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Fitting various medical and engineering applications using the new lifetime distribution with a mathematical framework

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Abstract: Sustainability considerations play a crucial role in informing the modeling and fitting of medical and engineering data, ensuring the development of robust and environmentally conscious solutions. This paper delves into the investigation of a novel continuous distribution, aiming to provide a thorough understanding of its various fundamental mathematical and statistical properties. The analysis encompasses an exploration of survival functions, hazard rate functions, quantile, skewness, kurtosis, moments, mean time to failure, mean time to repair, insurance pricing principles, availability, and mean residual (past) lifetime functions. The proposed model demonstrates versatility in modeling both asymmetric and symmetric data across various kurtosis shapes. It can effectively handle outlier observations and accommodate different shapes of failure rates, including unimodal, bathtub, increasing, or decreasing patterns. This makes the proposed model suitable for modeling data in diverse fields. The maximum likelihood approach is employed to estimate model parameters using complete and upper recorded values. A simulation study is conducted to evaluate the performance of the estimators under different sample sizes for both complete and upper recorded values. To further demonstrate the flexibility and effectiveness of the new model, two datasets from medical and engineering domains are utilized for validation and testing purposes.

Keywords: Statistical model, Failure analysis, Accessibility, Maximum likelihood method, Upper record values, Sustainability, Simulation, Statistics and numerical data.

1 Introduction

In the realm of lifetime data analysis, researchers frequently rely on a range of common distributions, such as the Weibull (W), exponential Weibull (EW), generalized exponential (GE), Rayleigh, Gompertz, Gumbel, Lomax and their extensions. These distributions are pivotal due to their varied hazard function characteristics. Specifically, the exponential distribution is known for its constant hazard function, contrasting sharply with the Rayleigh and GE distributions that display monotonic hazard functions, where the Rayleigh's hazard function increases and the GE's decreases over time. The pursuit of refining and expanding our toolkit for analyzing lifetime data has led to innovative approaches in recent years. The introduction of the EW family by [1] stands out as a significant milestone in this journey. They extended the Weibull family to include distributions that can model a wider array of failure rates, including the bathtub-shaped and unimodal, alongside expanding the scope of monotonic failure rates. This development marked a leap forward in the flexibility and applicability of lifetime distributions. Furthering this innovation, [2] introduced the exponentiated Weibull model. The exponentiated Weibull model exemplifies the ongoing effort to provide more comprehensive models that can capture the complexity of real-world data more accurately. These advancements highlight the dynamic nature of statistical modeling in lifetime data analysis. By building upon the foundational distributions with new formulations and combinations, researchers continue to

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enhance the precision and applicability of statistical methods to a wide range of disciplines and industries. [3] introduced the generalized exponential (GE) distributions characterized by three parameters. The applications of the EW distribution can be found in the work by [4] and [5]. These studies explore the utilization of this distribution within the context of reliability, where the CDF is a critical component. [6] introduced a generalized modified Weibull distribution that incorporates four parameters, enhancing its flexibility and applicability in various statistical analyses. This advanced version of the Weibull distribution, due to its four parameters, allows for a more precise modeling of data across a wide range of disciplines, including reliability engineering and life data analysis. Integrating sustainability principles into the analysis and modeling of lifetime data, particularly through innovative approaches like the integration of traditional and discrete distributions, holds promise for advancing both environmental consciousness and analytical capabilities in medical and engineering fields. Thus, in recent years, several probability models are discussed in the leteratre for this purpose, for more information, see [7-21], among others. This paper introduces and discusses a novel generalization of the GE model by incorporating an additional parameter, resulting in a probability distribution characterized by four parameters, including shape, location, and scale. Such a distribution significantly enhances modeling flexibility and accuracy for complex real-world data, allowing for a closer alignment with observed behaviors, such as unimodality, heavy tails, and skewness. This advancement is particularly beneficial across various applications in finance, engineering, and environmental science, where standard distributions fall short. It improves the precision in reliability and survival analysis by more accurately representing life durations and failure rates. Additionally, the four-parameter framework increases statistical power and efficiency, potentially reducing the required sample sizes for detecting significant effects, thereby offering a more cost-effective research approach. It also serves as a general model that includes simpler distributions as special cases, facilitating the theoretical study of their properties and interrelations. Importantly, this complex distribution is robust against imperfections in real-world data, such as outliers and missing values, leading to more reliable analyses. Despite the challenges associated with its complexity and computational demands, the benefits of applying a four-parameter distribution in terms of model fit, flexibility, and broad applicability significantly outweigh these challenges, making it a valuable tool in statistical analysis.

The subsequent sections of this paper are structured as follows: Section 2 introduces the new model. Section 3 focuses on deriving various statistical properties of this model. In Section 4, an in-depth discussion on two sub-models within the proposed model is presented. The estimation of the parameters of the proposed model using the maximum likelihood method is delineated in Section 5. Section 6 includes a simulation study demonstrating the application of these models. The effectiveness and significance of the proposed model are further demonstrated through the analysis of three real data sets in Section 7. Finally, Section 8 offers concluding remarks and reflections.

2 New Generalized Exponential Weibull Distribution

The random variable X is defined to have a new generalized exponential Weibull distribution (NGEWD) if its CDF is characterized by the following parameters: α, β, and θ are greater than 0, $0 < \gamma < x$, and $x > 0$ expressed as follows:

$$F_X(x) = \left[1 - exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\right]^{\alpha}, \quad (1)$$

where α and β represent shape parameters, γ denotes a location parameter, and θ signifies a scale parameter, this distribution will be denoted by NGEWD$(\alpha,\beta,\gamma,\theta)$. The hazard rate function (HRF) of the NGEWD$(\alpha,\beta,\gamma,\theta)$ exhibits increasing behavior for $\beta > 1$ and $\alpha\beta > 1$, decreasing behavior for $\beta < 1$ and $\alpha\beta < 1$, bathtub-shaped behavior for $\beta > 1$ and $\alpha\beta < 1$, and constant behavior for $\beta = 1$ and $\alpha = 1$. The probability density function, say $f_X(x)$, can be formulated as

$$f_X(x) = \frac{\alpha\beta(x-\gamma)^{\beta-1}}{\theta^{\beta}} exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\} \times \left[1 - exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\right]^{\alpha-1}. \quad (2)$$

The HRF can be reported as $h(x) = \frac{f_X(x)}{S_X(x)}$ where

$$S_X(x) = 1 - \left[1 - exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\right]^{\alpha}, \quad (3)$$

is the survival function of NGEWD$(\alpha,\beta,\gamma,\theta)$. In recent observations, the reversed hazard function has emerged as a significant factor in reliability analysis, as highlighted by [22]. The reversed hazard function, denoted as $r(x)$, for the NGEWD$(\alpha,\beta,\gamma,\theta)$, can be expressed as follows:

$$r(x) = \frac{\alpha\beta\theta^{-\beta}(x-\gamma)^{\beta-1} exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}}{\left[1 - exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\right]}. \quad (4)$$

The inverse (hazard) function is widely acknowledged as the primary factor influencing its associated probability density function. Figure 1 depicts the probability density functions (PDFs) of NGEWD$(\alpha,\beta,\gamma,\theta)$ for different parameter settings. Figure 1 also delves into the failure rate function of NGEWD$(\alpha,\beta,\gamma,\theta)$ across varying parameter values. These graphical representations demonstrate that the PDFs can display either decreasing or unimodal patterns, while the hazard function can take on unimodal, increasing, decreasing, or bathtub-shaped forms.

