Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180313

p-Laplacian functional differential equations with neutral delay arguments: new oscillation theorems

Noof Alamri[1], *Ali Allahem*[1], *Ali Muhib*[2], *and Osama Moaaz*[1,2,*]

[1]Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Received: 3 Nov. 2023, Revised: 11 Mar. 2024, Accepted: 15 Mar. 2024
Published online: 1 May 2024

Abstract: In this study, we are interested in investigating the oscillatory behavior of solutions to a general class of functional differential equations. We consider a neutral-type equation with multiple delays. We first test some monotonic properties of positive solutions to the studied equation. Then, we use some techniques to obtain criteria that guarantee the oscillation of all solutions. We obtain three different forms of oscillation criteria and compare them in terms of efficiency by applying them to a special case of the studied equation. The results obtained are an extension and generalization of previous results in the literature.

Keywords: Neutral differential equations; oscillatory behavior; second order.

1 Introduction

Equations essentially appear as a basic and universal tool for expressing the dynamics of the physical world, making them one of the most important and active research points in various eras until now. These applications extend across various scientific branches, from physics, medical and life sciences, environmental and climate sciences, and even computer sciences [1–3]. For example, it is used to predict the positions and velocities of celestial bodies, by combining the law of universal gravitation and Newton's second law, which enables scientists to develop accurate models to describe the movement of these bodies [4,5].

Neutral differential equations (NDEs) are considered one of the most important classes of differential equations (DEs) in which the highest-order derivative of the unknown function appears with and without delay. Which means that this type of equation takes into account the current and past state of the system. This provides an accurate and complete description of many mathematical models with delayed reactions. These equations are used to represent ecology models, to study the interactions between prey and their predators and to analyze the dynamics of their behavior [6]. And in neuroscience, in models of electrical networks, and models for reducing the process of loss in transmission lines and distribution networks in high-speed computers. See [7–9] for more details.

On the other hand, second-order DEs, are considered the most significant and active in the field of research due to their appearance in various applications and natural phenomena. As a result, there is a large amount of literature interested in this class of equations. We recommend to the reader the works of Agarwal et al. [8–10], Došly and Řehák [11], Györi and Ladas [12], and Erbe et al. [13].

In this study, we consider the following second-order NDEs with p-Laplacian like operators:

$$\left(r(t)\left|z'(t)\right|^{p-2}z'(t)\right)' + \sum_{i=1}^{m} q_i(t)\left|x(g_i(t))\right|^{p-2}x(g_i(t)) = 0, \tag{1}$$

where $t \in \mathbb{I} := [t_0, \infty)$, $z(t) = x(t) + b(t)x(\delta(t))$, and the following assumptions are satisfied:

(A_1) $p \in \mathbb{R}$ and $p > 1$;
(A_2) $r \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$, $r'(t) \geq 0$, and r satisfies that

$$\int_{t_0}^{\infty} r^{-1/(p-1)}(s)\,\mathrm{d}s < \infty; \tag{2}$$

(A_3) $b, q_i \in \mathbf{C}(\mathbb{I}, [0,\infty))$, $b(t) < 1$ for $i = 1, 2, ...m$, and b satisfies that

$$b(t)\int_{\delta(t)}^{\infty} r^{-1/(p-1)}(s)\,\mathrm{d}s \leq \int_{t_0}^{\infty} r^{-1/(p-1)}(s)\,\mathrm{d}s;$$

* Corresponding author e-mail: o_moaaz@mans.edu.eg

(A$_4$) $\delta, g_i \in \mathbf{C}(\mathbb{I},\mathbb{R})$, $\delta(t) \leq t$, $g_i(t) \leq t$, $\lim_{t\to\infty}\delta(t) = \infty$, $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1,2,...m$, and $g'_{\min}(t)$ and $g'_{\max}(t)$ are nondicreasing, where

$$g_{\min}(t) := \min\{g_i(t),\ i = 1,2,...,m\},$$
$$g_{\max}(t) := \max\{g_i(t),\ i = 1,2,...,m\}.$$

A solution of equation (1) is defined as a function $x \in \mathbf{C}(\mathbb{I},\mathbb{R})$ which has the property $r(z')^{p-1} \in \mathbf{C}^1(\mathbb{I},\mathbb{R})$ and satisfies equation (1) on $\mathbb{I}$. Our interest is directed to the solutions of equation (1) that satisfy the condition $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A nontrivial solution of (1) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory. The NDE (1) is said to be in canonical form if it satisfies that

$$\int_{t_0}^{\infty} r^{-1/(p-1)}(s)\,\mathrm{d}s = \infty,$$

while it is called in the noncanonical form if the equation achieves (2).

In the next sections, we provide new monotonic properties for the positive solutions of (1) and use them to obtain new oscillatory criteria. The provided results in general improve the previous ones.

2 Literature Review

In the following, we present a historical summary to illustrate the significant improvement in the paper's results and contents.

Many researchers have been interested in establishing sufficient criteria for the oscillation of different types of DEs over the past decades. Zhang [14], Muhib [15] and Hassan et al. [16] studied the oscillation of second-order nonlinear neutral equations in noncanonical form. in contrast, Dzurina [17] and Bazighifan [18] are interested in the canonical form. For the mixed-type NDDE Dzurina and others [19–21] established new criteria to ensure the oscillation of solutions. In a recent papers Agarwal et al. [22], Almarri et al. [23] and Moaaz et al. [24–26] interested in the oscillatory behavior of the even-order NDEs. Lastly, many researchers have been interested in obtaining sufficient conditions for the oscillation of solutions of different classes of NDEs, see for example [27–33].

