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# Generalized Higher Derivations and a Center Invariance Problem

*O. H. Ezzat*[1,2]

[1]Department of Mathematics, College of Sciences, University of Bisha, Bisha 61922, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt

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**Abstract:** The centrally–extended generalized higher derivations concept is introduced. We show that in a semiprime ring without any nonzero central ideals, this concept is aligned with a generalized higher derivation. In addition, we investigate how this type of derivation affects our ring center. Examples are presented to explore such new families and guarantee that the theorems' conditions are not needless.

**Keywords:** Rings, central subsets, centrally-extended maps, generalized derivations, center invariance problem.

## 1 Introduction

Assume $\mathscr{R}$ is a ring and $\mathscr{Z}$ contains all its central elements. We say that a map $d:\mathscr{R}\to\mathscr{R}$ is a *derivation*, when for each $u_1,u_2\in\mathscr{R}$ we get $d(u_1+u_2)-d(u_1)-d(u_2)=0$ and $d(u_1u_2)-d(u_1)u_2-u_1d(u_2)=0$. We know that $0\in\mathscr{Z}$. So, if $d$ is a derivation, then $d(u_1+u_2)-d(u_1)-d(u_2)\in\mathscr{Z}$ and $d(u_1u_2)-d(u_1)u_2-u_1d(u_2)\in\mathscr{Z}$. Bell and Daif [1] called the map satisfies these two conditions a *centrally–extended derivation* and proved that it is equivalent to a derivation in case that the ring is semiprime without any nonzero central ideals. They investigated an invariance problem for $\mathscr{Z}$ under their new concept and showed that if there are no common elements between $\mathscr{Z}$ and the nilpotent elements set of $\mathscr{R}$, then $\mathscr{Z}$ is invariant under the effect of every centrally–extended derivation.

Suppose $f_1$ and $f_2$ are mappings of $\mathscr{R}$. An additive mapping $h$ is said to be an *$(f_1,f_2)$–derivation* if $h(u_1u_2)=h(u_1)f_1(u_2)+f_2(u_1)h(u_2)$ for all $u_1,u_2\in\mathscr{R}$. An additive mapping $g$ is said to be a *generalized $(f_1,f_2)$–derivation*, associated with an $(f_1,f_2)$–derivation $h$, if $g(u_1u_2)=g(u_1)f_1(u_2)+f_2(u_1)h(u_2)$ for all $u_1,u_2\in\mathscr{R}$. If we let $f_1=f_2=id_{\mathscr{R}}$, then we call $g$ a *generalized derivation* with a derivation $h$.

Motivated by Bell and Daif results, Tammam El-Sayiad [2] presented the new concept of *centrally–extended generalized $(f_1,f_2)$–derivation* and proved analogous results to [1]. There are a lot of manuscripts related to these mappings and their extensions (see [3–5]).

Assume that $\mathbb{N}_0$ is the nonnegative integers set, $\mathscr{D}=(d_i)_{i\in\mathbb{N}_0}$ is an additive family in $\mathscr{R}$ with $d_0=id_{\mathscr{R}}$. $\mathscr{D}$ is said to be a *higher derivation* when for $n\in\mathbb{N}_0, d_n(u_1u_2)=\sum_{i+j=n}d_i(u_1)d_j(u_2)$ fulfills, $u_1,u_2\in\mathscr{R}$.

Generalizing theorems related to derivations to the case of higher derivations is an interested line of research. We mention one of them, Ferrero and Haetinger [6] extended Herstein Jordan derivation related theorem [7] to higher derivations. (See also [8–11]).

In [12], Ezzat gave the definition of a *centrally–extended higher derivation* by a family $\mathscr{D}=(d_n)_{n\in\mathbb{N}_0}$ of maps $d_n$ on $\mathscr{R}$, in which $d_0=id_{\mathscr{R}}$ and for each $u_1,u_2\in\mathscr{R},n\in\mathbb{N}_0$, that $d_n(u_1+u_2)-d_n(u_1)-d_n(u_2)\in\mathscr{Z}$, and $d_n(u_1u_2)-\sum\limits_{i+j=n}d_i(u_1)d_j(u_2)\in\mathscr{Z}$.