The NGEWD$(\alpha,\beta,\gamma,\theta)$ distribution yields several special cases, including: the Weibull distribution W(β,γ,θ) when α equals 1; the exponential E(θ) distribution when α and β all equal 1, with γ equaling 0; the exponential Weibull EW(α,β,θ) distribution when γ equals 0; the generalized exponential GE(α,θ) distribution when β equals 1 and γ equals 0; the Rayleigh R(b) distribution when α equals 1, β equals 2, γ equals 0, and θ equals the square root of $(2/b)$, where b is greater than 0; the generalized Rayleigh GR(α,b) distribution when β equals 2, γ equals 0, and θ equals the square root of $(2/b)$, where b is greater than 0; the generalized exponential GE(α,γ,θ) distribution when β equals 1; the Burr-type X BT(α,b) distribution when β equals 2, γ equals 0, and θ equals $(1/b)$, where b is greater than 0; and the generalized Burr-type X GBT(α,b,β) distribution when γ equals 0, and θ equals $(1/b)$, where b is greater than 0.

3 Certain Statistical Properties

3.1 Quartile, mode, and associated concepts

As anticipated, it is noted that the mean of NGEWD$(\alpha,\beta,\gamma,\theta)$ cannot be expressed explicitly. Instead, it can be derived through an infinite series expansion, allowing for the determination of various moments of NGEWD$(\alpha,\beta,\gamma,\theta)$ in general. However, the quantile x_q of NGEWD$(\alpha,\beta,\gamma,\theta)$ can be obtained in a closed form using the following equation

$$F_X\left(x_q;\alpha,\beta,\gamma,\theta\right)-q=0,$$

thus

$$x_q=\gamma+\theta\left(-Ln\left[1-(q)^{\frac{1}{\alpha}}\right]\right)^{\frac{1}{\beta}}. \tag{5}$$

So, the median $m(X)$ of NGEWD$(\alpha,\beta,\gamma,\theta)$ can be obtained at $q=\frac{1}{2}$

$$x_{0.5}=\gamma+\theta\left(-Ln\left[1-(0.5)^{\frac{1}{\alpha}}\right]\right)^{\frac{1}{\beta}}.$$

The effects of the shape parameters on the skewness and kurtosis can be studied by using quantile function. The Bowley skewness is one of the earliest skewness measures defined by $S=\frac{x_{3/4}+x_{1/4}-2x_{1/2}}{x_{3/4}-x_{1/4}}$. The Moors kurtosis is based on octiles, namely $K=\frac{x_{3/8}-x_{1/8}+x_{7/8}-x_{5/8}}{x_{6/8}-x_{2/8}}$. Furthermore, the mode of NGEWD$(\alpha,\beta,\gamma,\theta)$ can be derived by solving the subsequent non-linear equation. Consequently,

$$\frac{d}{dx}\left[\begin{array}{c}(x-\gamma)^{\beta-1}\left(1-exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\right)^{\alpha-1}\\ \times exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\end{array}\right]=0.$$

In the general case, obtaining an explicit solution is not feasible; thus, numerical methods must be employed. Explicit forms may be derived for various special cases.

3.2 Moments

Moments, mathematical descriptors of a probability distribution, find applications in fields such as statistics, physics, and engineering for summarizing and analyzing data distributions. The following lemma gives the r^{th} moment of NGEWD$(\alpha,\beta,\gamma,\theta)$, when $\alpha\geq 1$.

Lemma 1.

If X has NGEWD$(\alpha,\beta,\gamma,\theta)$, then the r^{th} moment of X, say $\Phi^{(r)}$, is given as follows for $\beta,\theta>0$, $\alpha\geq 1$ and $0<\gamma<x$

$$\Phi^{(r)}=\alpha\theta^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{r}(-1)^{i}\binom{\alpha-1}{i}\binom{r}{j}\left(\frac{\gamma}{\theta}\right)^{j}\times(1+i)^{\frac{j-r-\beta}{\beta}}\,\Gamma\left(\frac{r-j+\beta}{\beta},(1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}\right). \tag{6}$$

Proof.

$$\Phi^{(r)}=\int_{0}^{\infty}x^{r}f(x;\alpha,\beta,\gamma,\theta)dx,$$

thus,

$$\Phi^{(r)}=\frac{\alpha\beta}{\theta^{\beta}}\int_{o}^{\infty}x^{r}(x-\gamma)^{\beta-1}exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\times\left(1-exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}\right)^{\alpha-1}dx,$$

since

$$0<exp\left\{-\left(\frac{1}{\theta}(x-\gamma)\right)^{\beta}\right\}<1\ \text{ for } 0<\gamma<x\,,$$

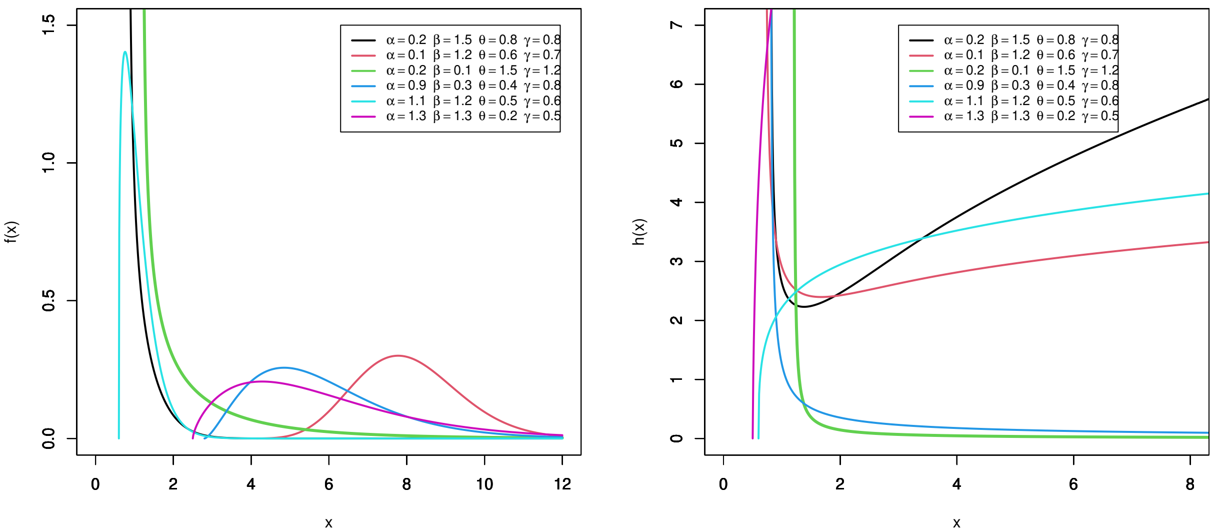

Fig 1. The PDF (left panel) and HRF (right panel) plots of the NGEWD$(\alpha,\beta,\gamma,\theta)$