In 1986, Koplatadze [34] studied the delay second order linear DE

$$x''(t) + q(t)x(g(t)) = 0, \tag{3}$$

where $r(t) = 1$ and $b(t) = 0$, he established that (3) is oscillatory if

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s)g(s)\,\mathrm{d}s > 1.$$

These results were obtained using the well-known comparison method with lower orders. [35, 36] used the same technique to provide improved results.

On other hand, Sun and Meng [37], used Riccati inequality technique to extend the work on the half linear equation

$$\left(r(t)\left(x'(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(g(t)) = 0, \tag{4}$$

which is oscillatory if

$$\int_{t_0}^{\infty}\left(R^{\alpha}(g(s))q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{g'(s)}{R(g(s))r^{1/\alpha}(s)}\right)\mathrm{d}s = \infty,$$

where α is a quotient of any two odd positive integers. Other authors have improved and generalized this results, see [38, 39].

Ladas et al. [40] studied the neutral differential equation

$$(x(t) + b(t)x(t-\delta))'' + q(t)x(t-g) = 0, \tag{5}$$

they proved that for $0 \leq b(t) \leq 1$, (5) is oscillatory if

$$\int_{t_0}^{\infty} q(s)(1 - b(s-g))\,\mathrm{d}s = \infty.$$

Please, see [41–43] and the references mentioned in them for more information.

For the noncanonical form, Dzurina and Jadlovská [44] recently established, in contrast to most known results, a one-condition oscillation criteria for the noncanonical form of (4). They showed in particular that (4) is oscillatory for $t_1 \geq t_0$, if

$$\limsup_{t\to\infty} \pi^{\alpha}(t)\int_{t_1}^{t} q(s)\,\mathrm{d}s > 1 \tag{6}$$

or

$$\int^{\infty}\frac{1}{r^{1/\alpha}(t)}\left[\int^{t}\pi^{\alpha}(\sigma(s))q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}t = \infty.$$

On other hand, Li and Han [45] considered the general second-order NDEs

$$\left(r(t)\left(x(t) + b(t)x(\delta(t))\right)'\right)' + q(t)x(g(t)) = 0, \tag{7}$$

they established a new oscillation criteria for (7), if

$$\int_{t_0}^{\infty} Q_*(s)\,\mathrm{d}s = \infty,$$

where

$$Q_*(t) = \min\{q(t), q(\delta(t))\}.$$

Many researches interested in the second order noncanonical NDDE

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(g(t)) = 0. \tag{8}$$

For more information see [46–49]. In 2017, Bohner et al. [50] improved Theorems in [46], [47] by giving a new sufficient condition ensures that (8) oscillates if

$$\limsup_{t\to\infty}\pi^{\alpha}(t)\int_{t_1}^{t}Q^{*}(s)\,\mathrm{d}s>1, \tag{9}$$

where

$$Q^{*}(t)=q(t)\left(1-b(g(t))\frac{\pi(\delta(g(t)))}{\pi(g(t))}\right)^{\alpha}.$$

Our main aim of this paper is to study the oscillatory and asymptotic properties of (1) in the noncanonical form further by providing some additional finite and infinite integral sufficient conditions. We present three different forms of criteria that ensure the oscillation of all solutions of the considered equation. This new results complements and extend a number of results reported in the literature. All functional inequalities are assumed to hold for all t large enough in following sections. Without losing of generality, we can only deal with the positive solutions to (1).

3 Main Results

To facilitate the presentation of results, we assume that

$$\phi(u)=\int_{u}^{\infty}r^{-1/(p-1)}(s)\,\mathrm{d}s$$

and

$$H(t)=\sum_{i=1}^{m}q_i(s)\left(1-b(g_i(s))\frac{\phi(\delta(g_i(s)))}{\phi(g_i(s))}\right)^{p-1}.$$

3.1 Monotonic properties of positive solutions

In this section, we are interested in investigating the classification of positive solutions as well as the monotonic properties of these solutions. In addition, we obtain some inequalities and relationships that help us prove the main results.

Lemma 1. *Assume that x is a positive solution of (1). Then, the following cases of z holds:*

$C_1: z(t)>0,\ z'(t)>0,$ *and* $\left(r(t)|z'(t)|^{p-2}z'(t)\right)'\leq 0;$

$C_2: z(t)>0,\ z'(t)<0,$ *and* $\left(r(t)|z'(t)|^{p-2}z'(t)\right)'\leq 0.$

Proof. Assume that x is a positive solution of (1). From (1), we have

$$\left(r(t)\left|z'(t)\right|^{p-2}z'(t)\right)'=-\sum_{i=1}^{m}q_i(t)\left|x(g_i(t))\right|^{p-2}x(g_i(t))\leq 0.$$

Therefore, we obtain that $r(t)|z'(t)|^{p-2}z'(t)$ is nonincreasing function, thus either $z'(t)>0$ or $z'(t)<0$, eventually. Thus, the proof is complete.

Notation 1. We denote the class of eventually positive solutions that have a corresponding function which satisfies the properties in C_1 (or C_2) by $\mathscr{S}_1$ (or $\mathscr{S}_2$).

