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Classical and Bayesian techniques for modelling engineering dataset using new generalized probability distribution with mathematical features

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Abstract: This paper delves into the investigation of a novel continuous distribution, aiming to provide a thorough understanding of its various fundamental properties. The analysis encompasses an exploration of quantiles, skewness, kurtosis, hazard rate function, moments, incomplete moments, mean deviations, coefficient of variation, mean time to failure, mean time between failure, availability, and reliability functions within the context of consecutive linear and circular systems. Both maximum likelihood and Bayesian methods are employed for parameter estimation to ensure a comprehensive approach. The performance of the estimators is rigorously evaluated through a detailed simulation study, which meticulously considers bias and mean square error metrics. Furthermore, the significance of the new distribution is substantiated through the analysis of real-world datasets, offering practical insights into its applicability and potential advantages in various scenarios. This comprehensive approach not only contributes to the understanding of the distribution itself but also provides valuable guidance for its practical implementation and utilization in statistical modeling and analysis.

Keywords: Statistical modeling, Hazard rate analysis, Sequential systems, Estimation via maximum likelihood, Bayesian estimation, Simulated experimentation, Statistical analysis and numerical results.

1 Introduction

In recent decades, numerous generalized distributions have been developed through various modification methods. These methods involve adding one or more parameters to a base model to enhance its adaptability in modeling real lifetime data. With advancements in computing technology, many of these techniques have become more accessible, even when analytical solutions are complex. The term ”distribution” holds different meanings across various fields, including mathematics, science, technology, computer science, and economics. In mathematics, distributions refer to generalized functions used to formulate solutions for partial differential equations, while in probability, distributions represent the likelihood of specific values or value ranges of a variable, with cumulative distribution functions indicating the probability of values not exceeding a certain value. In science, species distribution refers to the spatial arrangement of a species, while distribution in pharmacology pertains to the movement of a drug within the body. In technology and computer science, electric power distribution denotes the final stage of delivering electricity, distributed computing involves the coordinated use of physically dispersed computers for tasks or storage, software distribution refers to pre-compiled and configured software bundles, and digital distribution involves the digital publishing of media. In economics, income or output distribution concerns the allocation of resources among individuals or factors of production, distribution in kind involves the transfer of non-cash assets from a company to a shareholder, and distribution resource planning is a method used in business administration for planning orders within a supply chain. Many statisticians are interested in developing and discovering a new distribution that possess certain properties enabling them to predict and describe the different types of lifetime data set. For more details, someone can see Altun et al. [1], Handique et al. [2],

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Shah et al. [3], Morgenthaler [4], Kafadar [5], Reyes et al. [6], Almetwally et al. [7], Lange et al. [8], Ahmad et al. [9], Ahmed et al. [10], Khan et al. [11], Kashid and Kulkarni [12], Eliwa et al. ([13], [14]), Rogers and Tukey [15], Alotaibi et al. [16], Ferede et al. [17], Jehhan et al. [18], Eldeeb et al. [19], Afifyet al. [20], Ahmad et al. [21], Alizadeh et al. [22], El-Morshedy [23], Jamshidian [24], Eliwa and Ahmed [25], Haj Ahmad et al. [26], among others. This paper introduces and examines a novel and flexible extension of the Weibull distribution. The proposed model is derived using a recent distribution generator proposed by Eliwa et al. [27] known as the exponentiated odd Chen-H family of distributions. In this class, the cumulative distribution function (CDF) of the generator can be defined as follows:

$$F(x;\alpha,\beta,\theta,\phi)=\left[1-e^{-\alpha\left[e^{\left(\frac{H(x;\phi)}{1-H(x;\phi)}\right)^{\beta}}-1\right]}\right]^{\theta};\ x>0, \tag{1}$$

where $H(x;\phi)$ represents the CDF of a baseline model with parameter vector ϕ and $\alpha,\beta,\theta>0$ are shape parameters. The EOChW distribution is introduced based on the following motivations: Firstly, it aims to define new model that encompass all types of hazard rate functions (HRF). By including a wide range of HRFs, the EOChW distribution offers greater flexibility in modeling various types of data. Secondly, the EOChW distribution has different shapes, including symmetric, left-skewed, and right-skewed distributions. This allows for a more comprehensive representation of data patterns and characteristics. Thirdly, the EOChW distribution can effectively model under- or over-dispersed datasets. This is particularly useful when dealing with data that deviates from the assumptions of traditional distribution models, allowing for more accurate and realistic modeling. Lastly, the EOChW distribution strives to consistently provide better fits to data compared to other models based on the Weibull distribution, making it a valuable tool for statistical analysis. The rest of the paper is outlined as follows: Section 2 provides the definition of the EOChW distribution. Following that, Section 3 presents statistical and reliability properties associated with this proposed family. In Section 4, the model parameters are estimated using both maximum likelihood and Bayesian methods, and the performance of these estimators is evaluated through simulations. Furthermore, Section 5 demonstrates the versatility of the EOChW distribution by analyzing a real dataset. Finally, Section 6 concludes the paper.

2 The EOChW Distribution: A Mathematical Framework

Consider the CDF of the Weibull distribution is given by

$$H(x)=1-e^{-\left(\frac{x}{b}\right)^{a}};x>0,a,b>0, \tag{2}$$

where a is shape parameter and b is scale parameter. Then, the CDF of the EOChW distribution is given by using Equations (2) in Equation (1) as follows

$$F(x;a,b,\theta,\alpha,\beta)=\left[1-e^{-\alpha\left(e^{\left(e^{\left(\frac{x}{b}\right)^{a}}-1\right)^{\beta}}-1\right)}\right]^{\theta}, \tag{3}$$

where $x>0$ and a,θ,α,β are shape parameters and b is scale parameter. The corresponding PDF of Equation (3) is given as

$$f(x)=\frac{\theta\alpha\beta a}{b}\left(\frac{x}{b}\right)^{a-1}e^{\left(\frac{x}{b}\right)^{a}}\left(e^{\left(\frac{x}{b}\right)^{a}}-1\right)^{\beta-1}\times e^{\left(e^{\left(\frac{x}{b}\right)^{a}}-1\right)^{\beta}}e^{-\alpha\left(e^{\left(e^{\left(\frac{x}{b}\right)^{a}}-1\right)^{\beta}}-1\right)} \tag{4}$$

$$\times\left[1-e^{-\alpha\left(e^{\left(e^{\left(\frac{x}{b}\right)^{a}}-1\right)^{\beta}}-1\right)}\right]^{\theta-1}. \tag{5}$$

The reliability and the hazard rate functions of the EOChW distribution can be calculated by using the formulas $R(x)=1-F(x)$ and $h(x)=f(x)/(1-F(x))$.

Figure 1 shows the PDFs and HRFs of the EOChW for various values of the parameters.

Based on the observations depicted in Figure 1, it is evident that the PDF of the EOChW distribution exhibits diverse shapes corresponding to different parameter values. It can be utilized for analyzing unimodal datasets with asymmetry-shaped. Additionally, the hazard rate function (HRF) displays various patterns, including decreasing, increasing, bathtub, or J-shaped. These findings highlight the versatility of the EOChW distribution in effectively modeling a wide range of data types.