and using the binomial series expansion, we have

$$
\begin{aligned}
\Phi^{(r)} &= \frac{\alpha\beta}{\theta^{\beta}}\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha-1}{i}\int_{o}^{\infty}x^{r}(x-\gamma)^{\beta-1} \\
&\times e^{-(1+i)\left(\frac{x-\gamma}{\theta}\right)^{\beta}}dx \\
&= \frac{\alpha\beta}{\theta^{\beta}}\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha-1}{i}\frac{\theta^{\beta}\gamma^{r}}{\beta(1+i)} \\
&\times \int_{(1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}}^{\infty}\left[1+\frac{\theta}{\gamma}\left(\frac{y}{1+i}\right)^{\frac{1}{\beta}}\right]^{r}e^{-y}dy \\
&= \alpha\theta^{r}\sum_{i=0}^{\infty}\sum_{j=0}^{r}(-1)^{i}\binom{\alpha-1}{i}\binom{r}{j}\left(\frac{\gamma}{\theta}\right)^{j} \\
&\times (1+i)^{\frac{j-r-\beta}{\beta}}\Gamma\left(\frac{r-j+\beta}{\beta},(1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}\right).
\end{aligned}
$$

3.3 Mean time to failure (MTTF)

If the reliability function of a component is denoted by $R(t;.)$ and its probability density function by $f(t;.)$, then the expected failure time, representing the average duration for which the component is anticipated to function successfully, is expressed as:

$$
MTTF=\int_{0}^{\infty}t\,f(t;.)\,dt=\int_{0}^{\infty}R(t;.)dt. \qquad (7)
$$

The subsequent lemma calculates the mean time to failure for the random variable T, which follows the NGEWD$(\alpha,\beta,\gamma,\theta)$ distribution.

Lemma 2.

For a random variable T following the $NGEWD(\alpha,\beta,\gamma,\theta)$ distribution, where $\alpha\geq 1,\beta>0,\theta>0$ and $0<\gamma<x$, the MTTF is expressed as follows:

$$
\text{MTTF}=\alpha\theta\sum_{i=0}^{\infty}\Theta_{i}^{\alpha}\left\{\begin{array}{c}(1+i)^{\frac{-1}{\beta}-1}\\ \times\Gamma\left(\frac{\beta+1}{\beta},(1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}\right)\\ -\frac{\gamma e^{-(1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}}}{\theta(1+i)}\end{array}\right\},
$$

where $\Theta_{i}^{\alpha}=(-1)^{i}\binom{\alpha-1}{i}$.

proof. It is easy to prove this Lemma using the first moment about zero (see Lemma 1).

3.4 Mean time to repair (MTTR)

Let T denote the random variable of the time to repair or the total downtime. If the repair time T has a repair time density function $g(t;.)$, then the MTTR is the expected value of the random variable repair time, not failure time, and is given by

$$
MTTR=\int_{0}^{\infty}t\,g(t;.)\,dt. \qquad (8)
$$

In order to design and manufacture a maintainable system, it is necessary to predict the MTTR for various fault conditions that could occur in the system.

Lemma 3.

If T is a random variable with the PDF $g(t;\alpha_1,\beta_1,\gamma_1,\theta_1)$ of the NGEWD$(\alpha_1,\beta_1,\gamma_1,\theta_1)$ distribution, where $\alpha_1 \geq 1$, $\beta_1 > 0, \theta_1 > 0$ and $0 < \gamma_1 < x$, the MTTR is provided as follows:

$$\text{MTTR} = \alpha_1 \theta_1 \sum_{i=0}^{\infty} (-1)^i \binom{\alpha_1 - 1}{i} \times \left\{ \begin{array}{c} (1+i)^{\frac{-1}{\beta_1}-1} \\ \times \Gamma\left(\frac{\beta_1+1}{\beta_1}, (1+i)\left(\frac{-\gamma_1}{\theta_1}\right)^{\beta_1} \right) \\ - \frac{\gamma_1 e^{-(1+i)\left(\frac{-\gamma_1}{\theta_1}\right)^{\beta_1}}}{\theta_1 (1+i)} \end{array} \right\}.$$

3.5 Insurance pricing principle: The principle of expected value

Insurance premium methodologies are utilized to compute insurance premiums for diverse occurrences, taking into consideration the accompanying risk levels. Throughout the years, numerous premium methodologies have been formulated. This section introduces a selection of them, assuming a loss distribution adhering to the NGEWD$(\alpha,\beta,\gamma,\theta)$. In this context, let $\omega \geq 0$ represent the risk loading parameter. The principle of expected value (PEV) is a fundamental concept in insurance and risk management that guides the determination of insurance premiums. It asserts that the premium for insurance coverage ought to equate to the expected value of losses, modified by a risk loading factor. Thus, the PEV can be defined as:

$$\text{PEV}(\omega;.) = (1+\omega)\Phi^{(1)},$$

where PEV$(\omega;.)$ represents the insurance premium, ω denotes the risk loading factor, and $\Phi^{(1)} = E(X)$ denotes the expected value of losses (or the expected value of the distribution of losses). The term $(1+\omega)$ denotes the ”Risk Loading” factor, which is added to the expected value of losses to cover various expenses and provide the insurer with a profit margin. The value of ω depends on factors such as administrative costs, claims processing, underwriting, and the insurer’s desired profit level. The EVP of the NGEWD$(\alpha,\beta,\gamma,\theta)$ distribution can be derived using Lemma 1. The PEV stands as a fundamental principle in insurance pricing, applied across a spectrum of insurance categories such as property, liability, health, and life insurance. Insurers determine premiums by considering projected losses, with adjustments made for risk loading to ensure equity. This method aligns premiums with the anticipated costs of coverage, allowing insurers to manage expenses and generate profit, while also ensuring policyholders contribute appropriate premiums relative to the risks transferred. Economically, the PEV promotes risk transfer, facilitating efficiency through risk pooling and mitigating the financial ramifications of uncertainty. Nevertheless, it assumes known and accurately estimated loss distributions, disregarding factors such as moral hazard and market competition that can influence pricing and market dynamics.

3.6 Maintainability

Let T represent the random variable denoting the time to repair or the total downtime. If the repair time T follows a repair time density function $g(t;.)$, then maintainability $V(t;.)$ is defined as the probability of isolating and repairing a fault within a specified time frame in a system, or equivalently, the probability that the system will be restored to operation by time. If the repair time T is a random variable with a repair time density function $g(t;.)$ following the NGEWD$(\alpha_1,\beta_1,\gamma_1,\theta_1)$ distribution, then maintainability $V(t;.)$ is defined as:

$$V(t) = \sum_{i=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+m}\, i^m}{\theta_1^{\beta_1 m}\, m!} \binom{\alpha_1}{i} (t-\gamma_1)^{\beta_1 m}. \quad (9)$$

3.7 The mean residual (past) lifetime

In reliability theory and survival analysis, various measures have been defined to study the lifetime characteristics of living organisms, including the MRL and the mean past lifetime (MPL). Assuming that each component of the system has survived up to time t, the survival function of $T_i - t$ given that $T_i > t$, where $i = 1,2,\ldots,n$, is given as

$$S(x|t) = \frac{S(t+x;.)}{S(t;.)}.$$

This represents the corresponding conditional survival function of the components at age t. From the previous equation, we derive that the MRL $m(t;.)$ of each component is equal to

$$m(t;.) = \int_0^{\infty} S(x|t)\,dx = \frac{1}{S(t;.)}\int_t^{\infty} S(x;.)\,dx. \quad (10)$$

Lemma 4.