Lemma 2. *Assume that*

$$\lim_{t\to\infty}\int_{t_0}^{t}H(s)\,\mathrm{d}s=\infty. \tag{10}$$

Then, z is decreasing eventually.

Proof. Let $x\in\mathscr{S}_1$. By taking into account (A_4), that $x(\delta(t))$, $x(g_i(t))$ and z are eventually positive, for $i=1,2,..,m$. Then, $x(t)>(1-b(t))z(t)$, and so

$$\begin{aligned}0&\geq\left(r(t)(z'(t))^{p-1}\right)'+\sum_{i=1}^{m}q_i(t)(1-b(g_i(t)))^{p-1}z^{p-1}(g_i(t))\\&\geq\left(r(t)(z'(t))^{p-1}\right)'+z^{p-1}(g_{\min}(t))\sum_{i=1}^{m}q_i(t)(1-b(g_i(t)))^{p-1}.\end{aligned}\tag{11}$$

Integrating (11) from t_1 to t, we get

$$\begin{aligned}r(t)(z'(t))^{p-1}&\leq r(t_1)(z'(t_1))^{p-1}\\&\quad-\int_{t_1}^{t}z^{p-1}(g_{\min}(s))\sum_{i=1}^{m}q_i(s)(1-b(g_i(s)))^{p-1}\mathrm{d}s\\&\leq r(t_1)(z'(t_1))^{p-1}\\&\quad-z^{p-1}(g_{\min}(t_1))\int_{t_1}^{t}\sum_{i=1}^{m}q_i(s)(1-b(g_i(s)))^{p-1}\mathrm{d}s.\end{aligned}$$

Since

$$1-b(t)\geq 1-b(t)\frac{\phi(\delta(t))}{\phi(t)},$$

we have

$$\begin{aligned}&r(t)(z'(t))^{p-1}\\&\leq r(t_1)(z'(t_1))^{p-1}\\&\quad-z^{p-1}(g_{\min}(t_1))\int_{t_1}^{t}\sum_{i=1}^{m}q_i(s)\left(1-b(g_i(s))\frac{\phi(\delta(g_i(s)))}{\phi(g_i(s))}\right)^{p-1}\mathrm{d}s\\&=r(t_1)(z'(t_1))^{p-1}-z^{p-1}(g_{\min}(t_1))\int_{t_1}^{t}H(s)\,\mathrm{d}s,\end{aligned}$$

which with (10) contradicts to the positivity of $z'(t)$. Thus, the proof is complete.

Lemma 3. *Assume that $x\in\mathscr{S}_2$. Then, $z(t)\geq-r^{1/(p-1)}(t)z'(t)\phi(t)$ and $z(t)/\phi(t)$ is increasing.*

Proof. Assume that $x\in\mathscr{S}_2$. The monotonicity of $r^{-1/(p-1)}(t)z'(t)$ implies that

$$\begin{aligned}z(t)&\geq-\int_{t}^{\infty}\frac{r^{1/(p-1)}(s)z'(s)}{r^{1/(p-1)}(s)}\mathrm{d}s\\&\geq-r^{1/(p-1)}(t)z'(t)\int_{t}^{\infty}\frac{1}{r^{1/(p-1)}(s)}\mathrm{d}s\\&=-r^{1/(p-1)}(t)z'(t)\phi(t),\end{aligned}$$

which implies that

$$\left(\frac{z(t)}{\phi(t)}\right)'=\frac{r^{1/(p-1)}(t)\phi(t)z'(t)+z(t)}{r^{1/(p-1)}(t)\phi^{2}(t)}\geq 0.$$

Thus, the proof is complete.

Lemma 4. *Assume that (10) holds. Then, eventually,*

$$\left(r(t)\left|z'(t)\right|^{p-2}z'(t)\right)'+H(t)z^{p-1}(g_{\max}(t))\leq 0. \quad (12)$$

Proof. Let $x\in\mathscr{S}_1\cup\mathscr{S}_2$. From Lemma 2, z is decreasing eventually. Thus, $x\in\mathscr{S}_2$. It follows from Lemma 3 that $z(t)/\phi(t)$ is increasing. So,

$$z(\delta(t))\leq\frac{\phi(\delta(t))}{\phi(t)}z(t).$$

By using the definition of z, we obtain

$$\begin{aligned}x(t)&=z(t)-b(t)x(\delta(t))\\&\geq z(t)-b(t)z(\delta(t))\\&\geq z(t)-b(t)\frac{\phi(\delta(t))}{\phi(t)}z(t)\\&=\left(1-b(t)\frac{\phi(\delta(t))}{\phi(t)}\right)z(t).\end{aligned}$$

Substituting from the above inequality into (1), implies that

$$\left(r(t)\left|z'(t)\right|^{p-2}z'(t)\right)'$$
$$\leq-\sum_{i=1}^{m}q_i(t)\left(1-b(g_i(s))\frac{\phi(\delta(g_i(s)))}{\phi(g_i(s))}\right)^{p-1}z^{p-1}(g_i(t)).$$

Then,

$$\left(r(t)\left|z'(t)\right|^{p-2}z'(t)\right)'$$
$$\leq-z^{p-1}(g_{\max}(t))\sum_{i=1}^{m}q_i(t)\left(1-b(g_i(s))\frac{\phi(\delta(g_i(s)))}{\phi(g_i(s))}\right)^{p-1}$$
$$=-H(t)z^{p-1}(g_{\max}(t)).$$

Thus, the proof is complete.