3 Statistical Properties: Characteristics

3.1 Quantile function, skewness, and peakedness

For any $u\in(0,1)$, the qth quantile function $Q(u)$ of the EOChW is the solution of $F(Q(u))=u;\ Q(u)>0$. The

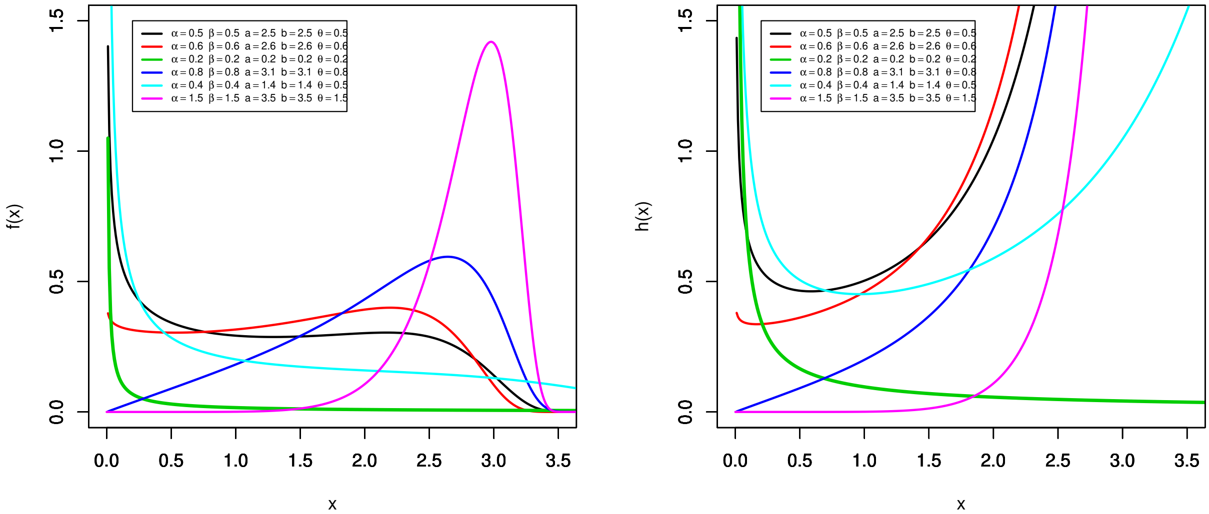

Figure 1. The plots of the PDFs (left panel) and HRFs (right panel) of the EOChW distribution for various values of the parameters.

simplified form can be written as

$$Q(u) = b\left[\log\left(1+\left\{\log\left(1-\log\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}\right)\right\}^{\frac{1}{\beta}}\right)\right]^{\frac{1}{a}},$$

Setting $q = 0.5$, we get the median of the EOChW distribution. The effects of the shape parameters on the skewness and kurtosis can be studied by using quantile function. The Bowley skewness (see, Kenney and Keeping [28]) is one of the earliest skewness measures defined by $S = \frac{Q(\frac{3}{4})+Q(\frac{1}{4})-2Q(\frac{1}{2})}{Q(\frac{3}{4})-Q(\frac{1}{4})}$. The Moors kurtosis (see, Moors [29]) is based on octiles, namely $K = \frac{Q(\frac{3}{8})-Q(\frac{1}{8})+Q(\frac{7}{8})-Q(\frac{5}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}$. The skewness and kurtosis of the EOChW distribution for selected choices of α, β, a and b as function of θ (theta) are displayed in Figure 2. We take $\beta = 1.1, a = 2$ and $b = 0.5$ to plot of Bowley skewness and Moors kurtosis.

The plots of skewness and kurtosis reveal that the shapes of the proposed distribution have strong dependence on the values of α and θ. Moreover, the EOChW distribution can be employed to model both positive and negative skewness, in addition to accommodating symmetric datasets across different forms of kurtosis.

3.2 Moments and incomplete moments

Moments and incomplete moments are employed in statistical analysis for characterizing the shape and variability of probability distributions. The rth moment $(\mu_r^{'})$ of the exponentiated odd chen-G family of distributions are defined by Eliwa et. al. [27] in the following form

$$\mu_r^{'} = \sum_{i,j=0}^{\infty}\sum_{k=0}^{j}\sum_{m,l=0}^{\infty} \Omega_{i,j,k}^{(m,l)} \mathbf{E}(Z_{\beta m+l}^{r}), \tag{6}$$

where $Z_{\beta m+l}^{r}$ has exponetial-G $(Exp-G)$ family with power parameter $\beta m+l$ that defined as

$$H_{\beta m+l}(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} e^{\left(\frac{x}{b}\right)^{a}\beta m+l}.$$

in addition,

$$\Omega_{i,j,k}^{(m,l)} = (-1)^{i+j+k}(\alpha i)^{j}(j-k)^{m}\binom{j}{k} \times \frac{\Gamma(\beta m+l)\Gamma(\theta+1)}{i!\ j!m!\ l!\ \Gamma(\beta m)\Gamma(\theta+1-i)}.$$

By setting $r = 1,2,3,4$ in Equation (6), we obtain the first four moments of the random variable X. For lifetime models, it is also of interest to obtain the incomplete moments. The incomplete moments play an important role for measuring inequality. The mth incomplete moment of X can be expressed as follows

$$M_{(m)}(t) = \sum_{i,j=0}^{\infty}\sum_{k=0}^{j}\sum_{m,l=0}^{\infty} \Omega_{i,j,k}^{(m,l)} M_{(m)}^{*}(t), \tag{7}$$

where $M_{(m)}^{*}(t) = \int_0^t x^m g_{\beta m+l}(x)dx$. Moreover, the primary utilization of the first incomplete moment is commonly associated with the Bonferroni and Lorenz curves. These curves find applications across various domains, including but not limited to reliability, economics, demography, medicine, and insurance.

3.3 Mean deviations and coefficient of variation

In the field of statistics, the mean deviations about the mean and median are utilized to quantify the dispersion or scatter within a population. When considering a random variable X following the EOChW $(a,b,\theta,\alpha,\beta)$ distribution, the mean deviations about the mean and median can be mathematically represented as follows:

$$\begin{aligned}\varepsilon_1 &= \int_0^\infty \left|x-\mu_1'\right| f(x;a,b,\theta,\alpha,\beta)dx \\ &= 2\mu_1' F(\mu_1';a,b,\theta,\alpha,\beta) - 2\mu_1' \\ &+ 2\int_{\mu_1'}^\infty x f(x;a,b,\theta,\alpha,\beta)dx \\ &= 2\mu_1' F(\mu_1') - 2M_{(1)}(\mu_1') \end{aligned} \quad (8)$$

and

$$\begin{aligned}\varepsilon_2 &= \int_0^\infty |x-Q(0.5)| f(x;a,b,\theta,\alpha,\beta)dx \\ &= -\mu_1' + 2\int_{Q(0.5)}^\infty x f(x;a,b,\theta,\alpha,\beta)dx \\ &= \mu_1' - 2M_{(1)}(Q(0.5)), \end{aligned} \quad (9)$$

respectively. Additionally, the coefficient of variation ($\mathbb{CV}$) serves as a metric to assess the variability within a dataset. Specifically, if X follows the $EOChW(a,b,\theta,\alpha,\beta)$ distribution, the $\mathbb{CV}$ can be expressed as $\mathbb{CV} = \sqrt{\mu_2' - \mu_1'^2}/\left|\mu_1'\right|$. A high $\mathbb{CV}$ value indicates a greater degree of variability and lower stability within the data, while a low $\mathbb{CV}$ value suggests lower variability and higher stability.