If T is a random variable following the NGEWD$(\alpha,\beta,\gamma,\theta)$ distribution, then the MRL is expressed as follows for $\alpha \geq 1, \beta > 0, \theta > 0$ and $0 < \gamma < x$,

$$m(t) = \frac{[I_1(\alpha,\beta,\gamma,\theta) - I_2(\alpha,\beta,\gamma,\theta)]}{S(t)},$$

where

$$I_1(\alpha,\beta,\gamma,\theta) = \alpha\theta \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \times \left\{ \begin{array}{c} (1+i)^{\frac{-1}{\beta}-1} \\ \times\Gamma\left(\frac{\beta+1}{\beta}, (1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}\right) \\ -\frac{\gamma e^{-(1+i)\left(\frac{-\gamma}{\theta}\right)^{\beta}}}{\theta(1+i)} \end{array} \right\},$$

$$I_2(\alpha,\beta,\gamma,\theta) = t - \sum_{i=0}^{\infty}\sum_{m=0}^{\infty} \Lambda_{m,i} \left[\begin{array}{c} (t-\gamma)^{\beta m+1} \\ -(-\gamma)^{\beta m+1} \end{array} \right],$$

$$S(t) = 1 - \sum_{i=0}^{\infty}\sum_{m=0}^{\infty} (\beta m+1)\Lambda_{m,i}(t-\gamma)^{\beta m},$$

and

$$\Lambda_{m,i} = \frac{(-1)^{i+m} i^m}{(\beta m+1)\,\theta^{\beta m} m!} \binom{\alpha}{i}.$$

The MRL function $m(t)$ is known to uniquely characterize the distribution function $F(t)$, as stated in [23]. On the other hand, the mean past lifetime (MPL) corresponds to the average time elapsed since the failure of T_i given that $T_i \le t$. In this scenario, the relevant random variable is $t - T_i$ where $T_i \le t$ and $i = 1,2,3,\ldots,n$. This conditional random variable represents the time elapsed since the failure of T_i given that it failed at or before time t. The expectation of this random variable provides the MPL denoted as $P(t)$, where

$$P(t;.) = E(t - T \le t) = \frac{1}{F(t;,)} \int_0^t F(x;.)dx. \quad (11)$$

Lemma 5.

If T is a random variable following the NGEWD$(\alpha,\beta,\gamma,\theta)$ distribution, then the MPL is given by the following expression for $\alpha,\beta,\theta > 0$ and $0 < \gamma < x$

$$P(t) = \frac{\sum_{i=0}^{\infty}\sum_{m=0}^{\infty} \Lambda_{m,i} \left[\begin{array}{c} (t-\gamma)^{\beta m+1} \\ -(-\gamma)^{\beta m+1} \end{array} \right]}{\sum_{i=0}^{\infty}\sum_{m=0}^{\infty} (\beta m+1)\Lambda_{m,i}\,(t-\gamma)^{\beta m}}. \quad (12)$$

The $P(t)$ also uniquely characterizes the underlying distribution, as demonstrated in [24].

4 Estimation for Different Sample Types

In this section, a comprehensive discussion has been provided on both complete and upper recorded data, offering detailed insights into their characteristics, implications, and applications.

4.1 Estimators derived from complete dataset

This section focuses on obtaining the maximum likelihood estimates for the unknown parameters α,β,γ and θ of the NGEWD$(\alpha,\beta,\gamma,\theta)$, utilizing a complete sample. Let's consider a random sample $X_1, X_2, \ldots, X_n$ drawn from the NGEWD$(\alpha,\beta,\gamma,\theta)$. The likelihood function for this sample is given by:

$$\begin{aligned} L_{Com}(\alpha,\beta,\gamma,\theta) = L_{Com} &= \prod_{i=1}^{n} f(x_i;\alpha,\beta,\gamma,\theta) \\ &= \left[\frac{\alpha\beta}{\theta^{\beta}}\right]^n exp\left(-\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}\right) \\ &\times \prod_{i=1}^{n}(x_i-\gamma)^{\beta-1}\left[1-e^{-\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}\right]^{\alpha-1}. \end{aligned} \quad (13)$$

The log-likelihood function becomes

$$\begin{aligned} Ln\ L_{Com} &= n\,(Ln\ \alpha\beta - \beta\ Ln\ \theta) - \theta^{-\beta}\sum_{i=1}^{n}(x_i-\gamma)^{\beta} \\ &+(\beta-1)\sum_{i=1}^{n} Ln\,(x_i-\gamma) \\ &+(\alpha-1)\sum_{i=1}^{n} Ln\left(1-e^{-\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}\right). \end{aligned} \quad (14)$$

So,

$$\frac{\partial LnL_{Com}}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} Ln\left[1-e^{-\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}\right],$$

$$\begin{aligned} \frac{\partial LnL_{Com}}{\partial\beta} &= n\left[\frac{1}{\beta} - Ln\ \theta\right] - \sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^{\beta} \\ &\times Ln\left(\frac{x_i-\gamma}{\theta}\right) + \sum_{i=1}^{n} Ln\,(x_i-\gamma) + (\alpha-1) \\ &\times \sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^{\beta} Ln\left(\frac{x_i-\gamma}{\theta}\right)\left[e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}-1\right]^{-1}, \end{aligned}$$

$$(\alpha-1)\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^{\beta} Ln\left(\frac{x_i-\gamma}{\theta}\right)\left[e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}-1\right]^{-1},$$

$$\begin{aligned} \frac{\partial LnL_{Com}}{\partial\gamma} &= \frac{\beta}{\theta^{\beta}}\sum_{i=1}^{n}(x_i-\gamma)^{\beta-1} - (\beta-1) \\ &\times \sum_{i=1}^{n}\frac{1}{x_i-\gamma} - \frac{(\alpha-1)^{\beta}}{\theta^{\beta}}\sum_{i=1}^{n}(x_i-\gamma)^{\beta-1} \\ &\times \left[e^{\left(\frac{x_i-\gamma}{\theta}\right)}-1\right]^{-1}, \end{aligned}$$

$$\frac{\partial LnL_{Com}}{\partial \theta} = \frac{-n\beta}{\theta} + \frac{\beta}{\theta^{\beta+1}} \sum_{i=1}^{n} (x_i - \gamma)^{\beta} - \frac{(\alpha-1)^{\beta}}{\theta^{\beta+1}} \sum_{i=1}^{n} (x_i - \gamma)^{\beta} \left[e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right]^{-1}.$$