A centrally–extended higher derivation is proven to be equivalent to a higher derivation in case of semiprime rings without nonzero central ideals. Also, he investigated the center preservation by centrally–extended higher derivations.

* Corresponding author e-mail: oahmed@ub.edu.sa, ohezzat@azhar.edu.eg

Inspired by results in [12] and [2], we present the new notion of a *centrally–extended generalized higher derivation*. Then prove its additive and multiplicative properties. Some related results to this new concept with the center invariance problem are also discussed.

## 2 Preliminaries

If we put $\beta = \alpha = id_{\mathscr{R}}$ in Theorem 3.1, Theorem 3.5 and Theorem 4.1 of [2], we can state the following lemmas

**Lemma 1.** *Any centrally–extended generalized derivation in an arbitrary ring $\mathscr{R}$, without any nonzero central ideals, is additive.*

**Lemma 2.** *Any centrally–extended generalized derivation in semiprime rings $\mathscr{R}$, without any nonzero central ideals, fulfils the multiplicative property.*

**Lemma 3.** *Assume the set $\mathscr{N}$ contains nilpotent elements of $\mathscr{R}$ and $\mathscr{Z} \cap \mathscr{N} = \{0\}$. Assume also $\mathscr{G}, \mathscr{H}$ are maps on $\mathscr{R}$ satisfying $\mathscr{G}(u_1u_2) - \mathscr{G}(u_1)u_2 - u_1\mathscr{H}(u_2) \in \mathscr{Z}$ and $\mathscr{G}(u_1u_2) - u_1\mathscr{G}(u_2) - \mathscr{H}(u_1)u_2 \in \mathscr{Z}$. Then $\mathscr{Z}$ is preserved by $\mathscr{G}$ and $\mathscr{H}$.*

The next definition presents the *centrally–extended generalized higher derivations* concept:

**Definition 1.** *A family $\mathscr{G} = (g_n)_{n\in\mathbb{N}_0}$ of maps $g_n : \mathscr{R} \to \mathscr{R}$, in which $g_0 = id_{\mathscr{R}}$, is called a* centrally–extended generalized higher derivation*, when there is a centrally–extended higher derivation $\mathscr{D} = (d_n)_{n\in\mathbb{N}_0}$ such that $g_n(u+v) - g_n(u) - g_n(v) \in \mathscr{Z}$ and $g_n(uv) - \sum_{i+j=n} g_i(u)d_j(v) \in \mathscr{Z}$ for each element $u$ and $v$ of $\mathscr{R}$ and $n \in \mathbb{N}_0$.*

In the next example, we assert that there is a generalized higher derivation but is not coincident with a centrally–extended generalized higher derivation.

*Example 1.* Suppose $\mathscr{A}$ is a nonzero central ideal of $\mathscr{R}$ and $\theta, \phi$ be two constant maps from $\mathscr{R}$ to $A$. Suppose $\mathscr{F}_1 = (f_n)_{n\in\mathbb{N}_0}$ is a generalized higher derivation of $\mathscr{R}$ with a higher derivation $\mathscr{D}_1 = (d_n)_{n\in\mathbb{N}_0}$. Now, assume we have two families $\mathscr{D}' = (d'_n)_{n\in\mathbb{N}_0}$ and $\mathscr{F}' = (f'_n)_{n\in\mathbb{N}_0}$ of maps defined on $\mathscr{R}$ by $d'_0 = id_{\mathscr{R}}$, $f'_0 = id_{\mathscr{R}}$, $d'_n(u_1) = d_n(u_1) + \theta(u_1)$ and $f'_n(u_1) = f_n(u_1) + \phi(u_1)$ for each $u_1 \in \mathscr{R}, n \in \mathbb{N}$. Then $\mathscr{F}'$ is not a generalized higher derivation. However, it is a centrally–extended generalized higher derivation associated with $\mathscr{D}'$.

## 3 Centrally–extended generalized higher derivations additivity

The centrally–extended generalized higher derivations additivity, for any ring without nonzero central ideals, is discussed in the next theorem.