3.4 Mean time to failure (MTTF), mean time between failure (MTBF) and availability (AvB)

MTTF, MTBF and AvB are reliability terms based on methods and procedures for lifecycle predictions for a product. Customers often must include reliability data when determining what product to buy for their application. MTTF, MTBF and AvB are ways of providing a numeric value based on a compilation of data to quantify a failure rate and the resulting time of expected performance. Also, In order to design and manufacture a maintainable system, it is necessary to predict the MTTF, MTBF and AvB. If $X \sim EOChW\ (a_1,b_1,\theta_1,\alpha_1,\beta_1)$, then the MTBF is given as

$$\text{MTBF} = \frac{-x}{\ln(1-F(x;a_1,b_1,\theta_1,\alpha_1,\beta_1))};\ x>0. \quad (10)$$

If $X \sim EOChW\ (a_2,b_2,\theta_2,\alpha_2,\beta_2)$, then the MTTF is given as

$$\text{MTTF } = \mathbf{E}(X) = \mu_1'|_{(a_2,b_2,\theta_2,\alpha_2,\beta_2)}, \quad (11)$$

where μ_1' denotes the first moment around zero, which can be derived from Equation (**??**) when $r=1$. The AvB is consider the probability that the component is successful at time x, i.e.

$$\text{AvB} = -\mu_1'|_{(a_2,b_2,\theta_2,\alpha_2,\beta_2)} \frac{\ln(1-F(x;a_1,b_1,\theta_1,\alpha_1,\beta_1))}{x}, \quad (12)$$

where AvB $=\frac{\text{MTTF}}{\text{MTBF}}$.

3.5 Reliability function of consecutive $k^* - out - of - n^*$: fails system

Redundancy is employed in the design process as a means to enhance the reliability of systems. A system that follows the consecutive $k^* - out - of - n^*$: fails ($k^* - out - of - n^* : F$) with total n^* components if and only if at least k^* consecutive components in the system have failed (see Chang et al. [30]). These systems are comprised of n^* components arranged either linearly (in a line) or circularly (in a circle), and they fail or function based on the occurrence of a specific pattern involving consecutively failed or working components. Such systems serve as models for various engineering systems, including microwave stations within a telecommunications network, oil pipeline systems, and vacuum systems in electron accelerators.

3.5.1 Reliability function for parallel and series systems

Let's consider a system comprising n^* independent components, where each component follows the EOChW distribution. When k^* equals n^*, the reliability of the parallel system (P) can be determined as follows:

$$R_P(x) = 1 - \left[1-\exp\left(-\alpha\left[e^{\left(e^{\left(\frac{x}{b}\right)^a}-1\right)^\beta}-1\right]\right)\right]^{\theta n^*}. \quad (13)$$

When $k^*=1$, the reliability of the series system (S) is given by

$$R_S(x) = \left[1-\left[1-\exp\left(-\alpha\left[e^{\left(e^{\left(\frac{x}{b}\right)^a}-1\right)^\beta}-1\right]\right)\right]^\theta\right]^{n^*}. \quad (14)$$

3.5.2 Reliability function of linear consecutive $k^* - out - of - n^* : F$ system

Assume $T \sim EOChW\ (a,b,\theta,\alpha,\beta)$, then the reliability formula for linear consecutive $k^* - out - of - n^* : F$ system, say $R_L(t;k^*,n^*)$, is given by

$$R_L(t) = \sum_{j^*=0}^{m^*}\sum_{i^*=0}^{j^*}(-1)^{i^*}N_L(j^*;k^*,n^*)\binom{j^*}{i^*}\times \left[1-\exp\left(-\alpha\left[e^{\left(e^{\left(\frac{x}{b}\right)^a}-1\right)^\beta}-1\right]\right)\right]^{\theta(n^*-j^*+i^*)}, \quad (15)$$

with

$$m^* = \begin{cases} n^* - \left\lfloor \frac{n^*+1}{k^*} \right\rfloor - 1 & ;\text{if } n^*+1 \text{ is amultiple of } k^* \\ n^* - \left\lfloor \frac{n^*+1}{k^*} \right\rfloor & ;\text{if } n^*+1 \text{ is not amultiple of } k^* \end{cases}$$

and

$$N_L(j^*;k^*,n^*) = \begin{cases} \sum_{l^*=0}^{\lfloor j^*/k^* \rfloor}(-1)^{j^*}\binom{n^*-j^*+1}{l^*} \times \binom{n^*-l^*k^*}{n^*-j^*} & ;k^* \le j^* \le n^* \\ 0 & ;\ j^* > m^* \\ \binom{n^*}{j^*} & ;\ 0 \le j^* \le k^*-1, \end{cases}$$

where $N_L(j^*;k^*,n^*)$ denotes the count of possible arrangements of j^* failed components in a linear configuration, ensuring that no more than k^*-1 failed components occur consecutively. Additionally, m^* represents the maximum allowable number of failed components in the system without triggering a complete system failure.

3.5.3 Reliability function of circular consecutive $k^* - out - of - n^* : F$ system

Assume $T \sim EOChW\ (a,b,\theta,\alpha,\beta)$, then the reliability formula for circular consecutive $k^* - out - of - n^* : F$ system, say $R_C(t;k^*,n^*)$, is given by

$$R_C(t) = \sum_{j^*=0}^{d}\sum_{i^*=0}^{j^*}(-1)^{i^*}N_C(j^*;k^*,n^*)\binom{j^*}{i^*}\times \left[1-\exp\left(-\alpha\left[e^{\left(e^{\left(\frac{x}{b}\right)^a}-1\right)^\beta}-1\right]\right)\right]^{\theta(n^*-j^*+i^*)} \quad (16)$$

with

$$d = \begin{cases} n^* - \frac{n^*}{k^*} & ;\text{if } n^* \text{ is amultiple of } k^* \\ n^* - \left\lfloor \frac{n^*}{k^*} \right\rfloor - 1 & ;\text{if } n^* \text{ is not amultiple of } k^*, \end{cases}$$

and

$$N_C(j^*;k^*,n^*) = \frac{n^*}{n^*-j^*}N_L(j^*;k^*,n^*) \ \text{ for } 0 \le j^* \le d,$$

where $N_C(j^*;k^*,n^*)$ denotes the count of possible arrangements of n^* components, including j^* failed ones, in a circular configuration, ensuring that no more than k^*-1 failed components occur consecutively. Additionally, d represents the maximum allowable number of failed components in the system without triggering a complete system failure.

3.5.4 Reliability measures for different systems

Consider four distinct systems, namely P, S, Lc, and Cc. Each of these systems is comprised of five components, where each component follows the EOChW$(0.2,0.5,\theta,0.5,1.1)$ distribution. In the case of the Lc and Cc systems, let's assume a $2-out-of-6:F$ configuration. Tables 1 and 2 present various computational statistics for these systems at a specific time point, $x=5$.

Based on Tables 1 and 2, with fixed values of α,β,a and b and as θ approaches infinity, the following trends emerge: in P and S systems, reliability, MTTF, MTBF, and AvB values exhibit an increase, while in Lc and Cc systems, these values experience a decrease.

4 Estimation Methods: Mathematical Formulations and Algorithms

4.1 Maximum likelihood estimation (MLE)

Numerous methods are available for parameter estimation, with the MLE method being the most prevalent. Hence, we employ the MLE method to estimate the parameters a,b,θ,α and β for the EOChW distribution. Suppose $X_1,X_2,...,X_n$ is a random sample of size n from the EOChW distribution. In this case, the log-likelihood function $L(a,b,\theta,\alpha,\beta)$ can be formulated as follows:

Table 1. Evaluation of reliability measures for P and S systems utilizing the EOChW model.

Parameter ↓ System →	P				S			
θ	R_p	MTTF	MTBF	AvB	R_S	MTTF	MTBF	AvB
1.5	0.4135	5.5615	5.6625	0.9821	3.79×10^{-7}	0.2372	0.3382	0.7013
5	0.8311	26.9374	27.0384	0.9962	0.0003	0.5115	0.6125	0.8351
10	0.9714	172.8020	172.9030	0.9994	0.0080	0.9348	1.0358	0.9024
15	0.9951	1036.3971	1036.4981	0.9999	0.0418	1.4738	1.5748	0.9358

Table 2. Evaluation of reliability measures for Lc and Cc systems utilizing the EOChW model.