The normal equations lack an explicit solution and are typically solved numerically. The MLE of α, denoted as $\widehat{\alpha}$, can be obtained as follows:

$$\widehat{\alpha} = -n \left(\sum_{i=1}^{n} Ln \left[1 - exp \left\{ - \left(\frac{1}{\theta} (x_i - \gamma) \right)^{\beta} \right\} \right] \right)^{-1}. \tag{15}$$

The MLEs of β, γ and θ are obtained by solving three non-linear equations. Additionally, in this section, we derive the asymptotic confidence intervals of these parameters when $\alpha, \beta, \theta > 0$ and $0 < \gamma < x$, since the MLEs of the unknown parameters β, γ and θ cannot be obtained in closed forms. This is done by utilizing the variance-covariance matrix (VCM). The derivatives in VCM are given as follows:

$$\frac{\partial^2 LnL_{Com}}{\partial \alpha^2} = \frac{-n}{\alpha^2}, \quad \frac{\partial^2 LnL_{Com}}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \frac{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta} Ln\left(\frac{x_i-\gamma}{\theta}\right)}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)},$$

$$\frac{\partial^2 LnL_{Com}}{\partial \alpha \partial \gamma} = \frac{-\beta}{\theta^{\beta}} \sum_{i=1}^{n} \frac{(x_i-\gamma)^{\beta-1}}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)},$$

$$\frac{\partial^2 LnL_{Com}}{\partial \alpha \partial \theta} = \frac{-\beta}{\theta^{\beta+1}} \sum_{i=1}^{n} \frac{(x_i-\gamma)^{\beta}}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)},$$

$$\frac{\partial^2 LnL_{Com}}{\partial \beta^2} = \frac{-n}{\beta^2} - \sum_{i=1}^{n} \left(\frac{x_i-\gamma}{\theta} \right)^{\beta} \left[Ln \left(\frac{x_i-\gamma}{\theta} \right) \right]^2 + (\alpha-1) \sum_{i=1}^{n} \frac{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta} \left[Ln\left(\frac{x_i-\gamma}{\theta}\right)\right]^2 e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} \times \left[1 - \left(\frac{x_i-\gamma}{\theta}\right)^{\beta} - e^{-\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}\right]}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)^2},$$

$$\frac{\partial^2 LnL_{Com}}{\partial \beta \partial \gamma} = \frac{1}{\theta^{\beta}} \sum_{i=1}^{n} (x_i-\gamma)^{\beta-1} \left[\beta Ln \left(\frac{x_i-\gamma}{\theta} \right) + 1 \right] - \sum_{i=1}^{n} \frac{1}{x_i-\gamma} - \frac{(\alpha-1)}{\theta^{\beta}} \times \left(\sum_{i=1}^{n} \left[\frac{(x_i-\gamma)^{\beta-1} \left[\beta Ln\left(\frac{x_i-\gamma}{\theta}\right) + 1 \right]}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)} + \frac{\frac{(x_i-\gamma)^{2\beta}}{\theta^{\beta}} \left[Ln\left(\frac{x_i-\gamma}{\theta}\right) e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} \right]}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)^2} \right] \right),$$

$$\frac{\partial^2 LnL_{Com}}{\partial \beta \partial \theta} = \frac{-n}{\theta} + \frac{1}{\theta^{\beta+1}} \sum_{i=1}^{n} (x_i-\gamma)^{\beta} \times \left[\beta \; Ln \left(\frac{x_i-\gamma}{\theta} \right) + 1 \right] + \frac{(\alpha-1)}{\theta^{\beta+1}} \left(\sum_{i=1}^{n} \left[\frac{(x_i-\gamma)^{\beta} \left[\beta Ln\left(\frac{x_i-\gamma}{\theta}\right) + 1 \right]}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)} + \frac{\frac{\beta(x_i-\gamma)^{2\beta}}{\theta^{\beta}} \left[Ln\left(\frac{x_i-\gamma}{\theta}\right) \right] e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}}{\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)^2} \right] \right),$$

$$\frac{\partial^2 LnL_{Com}}{\partial \gamma^2} = \frac{-\beta(\beta-1)}{\theta^{\beta}} \sum_{i=1}^{n} (x_i-\gamma)^{\beta-2} - (\beta-1) \sum_{i=1}^{n} \frac{1}{(x_i-\gamma)^2} + \frac{(\alpha-1)^{\beta}}{\theta^{\beta}} \times \sum_{i=1}^{n} (x_i-\gamma)^{\beta-2} \left[(\beta-1)\left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1\right) - \frac{\beta}{\theta^{\beta}} (x_i-\gamma)^{\beta} e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} \right] \times \left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)^{-2},$$

$$\frac{\partial^2 LnL_{Com}}{\partial \gamma \partial \theta} = \frac{-\beta^2}{\theta^{\beta+1}} \sum_{i=1}^{n} (x_i-\gamma)^{\beta-1} - \frac{(\alpha-1)\beta^2}{\theta^{2\beta+1}} \sum_{i=1}^{n} (x_i-\gamma)^{\beta-1} e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} \left[(x_i-\gamma) - \theta^{\beta} \left(1 - e^{-\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}}\right) \right] \left(e^{\left(\frac{x_i-\gamma}{\theta}\right)^{\beta}} - 1 \right)^{-2},$$

$$\frac{\partial^2 LnL_{Com}}{\partial \theta^2} = \frac{n\beta}{\theta^2} - \frac{\beta(\beta+1)}{\theta^{2\beta+2}} \sum_{i=1}^{n} (x_i-\gamma)^\beta + \frac{(\alpha-1)^\beta}{\theta^{2\beta+1}} \sum_{i=1}^{n} (x_i-\gamma)^\beta [\beta(x_i-\gamma) \times e^{\left(\frac{x_i-\gamma}{\theta}\right)^\beta} + (\beta+1)\theta^\beta (e^{\left(\frac{x_i-\gamma}{\theta}\right)^\beta} - 1)].$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters $\alpha, \beta, \gamma, \theta$ by using variance-covariance matrix as in the following forms

$$\widehat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{var(\widehat{\alpha})},\ \widehat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{var(\widehat{\beta})},$$
$$\widehat{\gamma} \pm Z_{\frac{\delta}{2}} \sqrt{var(\widehat{\gamma})},\ \widehat{\theta} \pm Z_{\frac{\delta}{2}} \sqrt{var(\widehat{\theta})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})th$ percentile of the standard normal distribution.

4.2 Estimators derived from upper record values

Upper record values play a significant role in various fields such as reliability theory, extreme value theory, and environmental studies. These values represent the maximum observed values in a dataset or sequence, providing critical insights into the tail behavior of a distribution. They are important for assessing the extreme behavior of a system or phenomenon, identifying potential outliers, and estimating extreme quantiles or probabilities. Additionally, upper record values are used in modeling extreme events such as floods, earthquakes, and financial crises, where understanding the tail behavior of the distribution is crucial for risk assessment and decision-making. Moreover, studying upper record values helps in designing reliable systems, optimizing resource allocation, and developing robust statistical models to handle extreme events effectively. Therefore, the analysis of upper record values is essential for understanding and managing risks associated with extreme events in various fields. When considering a sequence of independently and identically distributed random variables following the NGEWD$(\alpha, \beta, \gamma, \theta)$ and observing n upper record (UP-RC) values $X = \left\{X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}\right\}$, the likelihood function can be expressed as follows for the NGEWD$(\alpha, \beta, \gamma, \theta)$:

$$L_{UP-RC}(\alpha, \beta, \gamma, \theta | x) = f(x_{U(n)}; \alpha, \beta, \gamma, \theta) \times \prod_{i=1}^{n-1} \frac{f(x_{U(i)}; \alpha, \beta, \gamma, \theta)}{R(x_{U(i)}; \alpha, \beta, \gamma, \theta)}; 0 \leq x_{U(1)} < x_{U(2)} < .. < x_{U(n)}. \tag{16}$$

Differentiating Equation (16) with respect to α, β, γ and θ yields nonlinear likelihood equations. These equations necessitate an iterative procedure such as the Newton-Raphson method to solve them numerically.