3.2 Oscillatory behavior of solutions

After investigating the monotonic properties of the positive solutions, we are prepared to test the oscillation of all solutions of equation (1). We use several techniques to obtain different forms of the oscillation criteria for equation (1).

Theorem 1. *Assume that*

$$\int_{t_0}^{\infty}\left(\frac{1}{r(u)}\int_{t_0}^{u}H(s)\phi^{p-1}(g_{\max}(s))\,\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}u=\infty. \quad (13)$$

Then (1) is oscillatory.

Proof. On the basis of assuming the opposite of the results of this theorem, we assume that there is a positive solution to the studied equation. It is easy to see that conditions (13) and (A_2) necessarily lead to the fact that

$$\int_{t_0}^{t}H(s)\phi^{p-1}(g_{\max}(s))\,\mathrm{d}s$$

must be unbounded. Moreover, since $\phi'(s,\infty)<0$, we obtain that (10) holds. From Lemma 2, we have that z is decreasing eventually. Now, from Lemma 4, we get that (12) holds. Integrating (12) from t_1 to t, we deduce that

$$\begin{aligned}&r(t)\left|z'(t)\right|^{p-2}z'(t)\\&\leq r(t_1)\left|z'(t_1)\right|^{p-2}z'(t_1)-\int_{t_1}^{t}H(s)z^{p-1}(g_{\max}(s))\mathrm{d}s\\&\leq-\int_{t_1}^{t}H(s)z^{p-1}(g_{\max}(s))\mathrm{d}s. \quad (14)\end{aligned}$$

From Lemma 3, we see that

$$\begin{aligned}&z^{p-1}(g_{\max}(t))\\&\geq\left[-r^{1/(p-1)}(g_{\max}(t))z'(g_{\max}(t))\right]^{p-1}\phi^{p-1}(g_{\max}(t)),\end{aligned}$$

which with (14) gives

$$\begin{aligned}&-r(t)\left|z'(t)\right|^{p-2}z'(t)\\&\geq\int_{t_1}^{t}H(s)\left[-r^{1/(p-1)}(g_{\max}(s))z'(g_{\max}(s))\right]^{p-1}\\&\quad\times\phi^{p-1}(g_{\max}(s))\,\mathrm{d}s. \quad (15)\end{aligned}$$

Since $g_{\max}(t)$ is nondecreasing and $r(t)(-z'(t))^{p-1}$ is decreasing, we have

$$r^{1/(p-1)}(g_{\max}(s))z'(g_{\max}(s))\leq r^{1/(p-1)}(g_{\max}(t_1))z'(g_{\max}(t_1))$$

for all $s\geq t_1$. Thus, (15) becomes

$$\begin{aligned}z'(t)\leq&-\left[-r^{1/(p-1)}(g_{\max}(t_1))z'(g_{\max}(t_1))\right]\\&\times\left(\frac{1}{r(t)}\int_{t_1}^{t}H(s)\phi^{p-1}(g_{\max}(s))\,\mathrm{d}s\right)^{1/(p-1)}.\end{aligned}$$

Integrating once more from t_1 to t, we get

$$\begin{aligned}z(t)\leq&\,z(t_1)-\left[-r^{1/(p-1)}(g_{\max}(t_1))z'(g_{\max}(t_1))\right]\\&\times\int_{t_1}^{t}\left(\frac{1}{r(t)}\int_{t_1}^{t}H(s)\phi^{p-1}(g_{\max}(s))\,\mathrm{d}s\right)^{1/(p-1)}.\end{aligned}$$

Therefore, by (13) we have $\lim_{t\to\infty}z(t)=-\infty$, which contradicts with the fact that $z(t)>0$. The proof is complete.

Theorem 2. *Assume that*

$$\limsup_{t\to\infty}\left(\phi^{p-1}(t)\int_{t_0}^{t}H(s)\,\mathrm{d}s\right)>1. \quad (16)$$

Then (1) is oscillatory.

Proof. On the basis of assuming the opposite of the results of this theorem, we assume that there is a positive solution to the studied equation. It is easy to see that conditions (16) and (A_2) necessarily lead to (10) holds. Then, from

Lemma 2, we have that z is decreasing eventually. As in the proof of Theorem 1, we arrive at

$$\begin{aligned} r(t)\left|z'(t)\right|^{p-2} z'(t) &\leq -\int_{t_1}^{t} H(s) z^{p-1}(g_{\max}(s))\mathrm{d}s \\ &\leq -z^{p-1}(g_{\max}(t))\int_{t_1}^{t} H(s)\,\mathrm{d}s \\ &\leq -z^{p-1}(t)\int_{t_1}^{t} H(s)\,\mathrm{d}s. \end{aligned} \tag{17}$$

From Lemma 3, we get

$$z^{p-1}(t) \geq r(t)\left(-z'(t)\right)^{p-1}\phi^{p-1}(t),$$

which with (17) gives

$$-r(t)\left|z'(t)\right|^{p-2} z'(t) \geq r(t)\left(-z'(t)\right)^{p-1}\phi^{p-1}(t)\int_{t_1}^{t} H(s)\,\mathrm{d}s,$$

and so

$$1 \geq \phi^{p-1}(t)\int_{t_1}^{t} H(s)\,\mathrm{d}s.$$

By taking $\limsup_{t\to\infty}$ for both sides of the above inequality, yields a contradiction with (16). Hence, the proof is complete.