Parameter ↓ System →	Lc				Cc			
θ	R_L	MTTF	MTBF	AvB	R_C	MTTF	MTBF	AvB
1.5	0.8697	35.7297	35.8307	0.9971	0.6958	13.6849	13.7859	0.9926
5	0.6725	12.5029	12.6039	0.9919	0.5380	7.9655	8.0665	0.9874
10	0.3726	4.9642	5.0652	0.9800	0.2981	4.0303	4.1313	0.9755
15	0.1816	2.8306	2.9316	0.9655	0.1453	2.4914	2.5924	0.9610

$$
\begin{aligned}
L(a,b,\theta,\alpha,\beta) = {} & n\ln\left(\frac{a\alpha\beta\theta}{b}\right) + (a-1)\sum_{i=1}^{n}\ln\left[\frac{x_i}{b}\right] \\
& + \sum_{i=1}^{n}\left[\frac{x_i}{b}\right]^{a} + (\beta-1)\sum_{i=1}^{n}\ln\left[e^{\left(\frac{x_i}{b}\right)^{a}}-1\right] \\
& - \alpha\sum_{i=1}^{n}\ln\left[e^{\left(e^{\left(\frac{x_i}{b}\right)^{a}}-1\right)^{\beta}}-1\right] \\
& + (\theta-1)\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(e^{\left(e^{\left(\frac{x_i}{b}\right)^{a}}-1\right)^{\beta}}-1\right)}\right].
\end{aligned} \tag{17}
$$

To estimate the unknown parameters a,b,θ,α and β, we take the partial derivative of $L(a,b,\theta,\alpha,\beta)$ with respect to a,b,θ,α and β, and equate the result equation to 0. The solution can be reported by utilizing any numerical method such as the Newton-Raphson method using some R packages.

4.1.1 Simulation study using MLE methouds

In this subsection, we assess the efficacy of the MLE method relative to the sample size through a simulation study. To execute the simulation study, we follow these steps:

1. Generate 10000 samples of size $n = 20,100,250,350,450$ from Schema I: EOChW$(0.5,0.6,0.7,1.4,0.9)$, Schema II: EOChW $(0.9,0.8,0.7,0.9,0.8)$ and Schema III: EOChW$(1.5,1.6,1.7,0.6,0.7)$.
2. Calculate the MLEs for the 10000 samples, say $\widehat{a}_j,\widehat{b}_j,\widehat{\theta}_j,\widehat{\alpha}_j$ and $\widehat{\beta}_j$ for $j=1,2,...,10000$.

Calculate the biases and mean-squared errors (MSEs)., where

$$|\text{Bias}| = \left|\frac{1}{10000}\sum_{j=1}^{10000}\left(\widehat{\varphi}_j-\varphi\right)\right|,$$

and

$$MSE = \frac{1}{10000}\sum_{j=1}^{10000}\left(\widehat{\varphi}_j-\varphi\right)^2.$$

The results for this study are listed in Table 3.

Bias refers to the difference between the expected value of an estimator or model and the true value of the parameter being estimated or predicted. In simpler terms, it tells us if our estimator consistently overestimates or underestimates the true value. A biased estimator tends to consistently deviate from the true value in one direction. For example, if you have a biased scale that consistently adds 2 pounds to every measurement, it's biased upwards. Mean squared error (MSE), on the other hand, is a measure of the average squared difference between the estimated or predicted values and the true values. It takes into account both bias and variance (the variability of the estimator's predictions). A lower MSE indicates that the estimator or model is closer, on average, to the true value. However, MSE can be decomposed into bias squared and variance, which means reducing bias might increase variance and vice versa. In practical terms, bias and MSE are crucial in assessing the accuracy and reliability of statistical models or estimators. A model with low bias and low MSE is generally preferred because it indicates that the model is both accurate (low bias) and consistent (low variance). However, finding this balance can sometimes be challenging, as reducing bias can increase variance and vice versa, leading to a trade-off that needs to be carefully managed in model development and selection. Finding an estimator that balances consistency and low variance often involves trade-offs. For example, increasing the complexity of a model or estimator may reduce bias (improving consistency) but can also increase variance. On the other hand, simplifying the model can reduce variance but may introduce bias. In practice, techniques like regularization (which penalizes complexity in models) can help control variance while maintaining consistency to some extent. Cross-validation methods can also be used to assess the stability and generalization performance of an estimator, helping to select models that have a good balance between consistency and variance.

By examining Table 3, one can notice that the magnitude of the bias tends to zero as n approaches infinity. In addition, as n goes toward infinity, MSEs continually decrease to zero, demonstrating the consistency of the estimators. From these observations, we can conclude that the MLE method shows strong performance in accurately estimating model parameters.

4.2 Bayesian estimation (BSE)

This section is concerned with the Bayesian analysis of EOChW distribution. In particular, The Bayesian analysis uses the Bayes theorem to combine the prior information with the observed information. It is to be noted that prior can be noninformative in the sense that it provides less accurate information than the informative prior. The likelihood of the EOChW distribution can be written as

$$\begin{aligned} L(a,b,\theta,\alpha,\beta|\mathbf{x}) \propto & a^n\alpha^n\beta^n\theta^n \exp\big(-a\sum_{i=1}^n \ln(b/x_i)\big) \\ & \times \exp\Bigg(\sum_{i=1}^n \big[(-1 \\ & +\exp(x_i/b)^a)^\beta + (x_i/b)^a\big]\Bigg) \\ & \times \exp\Bigg(\beta\sum_{i=1}^n \ln\big(-1 \\ & +\exp(x_i/b)^a\big)\Bigg)\prod_{i=1}^n\Bigg(-1 \\ & +\exp(x_i/b)^a\Bigg)^{-1}\exp\Bigg(\theta \\ & \times\sum_{i=1}^n \ln\big[1-\exp\big(\alpha-\alpha\exp\big(\\ & -1+\exp(x_i/b)^a\big)^\beta\big)\big]\Bigg) \\ & \times\prod_{i=1}^n\Bigg(-1+\exp\big(\alpha\big(-1+\exp\big(\\ & -1+\exp\big(x_i/b\big)^a\big)^\beta\big)\big)\Bigg). \end{aligned} \tag{18}$$

For the sake of simplicity, assuming the independent gamma prior for a, b, θ, α and β, i.e., $a \sim G(a_1,b_1), \beta \sim G(a_2,b_2), \theta \sim G(a_3,b_3), \alpha \sim G(a_4,b_4)$ and $b \sim G(a_5,b_5)$ the joint posterior can be written as

$$\begin{aligned} P(a,b,\theta,\alpha,\beta|\mathbf{x}) \propto & a^{n+a_1-1}\exp\Bigg(-a\big(b_1 \\ & +\sum_{i=1}^n \ln(b/x_i)\big)\Bigg)\beta^{n+a_2-1} \\ & \times\exp\Bigg(-\beta\big(b_2-\sum_{i=1}^n \ln\big(-1 \\ & +\exp(x_i/b)^a\big)\Bigg)\theta^{n+a_3-1}\exp\Bigg(\\ & -\theta\Bigg(b_3-\sum_{i=1}^n \ln\big[1-\exp\big(\alpha \\ & -\alpha\exp\big(-1 \\ & +\exp(x_i/b)^a\big)^\beta\big)\big]\Bigg)\Bigg) \\ & \alpha^{n+a_4-1}\exp(-b_4\alpha)\prod_{i=1}^n\Bigg(-1 \\ & +\exp\big(\alpha\big(-1+\exp\big(-1 \\ & +\exp(x_i/b)^a\big)^\beta\big)\big)\Bigg) \\ & b^{n+a_5-1}\exp(-b_5 b)\prod_{i=1}^n\Bigg(-1 \\ & +\exp(x_i/b)^a\Bigg)^{-1} \\ & \times\exp\Bigg(\sum_{i=1}^n\big[\big(\\ & -1+\exp(x_i/b)^a\big)^\beta+(x_i/b)^a\big]\Bigg). \end{aligned} \tag{19}$$