5 Investigation into Estimator Properties via Simulation Analyses

In this section, we utilize the MLE approach to estimate the parameters α, β, γ and θ of the NGEWD. The population parameters are generated using the R software package. Sampling distributions are derived for various sample sizes, specifically n = 50, 100, 150, 250, 350, and 400, across N = 1000 replications. This research evaluates the characteristics of the MLE method regarding bias and mean square error (MSE). Table 1 presents the MLE outcomes for complete samples and UP-RC values, showcasing two scenarios: Case I with NGEWD parameters (0.2, 0.3, 0.8, 0.7) and case II with NGEWD parameters (0.1, 0.8, 0.6, 0.9). The findings are illustrated in Figures 2 and 3 for these respective cases.

Observations from Table 1, Figure 2, and Figure 3 suggest that as the sample size n increases towards infinity, both the bias and mean square errors tend to diminish, indicating the consistency of the estimators. This phenomenon underscores the effectiveness of MLE for data analysis.

6 Modeling and Analysis of Data

In this section, we demonstrate the practical significance of the NGEWD distribution through two real-world applications. Dataset I is examined using complete data, while data set II is analyzed based on UP-RC values. The comparison of fitted distributions employs various criteria, including the negative maximized log-likelihood $(-L_{Com})$, Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramér-Von Mises ($\mathbf{W}^*$), Anderson-Darling ($\mathbf{A}^*$) statistics. The key idea behind AIC is to penalize the complexity of the model to avoid overfitting, ensuring that the model is as simple as possible while still capturing the underlying data patterns. In practice, you compare the AIC values of different models; the model with the lowest AIC is generally preferred. AIC is more prone to favor complex models than BIC, as its penalty for additional parameters is less severe. This makes it suitable for models where the primary goal is predictive accuracy. BIC tends to select simpler models than AIC, especially as sample size grows, due to its stronger penalty on model complexity. This can make it more appropriate for models where understanding the underlying process in a parsimonious manner is important. Incorporating a detailed discussion on AIC and BIC in the data analysis section should emphasize their theoretical foundations, the practical implications of their penalty terms, and how they guide the selection of an optimal model in terms of balance between fit and complexity. This will not only illuminate the rationale behind model selection but also guide readers in applying these criteria to their own analyses effectively.

Table 1. The values of bias and MSE for the estimated parameters of NGEWD.

Method	Size	α		β		γ		θ	
	n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
Case I$_{Com}$	**50**	0.0441	0.0262	0.0665	0.0271	0.0436	0.0288	0.0413	0.0315
	100	0.0348	0.0244	0.0605	0.0212	0.0366	0.0214	0.0366	0.0263
	150	0.0254	0.0233	0.0493	0.0152	0.0273	0.0169	0.0222	0.0216
	250	0.0192	0.0226	0.0346	0.0126	0.0204	0.0113	0.0189	0.0137
	350	0.0121	0.0155	0.0181	0.0086	0.0139	0.0097	0.0097	0.0072
	400	0.007	0.0093	0.0085	0.0043	0.0094	0.0053	0.0036	0.0027
Case II$_{UP-RC}$	**50**	0.0468	0.0292	0.0697	0.0288	0.0452	0.0292	0.0445	0.0364
	100	0.0355	0.0275	0.0655	0.0246	0.0395	0.0255	0.0383	0.0292
	150	0.0276	0.0226	0.0523	0.0169	0.0363	0.0185	0.0286	0.0247
	250	0.0179	0.0187	0.0498	0.0157	0.0296	0.0109	0.0198	0.0199
	350	0.0167	0.0168	0.0339	0.0113	0.0169	0.0093	0.0111	0.0125
	400	0.0091	0.0137	0.0172	0.0104	0.0127	0.0078	0.0081	0.0083

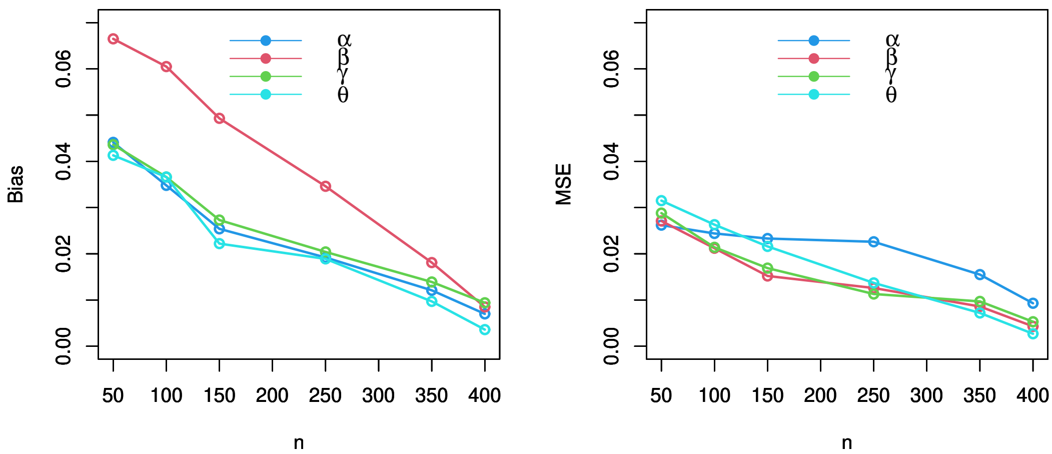

Fig 2. The bias and MSE values for the NGEWD estimated from complete samples.

Additionally, the Kolmogorov-Smirnov (K-S) statistic and its associated P-value are utilized for comparison purposes. The K-S test assesses the similarity between two empirical probability distributions or a sample distribution and a reference distribution by comparing their cumulative distribution functions.

6.1 Dataset I: Strengths of glass fibers

The dataset used in this study was previously utilized by [25]. The dataset comprises simulated strengths of glass fibers and is represented by the following values: 1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.460, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.670, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.910, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121. Figure 4 shows the initial visualization of this dataset using non-parametric plots. In relation to this dataset, we will assess the appropriateness of the NGEW distribution by comparing it with several alternative models, including exponential (E), Burr X (BX), Rayleigh (R), Burr-Hatke (BH), Lindley (L), Weibull (W), Gompertz (Go), odd Lindley Burr-Hatke (OLBH), and odd exponentiated Burr-Hatke (OEBH). The maximum likelihood estimators (MLEs) for the data and goodness-of-fit measures are presented in Tables 2 and 3, respectively.