It is known that studying the oscillatory behavior of differential equations in the canonical case is easier than studying them in the noncanonical case. Therefore, we find many studies that focus on investigating the oscillatory behavior of equations in the canonical case using many different techniques and methods. Based on this, we can say that coupling the oscillation of equation (1) with the oscillation of another equation, but in the canonical case, allows us to benefit from previous studies in finding many oscillation cariteria.

Theorem 3. *Assume that (10) holds, and the cononical delay DDE*

$$\begin{aligned} 0 = &\left(r^{1/(p-1)}(t)\,\phi^2(t)\,w'(t)\right)' \\ &+\frac{H(t)\,\phi^{p-1}(t)}{(p-1)}\phi\left(g_{\max}(t)\right)w\left(g_{\max}(t)\right) \end{aligned} \tag{18}$$

is oscillatory. Then (1) is oscillatory.

Proof. On the basis of assuming the opposite of the results of this theorem, we assume that there is a positive solution to the studied equation. From Lemma 2, we have that z is decreasing eventually. It follows from Lemma 4 that (12) holds.

Now, since $z'(t) < 0$, we get $z(g_{\max}(t)) \geq z(t)$. Hence, it follows from Lemma 3 that

$$z(g_{\max}(t)) \geq z(t) \geq -r^{1/(p-1)}(t)\,z'(t)\,\phi(t).$$

If $p \geq 2$, then we have

$$\left[\frac{r^{1/(p-1)}(t)\left(-z'(t)\right)}{z(g_{\min}(t))}\right]^{p-2} \leq \phi^{2-p}(t). \tag{19}$$

Moreover, from (12), we obtain

$$\begin{aligned} &-H(t)\,z^{p-1}(g_{\max}(t)) \\ &\geq -\left(r(t)\left|z'(t)\right|^{p-2}\left(-z'(t)\right)\right)' \\ &= -\left(\left[r^{1/(p-1)}(t)\left(-z'(t)\right)\right]^{p-1}\right)' \\ &= -(p-1)\left[r^{1/(p-1)}(t)\left(-z'(t)\right)\right]^{p-2}\left(r^{1/(p-1)}(t)\left(-z'(t)\right)\right)', \end{aligned}$$

and so,

$$\begin{aligned} &\left(r^{1/(p-1)}(t)\left(z'(t)\right)\right)' \\ &\leq -\frac{1}{(p-1)}\left[r^{1/(p-1)}(t)\left(-z'(t)\right)\right]^{2-p} H(t)\,z^{p-1}(g_{\max}(t)) \\ &= -\frac{1}{(p-1)}\left[\frac{z(g_{\max}(t))}{r^{1/(p-1)}(t)\left(-z'(t)\right)}\right]^{p-2} H(t)\,z(g_{\max}(t)). \end{aligned}$$

Using (19), we arrive at

$$\begin{aligned} &\left(r^{1/(p-1)}(t)\left(z'(t)\right)\right)' \\ &\leq -\frac{1}{(p-1)}\left[\frac{z(g_{\max}(t))}{r^{1/(p-1)}(t)\left(-z'(t)\right)}\right]^{p-2} H(t)\,z(g_{\max}(t)) \\ &\leq -\frac{1}{(p-1)}\phi^{p-2}(t)\,H(t)\,z(g_{\max}(t)). \end{aligned} \tag{20}$$

Next, we find

$$\begin{aligned} &\left(r^{1/(p-1)}(t)\,\phi^2(t)\left(\frac{z(t)}{\phi(t)}\right)'\right)' \\ &= \left(r^{1/(p-1)}(t)\left[\phi(t)\,z'(t)+r^{-1/(p-1)}(t)\,z(t)\right]\right)' \\ &= \left(\phi(t)\,r^{1/(p-1)}(t)\,z'(t)+z(t)\right)' \\ &= \phi(t)\left[r^{1/(p-1)}(t)\,z'(t)\right]' \\ &\quad -r^{-1/(p-1)}(t)\left[r^{1/(p-1)}(t)\,z'(t)\right]+z'(t) \\ &= \phi(t)\left[r^{1/(p-1)}(t)\,z'(t)\right]', \end{aligned}$$

which with (20) gives

$$\begin{aligned} 0 \geq &\left(r^{1/(p-1)}(t)\,\phi^2(t)\left(\frac{z(t)}{\phi(t)}\right)'\right)' \\ &+\frac{1}{(p-1)}\phi^{p-1}(t)\,H(t)\,z(g_{\max}(t)). \end{aligned}$$

Using Corollary 1 in [51], we get $w(t) = \frac{z(t)}{\phi(t)}$ is a positive solution of the equation

$$\begin{aligned} 0 = &\left(r^{1/(p-1)}(t)\,\phi^2(t)\,w'(t)\right)' \\ &+\frac{1}{(p-1)}H(t)\,\phi^{p-1}(t)\,\phi\left(g_{\max}(t)\right)w\left(g_{\max}(t)\right). \end{aligned}$$

This contradicts the assumptions of the theorem. Thus, the proof is complete.