The marginal distributions can be obtained as follows: $p(a|b,\mathbf{x}) \sim Gamma\Big(m + a_1, b_1 + \sum_{i=1}^n \ln(b/x_i)\Big)$, $p(\beta|a,b,\mathbf{x}) \sim Gamma\Big(n + a_2, b_2 - \sum_{i=1}^n \ln\big(-1 + \exp(x_i/b)^a\Big)$, $p(\theta|\alpha,a,b,\beta,\mathbf{x}) \sim Gamma\Big(n + a_3, b_3 - \sum_{i=1}^n \ln\big[1 - \exp\big(\alpha - \alpha\exp\big(-1 + \exp(x_i/b)^a\big)^\beta\big)\big]\Big)$, $p(\alpha|a,b,\beta,\mathbf{x}) \sim \alpha^{n+a_4-1}\exp(-b_4\alpha)\prod_{i=1}^n\Big(-1 + \exp\big(\alpha\big(-1 + \exp\big(-1 + \exp\big(x_i/b\big)^a\big)^\beta\big)\big)\Big)$, and $p(b|a,b,\beta) \sim b^{n+a_5-1}\exp(-b_5 b)\prod_{i=1}^n\Big(-1 +$

Table 3. The bias and MSE for the EOChW parameters.

		Schema I		Schema II		Schema III	
n	Parameter	|Bias|	MSE	|Bias|	MSE	|Bias|	MSE
20	a	0.03736622	0.00784582	0.02758462	0.00401946	0.10474862	0.01493446
	b	0.05473891	0.00846821	0.02770475	0.00374581	0.08463961	0.00648368
	θ	0.09947628	0.00845892	0.01870428	0.00253745	0.11736845	0.00946864
	α	0.06638912	0.00983614	0.01801047	0.00240364	0.04437525	0.00868652
	β	0.05993782	0.00645783	0.01050478	0.00135528	0.06648732	0.00947649
100	a	0.02884578	0.00184668	0.02115738	0.00294384	0.07749362	0.00204761
	b	0.04745681	0.00385691	0.02520482	0.00360364	0.00946846	0.00073458
	θ	0.03957815	0.00210578	0.01750047	0.00231817	0.04438927	0.00304871
	α	0.01873869	0.00473871	0.01104386	0.00145483	0.00985693	0.00054937
	β	0.01047934	0.00202487	0.00701364	0.00080463	0.02049855	0.00439471
250	a	0.01893689	0.00094767	0.01824689	0.00244492	0.01936374	0.00084749
	b	0.01185468	0.00088465	0.02390469	0.00331045	0.00113947	0.00089887
	θ	0.00224867	0.00059782	0.01615394	0.00214283	0.00802371	0.00088463
	α	0.00773826	0.00079761	0.00440346	0.00051018	0.00438046	0.00009571
	β	0.00849729	0.00088496	0.00704378	0.00082142	0.00847936	0.00019487
350	a	0.00834662	0.00025376	0.01110478	0.00149794	0.00974973	0.00093783
	b	0.00846816	0.00024974	0.02024367	0.00279479	0.00044384	0.00006648
	θ	0.00084648	0.00004977	0.01610390	0.00213310	0.00195727	0.00022949
	α	0.00103876	0.00013391	0.00004186	0.00008368	0.00074946	0.00000759
	β	0.00214970	0.00009458	0.00601135	0.00071163	0.00043846	0.00003397
450	a	0.00063538	0.00004757	0.00947636	0.00008692	0.00204876	0.00018461
	b	0.00043558	0.00002947	0.00184681	0.00048671	0.00008476	0.00000849
	θ	0.00026286	0.00000838	0.00554892	0.00046871	0.00064538	0.00003297
	α	0.00066386	0.00004363	0.00000194	0.00000745	0.00009486	0.00000039
	β	0.00073937	0.00002836	0.00074037	0.00001937	0.00007489	0.00000643

$\exp\left(x_i/b\right)^a\Big)^{-1} \exp\left(\sum_{i=1}^{n}\left[\left(-1 + \exp(x_i/b)^a\right)^{\beta} + (x_i/b)^a\right]\right)$.

Thus, a,β,θ can easily be generated from the gamma distributions as mentioned above, however, α,b cannot be generated directly. To generate these we use the Markove Chain Monte Carlo (MCMC) method named as the Metropolis Hastings sampling. To this end, the parameter γ is initialised with state γ_i and draw next state $\gamma^{'}$ with probability density $p(\gamma^{'}|\gamma_i)$, which is known as the transition kernel. Here, gamma distribution is used as the transition distribution and this choice is done purely for illustration purpose, and any other suitable distribution can be taken. Then, compute $\psi = \min(\psi_1\psi_2, 1)$, where $\psi_1 = \frac{p(\gamma^{'})}{p(\gamma_i)}$, which is the probability ratio between present and last sample γ_i and $\psi_2 = \frac{p(\gamma_i|\gamma^{'})}{p(\gamma^{'}|\gamma_i)}$. A new random number is rejected if $\psi = 1$, $\gamma_{i+1} = \gamma^{'}$, i.e.,

$$\gamma_{i+1} = \begin{cases} \gamma^{'} & \text{with probability } \psi \\ \gamma_i & \text{with probability 1-}\psi. \end{cases}$$

Then, the process is repeated until it forgot the initial state. Thus, to have more precise numbers, some initial state values can be discarded known as the burn-in period. After generating the marginal densities, the next step is to calculate the posterior summaries, $\mathbb{E}(\boldsymbol{\phi}|\boldsymbol{x}) = \int_{\boldsymbol{\phi}} \boldsymbol{\phi}\mathbb{P}(\boldsymbol{\phi}|\boldsymbol{x})$. For this, we propose the following steps.

Step 1:Take some initial guess values of a,b,θ,α and β, say $a_0,b_0,\theta_0,\alpha_0$ and β_0, respectively.

Step 2:To generate α and b proceed as follows:

1.To generate α and b evaluate the acceptance probability by $p(\alpha^{(i)},\alpha^{'}) = \min\left(1, \frac{p(\alpha^{(')}|\boldsymbol{x})p(\alpha^{(i)}|\alpha^{(')})}{p(\alpha^{(i)}|\boldsymbol{x})p(\alpha^{(')}|\alpha^{(i)})}\right)$, where $p(\alpha|b,a,\beta,\boldsymbol{x})$ has been defined above.

2.Generate a random numbers u from $Uniform(0,1)$.

3.If $p(\alpha^{(i)},\alpha^{(')}) \geq u$, $\alpha^{(i+1)} = \alpha^{(')}$, otherwise $\alpha^{(i+1)} = \alpha^{(i)}$. Similarly, generate b.

Step 3:Now generate a,θ and β from $p_{a|b,\boldsymbol{x}}$, $p_{\beta|a,b,\boldsymbol{x}}$ and $p_{\theta|\alpha,a,b,\beta,\boldsymbol{x}}$. Suppose at the ith step, α, β, θ, a and b take the values a_i,b_i, α_i, θ_i and β_i.

Step 4:Repeat the above step N times.

Step 4:Calculate the Bayes estimator of $h(a,b,\theta,\alpha,\beta)$ by $\frac{1}{N-M}\sum_{i=M+1}^{N} h(a_i,b_i,\theta_i,\alpha_i,\beta_i)$, where M denote the number of burn-in sample.