**Theorem 1.** *Assume $\mathscr{G} = (g_n)_{n\in\mathbb{N}_0}$ is a centrally–extended generalized higher derivation of $\mathscr{R}$. If we do not allow $\mathscr{R}$ to have any nonzero central ideals, then $\mathscr{G}$ is additive.*

*Proof.* Suppose the family of mappings $\mathscr{D} = (d_n)_{n\in\mathbb{N}_0}$ of $\mathscr{R}$ is the associated centrally–extended higher derivation of $G$.

By [12, Theorem 3.1],we can deduce that $\mathscr{D}$ is additive. From Definition 1, we have $g_0 = id_{\mathscr{R}}$ is additive and by Lemma 1, $g_1$ is additive.

For $m$ such that $m < n$, suppose that $g_m$ is additive, i.e.

$$g_m(u_1 + u_2) = g_m(u_1) + g_m(u_2), \text{for all } u_1, u_2 \in \mathscr{R}. \quad (1)$$

We have for any two elements $u_1, u_2 \in \mathscr{R}$.

$$g_n(u_1 + u_2) - g_n(u_1) - g_n(u_2) \in \mathscr{Z}. \quad (2)$$

That means we may pick $a \in \mathscr{Z}$ and write

$$g_n(u_1 + u_2) = g_n(u_1) + g_n(u_2) + a. \quad (3)$$

Let $u_3 \in \mathscr{R}$. We get

$$g_n((u_1 + u_2)u_3) - \sum_{i+j=n} g_i(u_1 + u_2)d_j(u_3) \in \mathscr{Z}. \quad (4)$$

Picking another element $b \in \mathscr{Z}$ and write

$$\begin{aligned} g_n((u_1 + u_2)u_3) &= \sum_{i+j=n} g_i(u_1 + u_2)d_j(u_3) + b \\ &= \sum_{\substack{i+j=n \\ i\neq n, j\neq 0}} g_i(u_1 + u_2)d_j(u_3) \\ &\quad + g_n(u_1 + u_2)u_3 + b. \end{aligned} \quad (5)$$

By (3), we get

$$\begin{aligned} g_n((u_1 + u_2)u_3) = &\sum_{\substack{i+j=n \\ i\neq n, j\neq 0}} g_i(u_1 + u_2)d_j(u_3) \\ &+ \big(g_n(u_1) + g_n(u_2) + a\big)u_3 + b. \end{aligned} \quad (6)$$

Using (1), (6) can be rewritten as

$$\begin{aligned} g_n((u_1 + u_2)u_3) = &\sum_{\substack{i+j=n \\ i\neq n, j\neq 0}} g_i(u_1)d_j(u_3) \\ &+ \sum_{\substack{i+j=n \\ i\neq n, j\neq 0}} g_i(u_2)d_j(u_3) \\ &+ g_n(u_1)u_3 + g_n(u_2)u_3 + au_3 + b. \end{aligned} \quad (7)$$

In (7). Gathering (first term to third term) and (second term to fourth term) gives

$$\begin{aligned} g_n((u_1 + u_2)u_3) = &\sum_{i+j=n} g_i(u_1)d_j(u_3) + \sum_{i+j=n} g_i(u_2)d_j(u_3) \\ &+ au_3 + b. \end{aligned} \quad (8)$$

We arrive now to $g_n(u_1u_3+u_2u_3) = g_n(u_1u_3)+g_n(u_2u_3)+c$, where $c$ is a central element of $\mathscr{R}$. Further, we can pick another $d, e \in \mathscr{Z}$ such that

$$g_n(u_1u_3+u_2u_3) = \sum_{i+j=n} g_i(u_1)d_j(u_3)+d + \sum_{i+j=n} g_i(u_2)d_j(u_3)+e+c. \quad (9)$$

Subtracting (8) and (9) gives $d+e+c = au_3+b$. So, we have $au_3 \in \mathscr{Z}$ for each $u_3 \in \mathscr{R}$. Hence, $a\mathscr{R} \subseteq \mathscr{Z}$. But $a\mathscr{R}$ is a central ideal in $\mathscr{R}$ which is without any nonzero central ideals. Therefore $a\mathscr{R} = \{0\}$.