Corollary 1. *Assume that (10) holds, and*

$$\liminf_{t\to\infty}\int_{g_{\max}(t)}^{t} H(u)\,\phi^{p-1}(u)\,\phi(g_{\max}(u)) \times\left(\int_{t_1}^{g_{\max}(u)}\frac{r^{-1/(p-1)}(s)}{\phi^2(s)}\mathrm{d}s\right)\mathrm{d}u > \frac{p-1}{\mathrm{e}} \quad (21)$$

for some $t_1 \ge t_0$. Then (1) is oscillatory.

Proof. Using Theorem 4 in [52], we find that condition (21) guarantees the oscillation of equation (18). From Theorem 3, we conclude that equation (1) is also oscillatory.

3.3 Examples and discussion

Here, we verify the efficiency of previous theorems in testing oscillation for a special case of the studied equation.

Example 1. Consider the second-order NDE

$$\left(t^p\left|z'(t)\right|^{p-2}z'(t)\right)' + q_0\sum_{i=1}^{m}\left|x(\lambda_i t)\right|^{p-2}x(\lambda_i t) = 0, \quad (22)$$

where $t \ge 1$

$$z(t) = x(t) + b_0 x(\delta_0 t),$$

$p \ge 2$, $q_0 > 0$, $\delta_0 \in (0,1)$, $\lambda_i \in (0,1)$, for $i = 1,2,\dots,m$. It is easy to see that $r(t) = t^p$, $b(t) = b_0$, $q(t) = q_0$, $\delta(t) = \delta_0 t$, $g_i(t) = g_i t$, and $g_{\max}(t) = \overline{\lambda}t$, where

$$\overline{\lambda} := \max\{\lambda_i,\ i = 1,2,\dots,m\}$$

Then, we have

$$\phi(t) = \int_t^\infty s^{-p/(p-1)}(s)\,\mathrm{d}s = \frac{(p-1)}{t^{1/(p-1)}},$$

and $\phi(t_0) < \infty$ (noncanonical case). Moreover,

$$H(t) = mq_0\left(1 - \frac{b_0}{\delta_0^{1/(p-1)}}\right)^{p-1} := H_0.$$

Now, we apply the theorem to obtain the oscillation criteria for equation (22).

For Theorem 1, we see that

$$\int_{t_0}^{\infty}\left(\frac{1}{r(u)}\int_{t_0}^{u} H(s)\,\phi^{p-1}(g_{\max}(s))\,\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}u$$
$$= \left(\frac{H_0(p-1)^{p-1}}{\overline{\lambda}}\right)^{1/(p-1)}\int_{t_0}^{\infty}\left(\frac{\ln u}{u^p}\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}u$$
$$= \left(\frac{H_0(p-1)^{p-1}}{\overline{\lambda}}\right)^{1/(p-1)}(p-1)^{1/(p-1)}\Gamma\left(\frac{1}{p-1}\right)$$
$$< \infty.$$

So condition (13) is not satisfied. Therefore, Theorem 1 fails to verify the oscillation of equation (22)

Using Theorem 2, we get

$$\limsup_{t\to\infty}\left(\phi^{p-1}(t)\int_{t_0}^{t} H(s)\,\mathrm{d}s\right)$$
$$= H_0(p-1)^{p-1}\limsup_{t\to\infty}\left(\frac{1}{t}\int_{t_0}^{t}\mathrm{d}s\right)$$
$$= H_0(p-1)^{p-1}.$$

Thus, condition (16) is satisfied if

$$H_0(p-1)^{p-1} > 1. \quad (23)$$

Now, for Corollary 1, we have

$$\liminf_{t\to\infty}\int_{\overline{\lambda}t}^{t} H(u)\,\phi^{p-1}(u)\,\phi\left(\overline{\lambda}u\right)\left(\int_{t_1}^{\overline{\lambda}u}\frac{r^{-1/(p-1)}(s)}{\phi^2(s)}\mathrm{d}s\right)\mathrm{d}u$$
$$= H_0\frac{(p-1)^p}{\overline{\lambda}^{1/(p-1)}}\liminf_{t\to\infty}\int_{\overline{\lambda}t}^{t}\frac{1}{u}\frac{1}{u^{1/(p-1)}}\left(\frac{1}{\phi\left(\overline{\lambda}u\right)} - \frac{1}{\phi(t_1)}\right)\mathrm{d}u$$
$$= H_0(p-1)^{p-1}\liminf_{t\to\infty}\int_{\overline{\lambda}t}^{t}\frac{1}{u}\mathrm{d}u$$
$$= H_0(p-1)^{p-1}\ln\frac{1}{\overline{\lambda}}.$$

Thus, condition (21) is satisfied if

$$H_0(p-1)^{p-1}\ln\frac{1}{\overline{\lambda}} > \frac{p-1}{\mathrm{e}}. \quad (24)$$

Hence, equation (22) is oscillatory if (23) or (24) hold.

Remark. To determine which criteria (23) and (24) are most efficient in testing the oscillation of equation (22), we apply them to the following special case:

$$\left(t^2\left[x(t) + \frac{1}{4}x\left(\frac{1}{2}t\right)\right]'\right)' + q_0\sum_{k=1}^{10} x\left(\frac{1}{k+1}t\right) = 0,$$

where $t \ge 1$ and $q_0 > 0$. We note that $\overline{\lambda} = 1/2$ and $m = 10$, and so $H_0 = 5q_0$. Thus, criteria (23) and (24) reduce to

$$q_0 > \frac{1}{5}$$

and

$$q_0 > \frac{1}{5\mathrm{e}\ln 2}$$

respectively. This means that condition (24) guarantees the oscillation of the equation

$$\left(t^2\left[x(t) + \frac{1}{4}x\left(\frac{1}{2}t\right)\right]'\right)' + \frac{1}{6}\sum_{k=1}^{10} x\left(\frac{1}{k+1}t\right) = 0,$$

while condition (23) fails.