To calculate the posterior summaries, we generated sample of different sizes like $n = 25,150,250,350$ assuming $(a,b,\theta,\alpha,\beta)$ as mentioned in the previous section for different distributions. Further, we considered two different types of priors, i.e., informative prior (IP) and noninformative prior (NIP). For the case of IP, we selected hyperparameters values that yield mean approximately equal to the nominal value of parameter with variance 0.5. However, to have the case of NIP, we considered the hyperparameters values which produce mean equal to the nominal value but variance 2.0. It is worth mentioning that in the case of IP and NIP, the means of the prior distributions are equal to the nominal value of the parameters but variances have different values, i.e., variance is larger than the case of IP for the NIP case. We generated 60000 MCMC samples and 10000 used as the burn-in period to have stable posterior summaries. The resulting study is tabulated in Tables 4 and 5. From the Tables 4 and 5, it is observed that although the posterior means are very close to the nominal parameter values, the standard deviation and MCMC error are very different for each case. It is also noticed that the standard deviation and MCMC error decrease by increasing the sample size and IP has the least standard deviation as compared to the NIP cases. Also, credible intervals for the IP are narrower than the NIP cases.

5 Analysis of Glass Fiber Data

In this section, we illustrate the empirical importance of the EOChW distribution using one complete real dataset. The fitted distributions are compared using some criteria namely, the negative maximized log-likelihood ($-L$), Cramér-von Mises (CvM), Anderson-Darling (AD) statistics, and Kolmogorov-Smirnov (KS) statistic and its p-value. The CvM and AD statistics, along with the KS statistic, serve as goodness-of-fit tests in statistical analysis, yet they differ in their approaches and sensitivity to specific types of deviations between observed and theoretical distributions. CvM and AD statistics focus on assessing discrepancies in the CDF of the data, with CvM being more sensitive to differences in the tails of the distribution, while AD provides increased sensitivity to deviations in the tails as well as in the center of the distribution. In contrast, the KS statistic measures the maximum vertical distance between the empirical distribution function of the sample and the theoretical distribution, making it sensitive to differences across the entire range of the distribution but less so in the tails compared to CvM and AD. Thus, while all three statistics serve similar purposes, their varying sensitivities make them suitable for different scenarios, with researchers often selecting the most appropriate test based on the specific characteristics of their data and the type of deviations they aim to detect. This data is reported in Smith and Naylor [31], which consists of 63 observations of the strengths of 1.5 cm glass fiber. Unfortunately, the units of measurement are not given in the paper. Various nonparametric visualizations, including QQ plots, box plots, histogram plots, TTT plots, violin plots, and kernel plots, were utilized to examine the inherent distribution of the raw fiber glass data. Based on these nonparametric plots, it was observed that the fiber glass data exhibited asymmetry and contained certain extreme values (refer to Figure 3). For this data set, we shall compare the fits of the EOChW distribution with some competitive models like odd log-logistic W (OLoLW), odd flexible Weibull W (OFWW), Topp-Leaon W (ToLeW), Kumaraswamy W (KuW), Gompertz W (GoW), transmuted W (TrW), generalized W (GW), exponentiated W (EW) and W. Tables 6 and 7 provide the MLEs for glass fiber data. and goodness-of-fit measures respectively. Regarding Table 7, it is clear that, the EOChW model provides the best fit to this data among all tested models because it has the smallest value among $-L$, CvM, AD, KS as well as it has the highest p-value. The empirical PDFs, CDFs and P-P plots for glass fiber data are displayed in Figures 4 and 5 respectively, which support the results of Table 7.

It is clear that the data set plausibly came from all tested models. But, the EOChW model is the best. In Table 8 the results of Bayesian estimation for glass fiber data are listed. The results presented in Table 8 are very similar to the MLE results. It is also noticed that the IP is more efficient than the NIP because of smaller associated standard deviation. The hyperparameters for the real data are selected in a similar way as described in the simulation study section, where the MLEs are treated as the nominal values. To assess the convergence visually, we depicted the density, history and trace plots in Figures 6, 7 and 8 respectively. Figure 9 shows the RF, HRF, AvB, MTTF and MTTR for glass fiber data using the EOChW model model based on ML estimators.

Some statistics for glass fiber data are reported in Table 9 using the EOChW model. The analysis reveals that the glass fiber data exhibits characteristics of under-dispersion, signifying that the variance is smaller than the mean. Additionally, the data displays a moderate left skewness, indicating a longer left tail with the majority of the distribution concentrated towards the right, accompanied by platykurtosis.

6 Concluding Reflections and Prospects

This study presents a comprehensive examination of the EOChW distribution, thoroughly exploring its mathematical and statistical characteristics. By investigating its hazard rate function, which exhibits diverse patterns including decreasing, increasing, bathtub, or J-shaped, the study underscores the EOChW distribution's adaptability in effectively modeling a wide array of data types. Moreover, the distribution demonstrates remarkable versatility by accommodating both positive and negative skewness, as well as symmetric datasets with varying forms of kurtosis. Parameter estimation techniques, including Bayesian and maximum likelihood methods, were employed to estimate model parameters, with simulation results showcasing the efficacy of both approaches. It was observed that although

Table 4. Posterior summaries for the EOChW distribution.

Prior	n	Parameters	Mean	SD	MC Error	2.5%	Median	97.5%
IP	50	a	0.60462784	0.71469564	0.00654485	0.00450463	0.35400485	2.58500475
		b	0.69410471	0.69560453	0.00653735	0.01590490	0.47874386	2.47402174
		θ	1.69806584	0.70731863	0.00695589	0.61331326	1.59400093	3.30300582
		α	1.51000475	0.71680474	0.00727494	0.44689642	1.39902178	3.22208478
		β	1.58502947	0.69671743	0.00621048	0.54518567	1.48400474	3.21603255
	150	a	0.60222746	0.71124956	0.00474639	0.00454405	0.35632153	2.55300583
		b	0.69580058	0.69468264	0.00509275	0.01692784	0.48140384	2.43604388
		θ	1.69806274	0.70465937	0.00510376	0.61432735	1.59805438	3.27409475
		α	1.50401048	0.71312947	0.00505257	0.44934865	1.39200484	3.19302477
		β	1.59506391	0.68585594	0.00469563	0.54680575	1.49103225	3.21204439
	250	a	0.60170464	0.70621047	0.00419852	0.00491325	0.35404689	2.54700485
		b	0.69787459	0.69365057	0.00403243	0.01734495	0.48153294	2.42600954
		θ	1.69800458	0.70455587	0.00415237	0.61470474	1.60009574	3.22703265
		α	1.50301184	0.71305875	0.00416355	0.44964264	1.39203226	3.19108004
		β	1.59500575	0.68430477	0.00380489	0.54849476	1.49009474	3.20200047
	350	a	0.60006483	0.70545478	0.00344343	0.00542625	0.35369474	2.54600463
		b	0.69810946	0.67010264	0.00369485	0.01881288	0.47921454	2.35501746
		θ	1.69905648	0.70422547	0.00360057	0.62050494	1.60206495	3.22604742
		α	1.50204654	0.71238465	0.00370575	0.44983326	1.39200487	3.19004447
		β	1.59506495	0.67320465	0.00350475	0.55289475	1.49005365	3.12284495

Table 5. Posterior summaries for the EOChW distribution.

