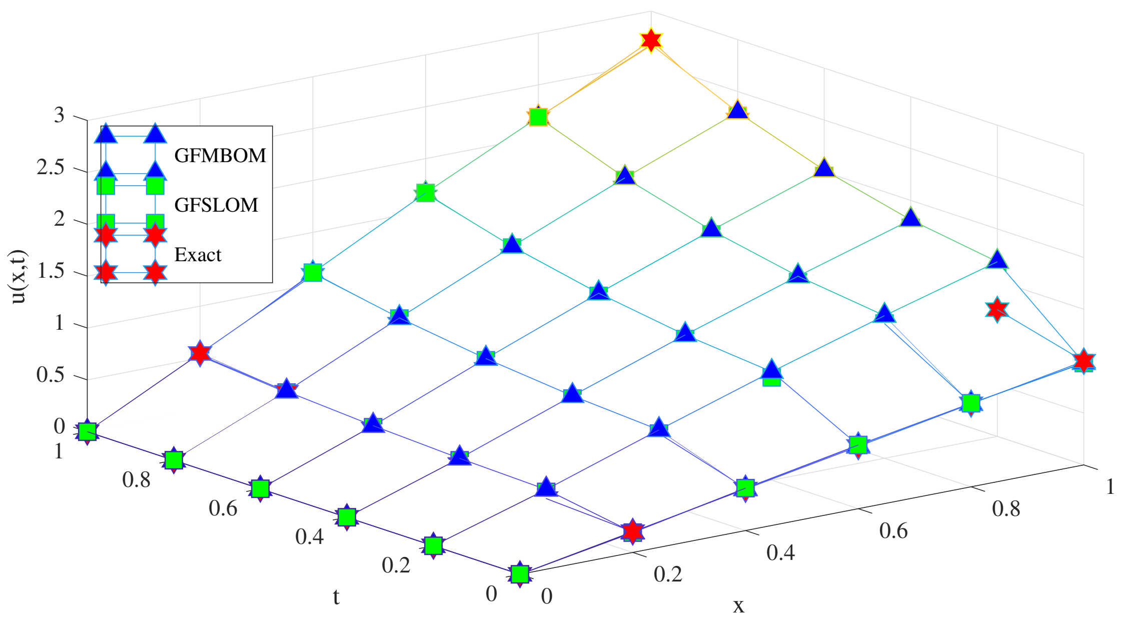

Fig. 7: Approximate and exact solution of example 3 at $\mu = 0.6$ of **case 2**.

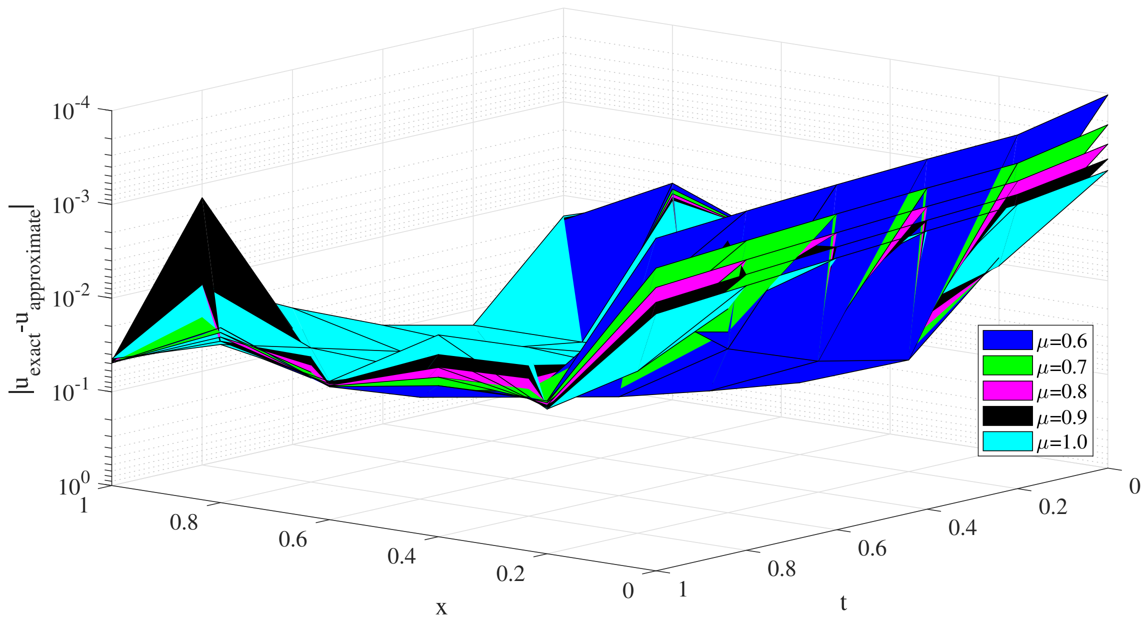

Fig. 8: Absolute error of example 3 at different μ by GFSLOM of **case 2**.