4 Conclusion

The study of the oscillation of neutral differential equations depends mainly on the monotonic properties of the positive solutions as well as the relationships between the derivatives of these solutions. Therefore, improving these characteristics and relationships necessarily leads to improving oscillation crtieria. In this article, we investigated the monotonic properties of positive solutions to equation (1), which is one of the neutral equations with multiple delays. Then, we obtained three different forms of criteria that guarantee the oscillation of all solutions to the studied equation. Finally, we verified the efficiency of these criteria through application to a special case of the studied equation, and it became clear that Theorem 3 provides us with the most efficient criteria. It would be an interesting research point to use the improved relationships between the solution and its corresponding function, as in [53], to obtain new oscillation criteria for equation (1).

Conflicts of Interest

The author declares no conflict of interests.

References

[1] M. J. Beira and P. J. Sebastiào, A differential equations model-fitting analysis of covid-19 epidemiological data to explain multi-wave dynamics, *Scientific Reports* **11**(1) (2021).

[2] S. Kumar, P. K. Shaw, A.-H. Abdel-Aty and E. E. Mahmoud, A numerical study on fractional differential equation with population growth model, *Numerical Methods for Partial Differential Equations* **40**(1) (2020).

[3] E. E. Holmes, M. A. Lewis, J. E. Banks and R. R. Veit, Partial differential equations in ecology: Spatial interactions and population dynamics, *Ecology* **75**(1) (1994) 17–29.

[4] J. D. Lambert, *Numerical Methods for Ordinary Differential Systems* (Wiley-Blackwell, 1991).

[5] G. Teschl, *Ordinary Differential Equations and Dynamical Systems* (American Mathematical Soc., 2012).

[6] M. Braun, *Differential Equations and Their Applications* (Springer Science & Business Media, 1992).

[7] D. Bainov and D. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay* (CRC Press, 1991).

[8] R. P. Agarwal, S. R. Grace and D. O'Regan, *Oscillation Theory for Second Order Dynamic Equations* (CRC Press, Nov 2002).

[9] R. P. Agarwal, M. Bohner, T. Li and C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, *Applied Mathematics and Computation* **225** (2013) 787–794.

[10] R. Agarwal, S. R. Grace and D. O'Regan, *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations* (Springer Science & Business Media, 2013).

[11] O. Dosly and P. Rehak, *Half-Linear Differential Equations* (Elsevier, 2005).

[12] I. Győri and G. E. Ladas, *Oscillation Theory of Delay Differential Equations* (Clarendon Press, 1991).

[13] L. Erbe, *Oscillation Theory for Functional Differential Equations* (Routledge, 2017).

[14] C. Zhang, R. P. Agarwal, M. Bohner and T. Li, Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators, *Bulletin of the Malaysian Mathematical Sciences Society* **38**(2) (2014) 761–778.

[15] A. Muhib, On oscillation of second-order noncanonical neutral differential equations, *Journal of Inequalities and Applications* **2021**(1) (2021).

[16] T. Hassan, O. Moaaz, A. Nabih, M. Mesmouli and A. El-Sayed, New sufficient conditions for oscillation of second-order neutral delay differential equations, *Axioms* **10**(4) (2021) p. 281.

[17] B. Baculkov and J. Džurina, Oscillatory criteria via linearization of half-linear second order delay differential equations, *Opuscula Mathematica* **40**(5) (2020) 523–536.

[18] O. Bazighifan, Some new oscillation results for fourth-order neutral differential equations with a canonical operator, *Mathematical Problems in Engineering* **2020** (2020) 1–7.

[19] T. Li, B. Baculkov and J. Džurina, Oscillation results for second-order neutral differential equations of mixed type, *Tatra Mountains Mathematical Publications* **48**(1) (2011) 101–116.

[20] Z. Han, T. Li, C. Zhang and Y. Sun, Oscillation criteria for certain second-order nonlinear neutral differential equations of mixed type, *Abstract and Applied Analysis* **2011** (2011) 1–9.

[21] O. Moaaz, A. Muhib and S. S. Santra, An oscillation test for solutions of second-order neutral differential equations of mixed type, *Mathematics* **9**(14) (2021) p. 1634.

[22] R. P. Agarwal, M. Bohner, T. Li and C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, *Applied Mathematics and Computation* **225** (2013) 787–794.

[23] B. Almarri, H. Ramos and O. Moaaz, New monotonic properties of the class of positive solutions of even-order neutral differential equations, *Mathematics* **10**(9) (2022) p. 1470.

[24] O. Moaaz, B. Almarri, F. Masood and D. Atta, Even-order neutral delay differential equations with noncanonical operator: New oscillation criteria, *Fractal and Fractional* **6**(6) (2022) p. 313.

[25] O. Moaaz, C. Cesarano and A. Muhib, Some new oscillation results for fourth-order neutral differential equations, *European Journal of Pure and Applied Mathematics* **13**(2) (2020) 185–199.