Prior	n	Parameters	Mean	SD	MC Error	2.5%	Median	97.5%
NIP	50	a	0.61283947	1.47609937	0.01427468	0.00000465	0.04393325	4.88809749
		b	0.69990374	1.38300387	0.01468757	0.00000947	0.11850489	4.97604368
		θ	1.70206394	1.40600047	0.01432293	0.12171133	1.33000357	5.38004947
		α	1.48400375	1.40503374	0.01382164	0.05490457	1.07501438	5.20309946
		β	1.60505643	1.42000464	0.01470434	0.08487473	1.21000474	5.38102144
	150	a	0.61270294	1.47403362	0.01025532	0.00000353	0.04435547	4.88800477
		b	0.70552143	1.37405483	0.01006483	0.00005647	0.12119473	4.94502273
		θ	1.69902739	1.40007346	0.01050464	0.12270475	1.32505374	5.38003530
		α	1.48402273	1.40103254	0.00898496	0.05526748	1.07509367	5.20109862
		β	1.59409934	1.41300475	0.01104437	0.08569774	1.19904536	5.37408946
	250	a	0.61545273	1.47305438	0.00798468	0.00000479	0.04610374	4.83108321
		b	0.70433326	1.37204374	0.00813308	0.00000087	0.12215436	4.92213872
		θ	1.70200485	1.39700048	0.00825484	0.12260415	1.33209464	5.28400469
		α	1.48708346	1.39301434	0.00820465	0.05540475	1.07505437	5.12805437
		β	1.60100475	1.40100475	0.00830374	0.08630462	1.20608564	5.34810464
	350	a	0.61532354	1.46703254	0.00714752	0.00000036	0.04611328	4.82515473
		b	0.70419477	1.36308745	0.00704766	0.00000090	0.12261338	4.89104846
		θ	1.70000948	1.38419254	0.00708742	0.12640473	1.33200474	5.27407347
		α	1.48909937	1.38404432	0.00729937	0.05627468	1.07504746	5.02700379
		β	1.60601244	1.39102239	0.00734563	0.08649466	1.21305363	5.34200453

Table 6. The MLEs for glass fiber data.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$\widehat{\theta}$
EOChW	0.147	0.184	0.991	0.285	4.619
OLoLW	0.944	–	6.029	1.624	–
OFWW	2.215	16.865	0.263	0.559	–
ToLeW	1.839	0.413	6.450	1.821	–
KuW	0.708	23.436	8.011	2.851	–
GoW	0.748	0.031	5.615	1.554	–
TrW	0.925	–	5.977	1.809	–
GW	0.312	–	5.783	1.331	–
EW	0.671	–	7.291	1.718	–
W	–	–	5.783	1.628	–

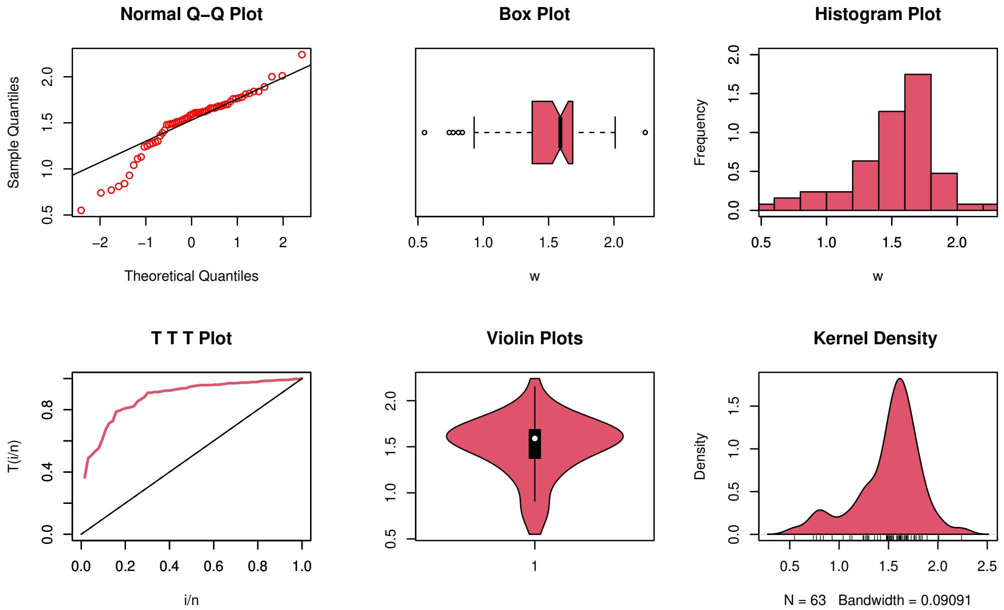

Figure 3. Some non-parametric plots of fiber glass data.

Table 7. Goodness-of-fit statistics for glass fiber data.

Model	$-L$	AD	CvM	KS	p-value
EOChW	14.151	0.895	0.159	0.129	0.246
OLoLW	15.179	1.291	0.235	0.154	0.103
OFWW	14.974	1.251	0.228	0.150	0.117
ToLeW	14.556	1.099	0.198	0.146	0.138
KuW	15.152	1.292	0.235	0.152	0.109
GoW	15.181	1.291	0.235	0.152	0.109
TrW	15.136	1.311	0.239	0.152	0.110
GW	15.199	1.312	0.239	0.152	0.109
EW	14.665	1.119	0.202	0.146	0.136
W	15.199	1.312	0.239	0.152	0.109

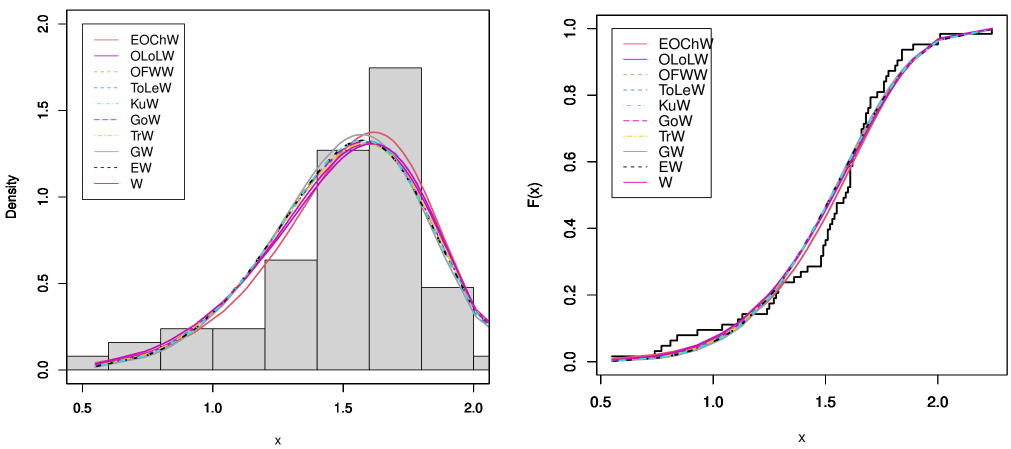

Figure 4. Fitted PDFs (left panel) and estimated CDFs (right panel) for glass fiber data.

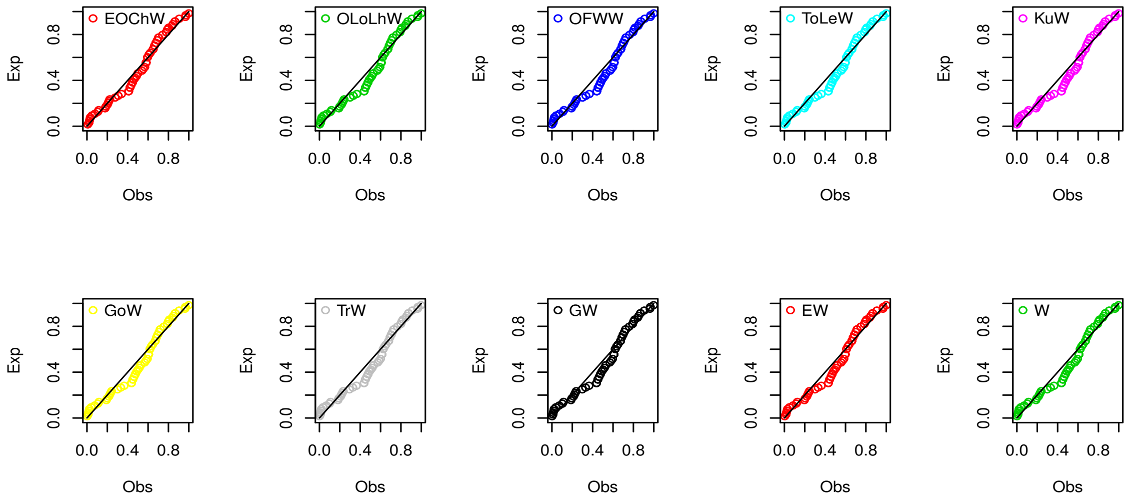

Figure 5. P-P plots for glass fiber data.