The estimated CDF and PDF plots of the best seven fitted models are displayed in Figure 5, whereas the PP plots of all models are depicted in Figure 6. The values in Table 3 are supported by the plots in Figures 5 and 6, showing that the NGEWD provides the best fit for the dataset I.

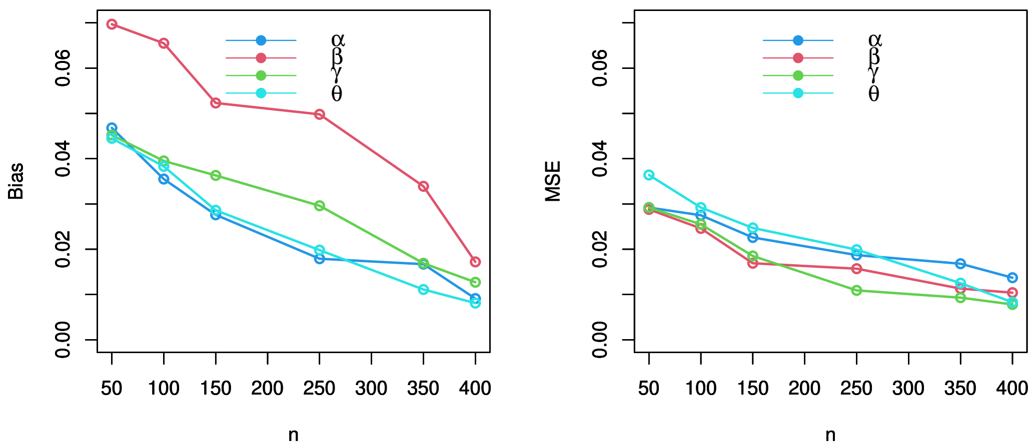

Fig 3. The bias and MSE values for the NGEWD estimated from UP-RC values.

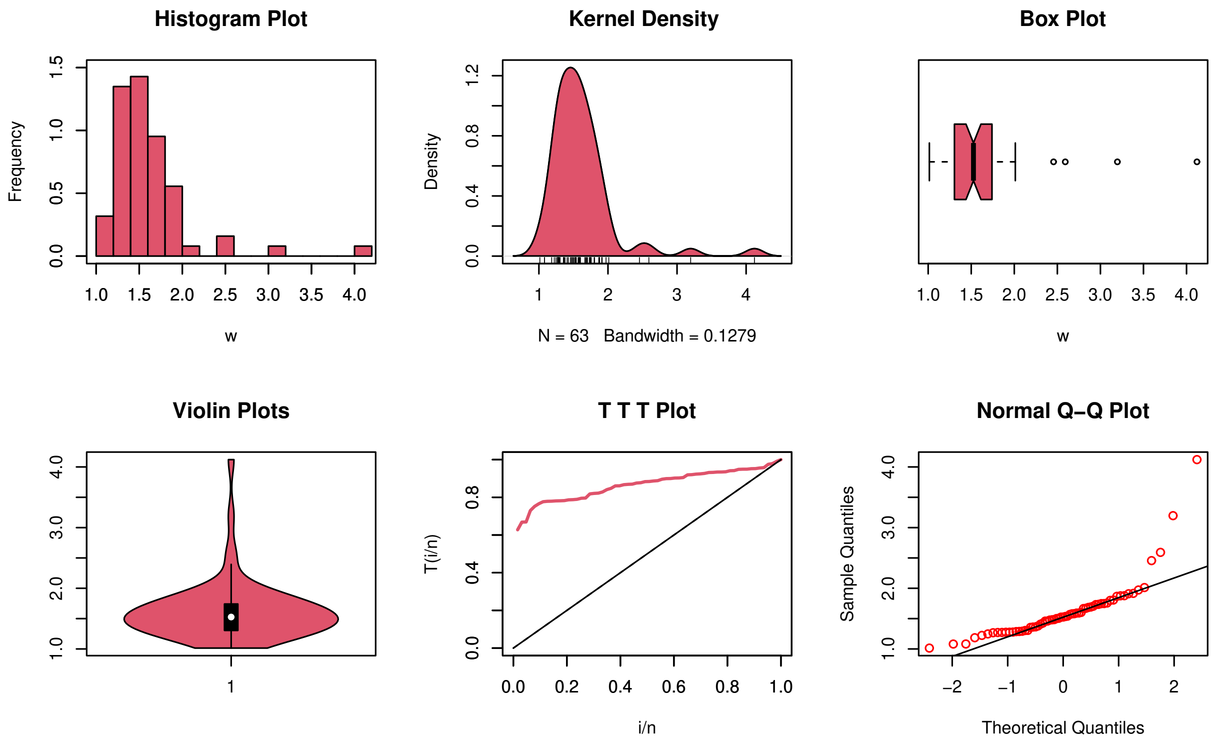

Fig 4. Non-parametric plots for dataset I.

Table 2. The MLEs with their corresponding SE for dataset I.

Model	α		β		γ		θ	
	MLE	SE	MLE	SE	MLE	SE	MLE	SE
E	0.619	0.078	–	–	–	–	–	–
BX	8.047	1.014	–	–	–	–	–	–
R	0.352	0.044	–	–	–	–	–	–
BH	0.233	0.078	–	–	–	–	–	–
L	0.938	0.089	–	–	–	–	–	–
W	3.062	0.240	1.788	0.078	–	–	–	–
Go	0.179	0.0408	1.051	0.117	–	–	–	–
OLBH	0.626	0.082	0.246	0.049	–	–	–	–
OEBH	0.824	0.108	0.085	0.025	–	–	–	–
NGEW	261.979	0.543	0.542	0.107	0.608	0.308	0.034	0.033

Table 3. The goodness-of-fit test for the dataset I.

Models	Statistic								
	$-L_{Com}$	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value
E	93.223	188.446	188.511	190.589	189.289	0.305	2.103	0.472	2.08×10^{-13}
BX	30.752	63.503	63.569	65.646	64.346	0.267	1.877	0.109	0.409
R	56.847	115.694	115.760	117.838	116.537	0.470	3.050	0.346	3.03×10^{-7}
BH	113.364	228.729	228.794	230.872	229.572	0.235	1.685	0.610	2.22×10^{-16}
L	85.476	172.952	173.018	175.095	173.795	0.333	2.269	0.435	2.41×10^{-11}
W	46.367	96.734	96.934	101.020	98.419	0.708	4.325	0.205	0.008
Go	64.384	132.768	132.968	137.054	134.454	1.181	6.699	0.296	2.11×10^{-5}
OLBH	56.532	117.064	117.264	121.351	118.750	1.009	5.854	0.246	0.0007
OEBH	62.545	129.091	129.291	133.377	130.777	1.137	6.481	0.287	4.23×10^{-5}
NGEW	20.377	48.754	49.444	57.326	52.125	0.071	0.564	0.078	0.814