Now, we have $a$ is an element of the annihilator $\mathrm{Ann}(\mathscr{R})$ of $\mathscr{R}$. Hence, $a = 0$ and (3) simplified to $g_n(u_1+u_2) = g_n(u_1)+g_n(u_2)$ which proves the desired result.

## 4 Centrally–extended generalized higher derivations multiplicativity

We turn the direction into investigate the multiplicative property of centrally–extended generalized higher derivations. In the previous section we study the additivity nd require the ring to be without nonzero central ideals. Here we shall require the semiprimeness of $\mathscr{R}$.

**Theorem 2.** *Suppose that $\mathscr{G} = (g_n)_{n\in\mathbb{N}_0}$ is a centrally–extended generalized higher derivation of $\mathscr{R}$. $\mathscr{G}$ is a generalized higher derivation when $\mathscr{R}$ is a semiprime ring which does not contain any nonzero central ideals, then .*

*Proof.* Assume the associated centrally–extended higher derivation of $\mathscr{G}$ is $\mathscr{D} = (d_n)_{n\in\mathbb{N}_0}$.

By [12, Theorem 4.1], we get $\mathscr{D}$ is a higher derivation. Hence, we have

$$d_n(vw) = \sum_{i+j=n} d_i(v)d_j(w) \text{ for each } v, w \in \mathscr{R}. \quad (10)$$

By Lemma 2, $g_1$ is a generalized derivation.

We suppose now $g_m(uv) = \sum_{i+j=m} g_i(u)d_j(v)$, for each $u, v \in \mathscr{R}$, for each positive integer $m$ and $m < n$.

Let $u \in \mathscr{R}$, and $v \in \mathscr{R}$, we have

$$g_n(uv) = \sum_{i+j=n} g_i(u)d_j(v)+z_1, z_1 \in \mathscr{Z}. \quad (11)$$

Use (11) and (10) to obtain

$$\begin{aligned} g_n((uv)w) &= \sum_{i+j=n} g_i(uv)d_j(w)+z_2, z_2 \in \mathscr{Z} \\ &= g_n(uv)w + \sum_{\substack{i+j=n \\ i\neq n, j\neq 0}} g_i(uv)d_j(w)+z_2 \\ &= \sum_{i+j=n} g_i(u)d_j(v)w + z_1w \\ &\quad + \sum_{\substack{l+k+j=n \\ l+k\neq n, j\neq 0}} g_l(u)d_k(v)d_j(w)+z_2 \\ &= \sum_{l+k+j=n} g_l(u)d_k(v)d_j(w)+z_1w+z_2. \end{aligned} \quad (12)$$

and

$$\begin{aligned} g_n(u(vw)) &= \sum_{i+j=n} g_i(u)d_j(vw)+z_3, z_3 \in \mathscr{Z} \\ &= \sum_{i+r+s=n} g_i(u)d_r(v)d_s(w)+z_3. \end{aligned} \quad (13)$$

From (12) and (13), $z_1w+z_2 = z_3$. Hence $z_1w \in \mathscr{Z}$ for each $w \in \mathscr{R}$. Then $z_1\mathscr{R} \subseteq \mathscr{Z}$. But $z_1\mathscr{R}$ is a central ideal in $\mathscr{R}$ which is without any nonzero central ideals. Therefore $z_1\mathscr{R} = \{0\}$.

Now, we have $z_1$ is an element of the annihilator $\mathrm{Ann}(\mathscr{R})$ of $\mathscr{R}$. Hence, $z_1 = 0$ and (11) reduced to

$$g_n(uv) = \sum_{i+j=n} g_i(u)d_j(v). \quad (14)$$

The following example which asserts that the ring must be without nonzero central ideals cannot be relaxed.