[26] O. Moaaz, A. Muhib, T. Abdeljawad, S. S. Santra and M. Anis, Asymptotic behavior of even-order noncanonical neutral differential equations, *Demonstratio Mathematica* **55**(1) (2022) 28–39.

[27] J. Yan, Oscillations of higher order neutral differential equations of mixed type, *Israel Journal of Mathematics* **115**(1) (2000) 125–136.

[28] A. Muhib, T. Abdeljawad, O. Moaaz and E. M. Elabbasy, Oscillatory properties of odd-order delay differential equations with distribution deviating arguments, *Applied Sciences* **10**(17) (2020) p. 5952.

[29] J. Džurina, Oscillation theorems for second order advanced neutral differential equations, *Tatra Mountains Mathematical Publications* **48**(1) (2011) 61–71.
[30] O. Moaaz, J. Awrejcewicz and A. Muhib, Establishing new criteria for oscillation of odd-order nonlinear differential equations, *Mathematics* **8**(6) (2020) p. 937.
[31] H. Ramos, O. Moaaz, A. Muhib and J. Awrejcewicz, More effective results for testing oscillation of non-canonical neutral delay differential equations, *Mathematics* **9**(10) (2021) p. 1114.
[32] K. Gopalsamy, B. S. Lalli and B. G. Zhang, Oscillation of odd order neutral differential equations, *Czechoslovak Mathematical Journal* **42**(2) (1992) 313–323.
[33] R. Agarwal and S. Grace, Oscillation of higher-order nonlinear difference equations of neutral type, *Applied Mathematics Letters* **12**(8) (1999) 77–83.
[34] R. G. Koplatadze, Criteria for the oscillation of solutions of differential inequalities and second-order equations with retarded argument, *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **17** (1986) 104–121.
[35] Z. C. Wang, I. P. Stavroulakis and X. Z. Qian, A survey on the oscillation of solutions of first order linear differential equations with deviating arguments, *Appl. Math. E-Notes* **2** (2002) 171–191.
[36] R. Koplatadze, G. Kvinikadze and I. Stavroulakis, Oscillation of second order linear delay differential equations, *Funct. Differ. Equ.* **7**(1-2) (2000) 121–145.
[37] Y. G. Sun and F. W. Meng, Note on the paper of džurina and stavroulakis, *Applied Mathematics and Computation* **174**(2) (2006) 1634–1641.
[38] S. R. Grace and B. S. Lalli, Oscillation of nonlinear second order neutral delay differential equations, *Rad. Mat* **3**(1) (1987) 77–84.
[39] L. Erbe, T. S. Hassan and A. Peterson, Oscillation of second order neutral delay differential equations, *Advances in Dynamical Systems and Applications* **3**(1) (2008) 53–71.
[40] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillations of second order neutral delay differential equations, *Rad. Mat* **1**(2) (1985) 267–274.
[41] B. Baculkov and J. Džurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Computers & Mathematics with Applications* **62**(12) (2011) 4472–4478.
[42] S. Fišnarov and R. Mařk, Oscillation of second order half-linear neutral differential equations with weaker restrictions on shifted arguments, *Mathematica Slovaca* **70**(2) (2020) 389–400.
[43] S. R. Grace, J. Džurina, I. Jadlovsk and T. Li, An improved approach for studying oscillation of second-order neutral delay differential equations, *Journal of Inequalities and Applications* **2018**(1) (2018).
[44] J. Džurina and I. Jadlovsk, A note on oscillation of second-order delay differential equations, *Applied Mathematics Letters* **69** (2017) 126–132.
[45] Z. Han, T. Li, S. Sun and W. Chen, Oscillation criteria for second-order nonlinear neutral delay differential equations, *Advances in Difference Equations* **2010** (2010) 1–24.
[46] Z. Han, T. Li, S. Sun and Y. Sun, Remarks on the paper [appl. math. comput. 207 (2009) 388–396], *Applied Mathematics and Computation* **215**(11) (2010) 3998–4007.
[47] R. P. Agarwal, C. Zhang and T. Li, Some remarks on oscillation of second order neutral differential equations, *Applied Mathematics and Computation* **274** (2016) 178–181.
[48] T. Li, Z. Han, C. Zhang and S. Sun, On the oscillation of second-order emden-fowler neutral differential equations, *Journal of Applied Mathematics and Computing* **37**(1–2) (2011) 601–610.
[49] T. Li, Y. V. Rogovchenko and C. Zhang, Oscillation results for second-order nonlinear neutral differential equations, *Advances in Difference Equations* **2013**(1) (2013).
[50] M. Bohner, S. Grace and I. Jadlovsk, Oscillation criteria for second-order neutral delay differential equations, *Electronic Journal of Qualitative Theory of Differential Equations* (60) (2017) 1–12.
[51] T. Kusano and M. Naito, Comparison theorems for functional differential equations with deviating arguments, *Journal of the Mathematical Society of Japan* **33**(3) (1981).
[52] J. Džurina, Comparison theorem for third-order differential equations, *Czechoslovak Mathematical Journal* **44**(2) (1994) p. 357–366.
[53] O. Moaaz, C. Cesarano and B. Almarri, An improved relationship between the solution and its corresponding function in fourth-order neutral differential equations and its applications, *Mathematics* **11**(7) (2023) p. 1708.