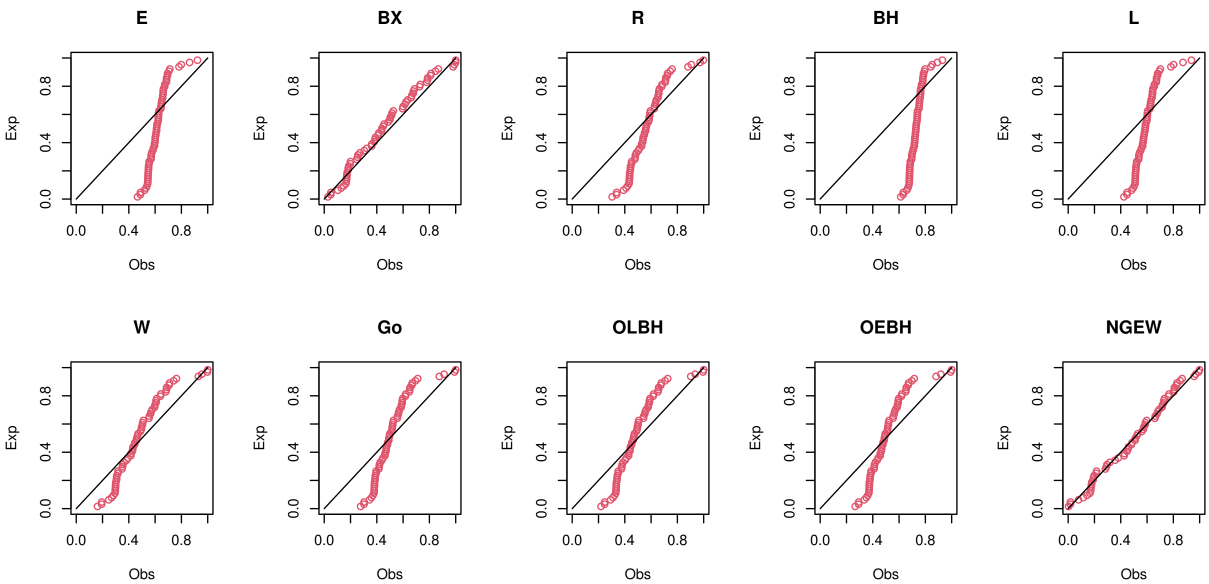

Fig 5. The PP plots of the fitted models using the dataset I.

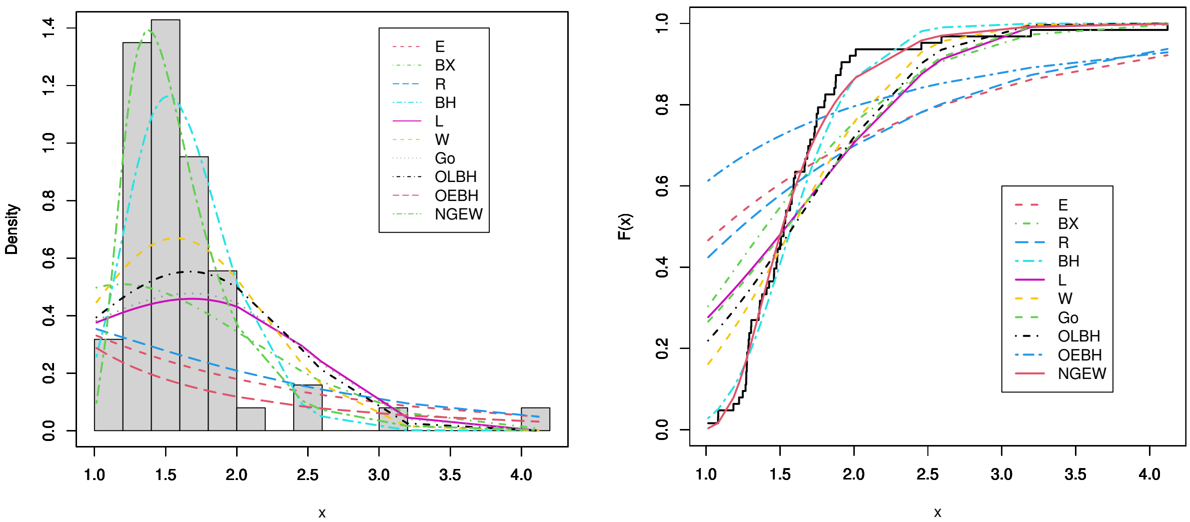

Fig 6. Estimated PDFs (left panel) and CDFs (right panel) for the dataset I.

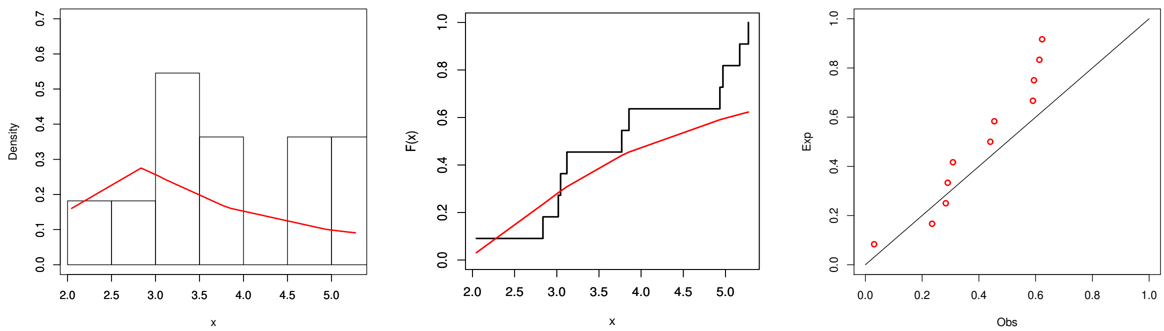

Fig 7. The fitted PDF, the estimated CDF and PP plot of NGEWD using UP-RC values from dataset II.

6.2 Dataset II: Bladder cancer patients

The data is reported in [26] which represents remission times (in months) of 90 bladder cancer patients. The MLEs, KS and its P-value are listed in Table 5. The MLEs, KS and P-values using UP-RC values from dataset II based on the NGEWD can be reported as $\widehat{\alpha} = 1.70$, $\widehat{\beta} = 0.24$, $\widehat{\gamma} = 1.56$, $\widehat{\theta} = 1.64$ with P-value more than 0.05. Thus, the NGEW model provides a good fit for this data. The empirical PDF, CDF and PP plots are displayed in Figure 7 which supports the results.

7 Reflections and Future Prospects

This study undertook an extensive analysis of the NGEWD, thoroughly examining its mathematical and statistical characteristics. Through an investigation of its HRF, which displayed a variety of patterns such as decreasing, increasing, bathtub, or unimodal shapes, the study emphasized the NGEWD's adaptability in effectively modeling diverse data types. Additionally, the distribution showcased significant versatility by accommodating positive and negative skewness, as well as symmetric datasets with varying forms of kurtosis. The study utilized the maximum likelihood method to estimate the model parameters, and simulation results illustrated the effectiveness of approaches based on complete and upper record samples. It was noted that both bias and MSE decreased as the sample size increased. Furthermore, an analysis of a real-world dataset provided a compelling demonstration of the NGEWD's importance and flexibility, highlighting its potential practical utility in various statistical applications.

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