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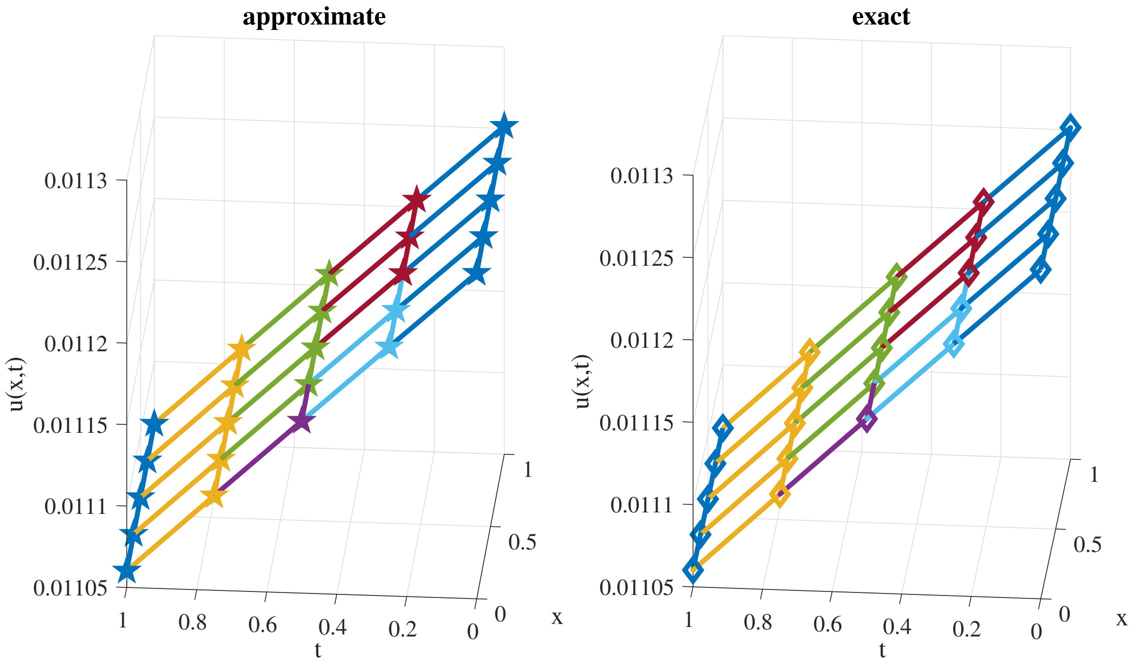

Fig. 9: Approximate and exact solution by GFLOM of example (3) **case 3**.

7 Conclusion

We test the accuracy and stability of the suggested numerical technique via different tested examples of partial differential equation and applied (KdV equation, Burgers equation, KdV-Burgers equation). We also study the applicability of operational matrices (OM) of fractional derivatives operators based on generalized fractional Laguerre(GFL), generalized fractional shifted Legendre(GFSL) and generalized fractional modified Bernstein(GFMB) approximations in the Caputo senses. The method is applied to a wide range of situations, and the results are achieved by simple manipulation in the MATLAB software. The numerical examples used to illustrate this paper's numerical findings demonstrate that the current methods produces accurate results and offers the following benefits:

– Both linear and nonlinear fractional partial differential equation can be solved using the GFLOM, GFSLOM and GFMBOM.
– We find a large equivalence between the approximate solutions and the exact solutions, even with the different value of the fractional order, and we can see this in figures (1, 3, 5, 7, 9).
– We also showed the method's effectiveness of different fractional order by calculating the amount of absolute error see figures (2, 4, 6, 8), table (2) and observing that as the scale level was increased, the amount of absolute error remained low table (3).
– The CPU time of the present approach is low (as shown in table 3, 4), and it is beneficial in obtaining results fast, saving us effort and time.

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