*Example 2.* Suppose $\mathscr{R}_1$ is a prime ring which is not commutative and admits a generalized higher derivation $\mathscr{H} = (h_n)_{n\in\mathbb{N}_0}$ with $\mathscr{D} = (d_n)_{n\in\mathbb{N}_0}$ as an associated higher derivation. Assume also that $\mathscr{R}_2$ is an integral domain. Let $\mathscr{T} = \mathscr{R}_1 \oplus \mathscr{R}_2$. Let $\mathscr{G} = (g_n)_{n\in\mathbb{N}_0}$ in which $g_n$ defined on $\mathscr{T}$, with $g_0 = id_{\mathscr{T}}$ and $g_n((u_1,u_2)) = (f_n(u_1), g(u_2))$, for all $u_1 \in \mathscr{R}_1$ and $u_2 \in \mathscr{R}_2$, such that $g : \mathscr{R}_2 \to \mathscr{R}_2$ is any mapping other than a generalized derivation. Then the semiprimeness of $\mathscr{T}$ is clear and $\mathscr{H}$ turns to be a proper centrally–extended generalized higher derivation. In addition to that, $\{0\} \oplus \mathscr{R}_2$ is a central ideal in $\mathscr{R}$.

## 5 The invariance of Z

Suppose that $\mathscr{E}$ is a subset in $\mathscr{R}$. A mapping $h : \mathscr{R} \to \mathscr{R}$ satisfying $h(\mathscr{E}) \subseteq \mathscr{E}$, is said to preserve $\mathscr{E}$. In the following result, we discuss the conditions in which centrally–extended generalized higher derivations preserve the center.

**Theorem 3.** *Suppose $\mathscr{O}$ is the set of all nilpotents in $\mathscr{R}$. Let $\mathscr{F} = (f_n)_{n\in\mathbb{N}_0}$ and $\mathscr{D} = (d_n)_{n\in\mathbb{N}_0}$ be two families of maps of $\mathscr{R}$ with $f_0 = d_0 = id_R$, $f_n(uv) - \sum_{i+j=n} f_i(u)d_j(v) \in \mathscr{Z}$,* *and* $f_n(uv) - \sum_{i+j=n} d_i(u)f_j(v) \in \mathscr{Z}$ *for all* $u,v \in \mathscr{R}, n \in \mathbb{N}_0$. *If* $\mathscr{N} \cap \mathscr{Z} = \{0\}$*, then* $\mathscr{F}$ *and* $\mathscr{D}$ *preserve* $\mathscr{Z}$.

*Proof.* A direct application of Lemma 3, $f_1$ and $d_1$ preserve $\mathscr{Z}$. Assume all $m < n$, satisfy $f_m$ preserves $\mathscr{Z}$. For an arbitrary element $z$ of the center and $u \in \mathscr{R}$, we get

$$f_n(uz) - \sum_{i+j=n} d_i(u)f_j(z) \in \mathscr{Z},$$

$$f_n(uz) - d_n(u)z - uf_n(z) - \sum_{\substack{i+j=n \\ i\neq n, j\neq n}} d_i(u)f_j(z) \in \mathscr{Z}, \quad (15)$$

and for $f_n(zu)$, we have

$$f_n(zu) - \sum_{i+j=n} f_i(z)d_j(u) \in \mathscr{Z},$$

$$f_n(zu) - f_n(z)u - zd_n(u) - \sum_{\substack{i+j=n \\ i\neq n, j\neq n}} f_i(z)d_j(u) \in \mathscr{Z}. \quad (16)$$

From (15) and (16), we get

$$[f_n(z), u] \in \mathscr{Z} \text{ for each } u \in \mathscr{R}. \quad (17)$$

Replace $u$ by $uf_n(z)$ in (17) to get

$$[[f_n(z), u]f_n(z), u] = 0. \quad (18)$$

Now $[f_n(z), u]^2 = 0$ for each $u \in \mathscr{R}$, so that $[f_n(z), u]$ is a common element in the center and $\mathscr{O}$. So, $[f_n(z), u] = 0$ for all $u \in \mathscr{R}$. We can prove a similar result for $\mathscr{D}$.

## Data Availability Statement

No data were used to support the findings of this study.

## Conflicts of Interest

The author declares no conflict of interests.

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