Table 8. Posterior summaries for glass fiber data.

Prior	Parameters	Mean	SD	MC Error	2.5%	Median	97.5%
IP	a	0.8221	0.6473	0.0034	0.0712	0.6599	2.4820
	α	0.0799	0.1985	0.0010	0.0000	0.0044	0.6628
	b	0.1547	0.2779	0.0015	0.0000	0.0398	0.9680
	β	0.1708	0.2895	0.0015	0.0000	0.0522	1.0070
	θ	3.7530	1.3740	0.0069	1.5700	3.586	6.8960
NIP	a	0.8142	0.9010	0.0049	0.0103	0.5159	3.2910
	α	0.0803	0.2814	0.0015	0.0000	0.0001	0.8401
	b	0.1507	0.3798	0.0019	0.0000	0.0065	1.2470
	β	0.1732	0.4082	0.0022	0.0000	0.0125	1.3900
	θ	3.7380	1.9280	0.0099	0.9610	3.4130	8.3800

Table 9. Some statistics for glass fiber data.

Method ↓ Measure →	Mean	Variance	Skewness	Kurtosis
MLE	1.3102	0.0905	−0.0819	1.2603

the posterior means were very close to the nominal parameter values, the standard deviation and MCMC error varied significantly for each case. Additionally, it was noticed that the standard deviation and MCMC error decreased as the sample size increased, with IP having the lowest standard deviation compared to the NIP cases. Furthermore, credible intervals for the IP were narrower than those for the NIP cases. Finally, the analysis of a real-world dataset serves as a compelling illustration of the significance and flexibility of the EOChW distribution model, emphasizing its potential practical utility in various statistical applications. Future research could further delve into the practical applications and extensions of the EOChW distribution explored in this study. Investigating its performance in modeling real-world datasets across diverse fields such as engineering, finance, and healthcare could provide valuable insights into its applicability and effectiveness in various statistical scenarios. Additionally, exploring alternative parameter estimation techniques and assessing their robustness under different data conditions could enhance our understanding of the distribution's behavior and improve estimation accuracy. Moreover, considering the EOChW distribution's ability to accommodate asymmetry and various forms of kurtosis, future work could focus on developing tailored inferential methods to capitalize on these properties and optimize statistical analyses. Furthermore, extending the study to investigate multivariate versions or hierarchical structures of the EOChW distribution could broaden its scope and utility in complex data modeling tasks. Overall, continued research into the EOChW distribution holds promise for advancing statistical theory and methodology.

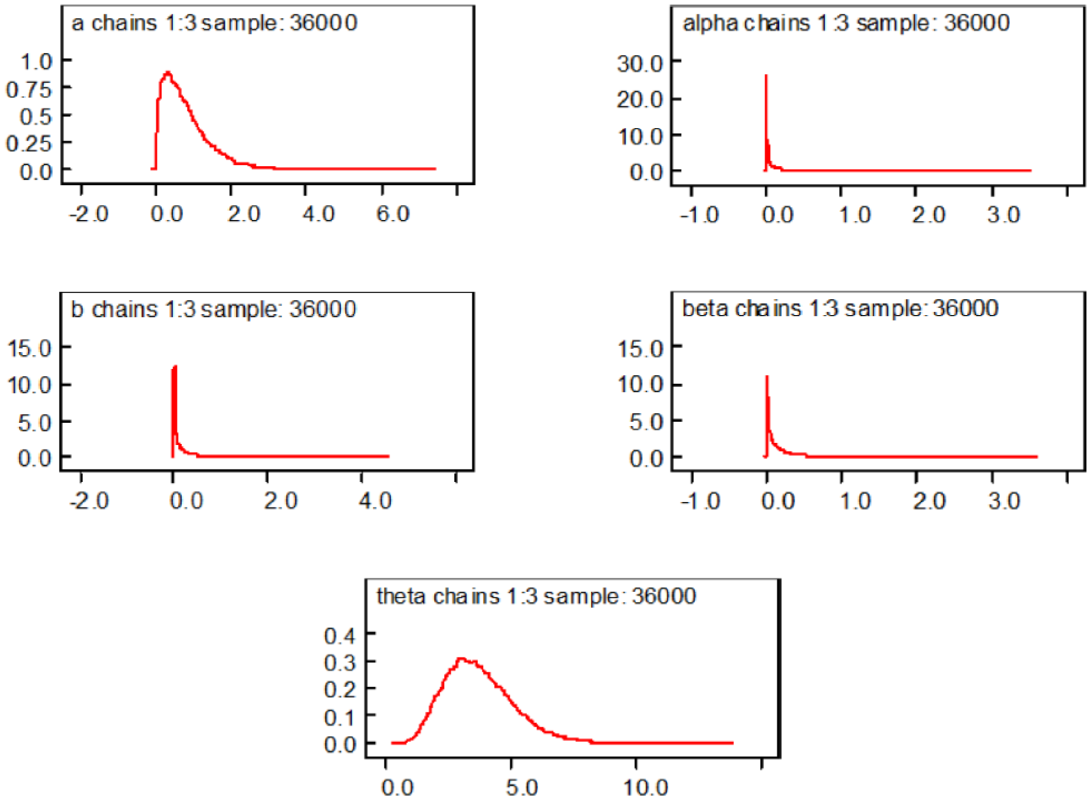

Figure 6. Density plots for glass fiber data.

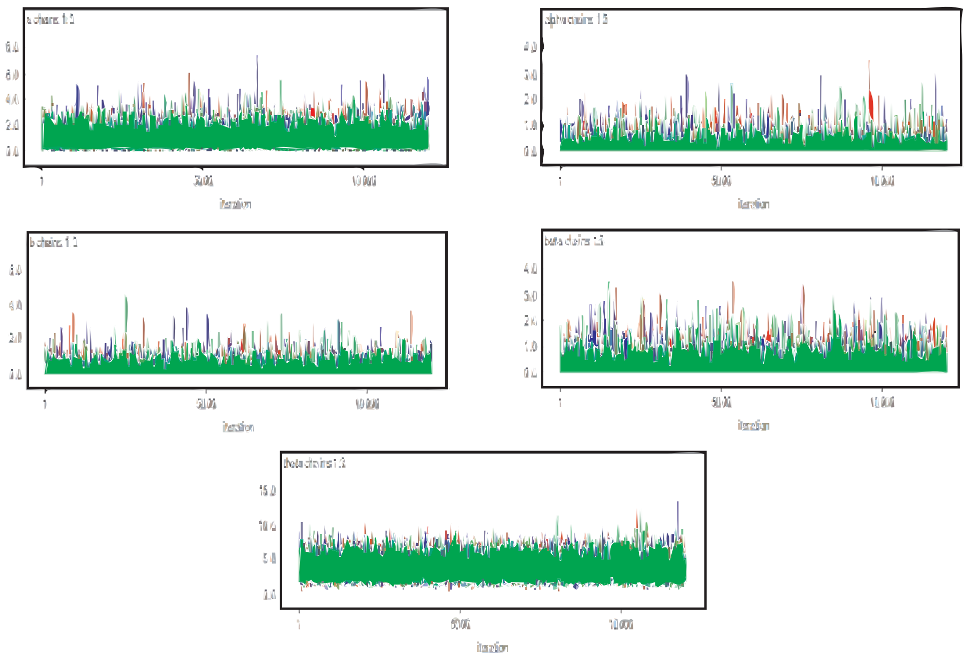
Figure 7. History plots for glass fiber data.

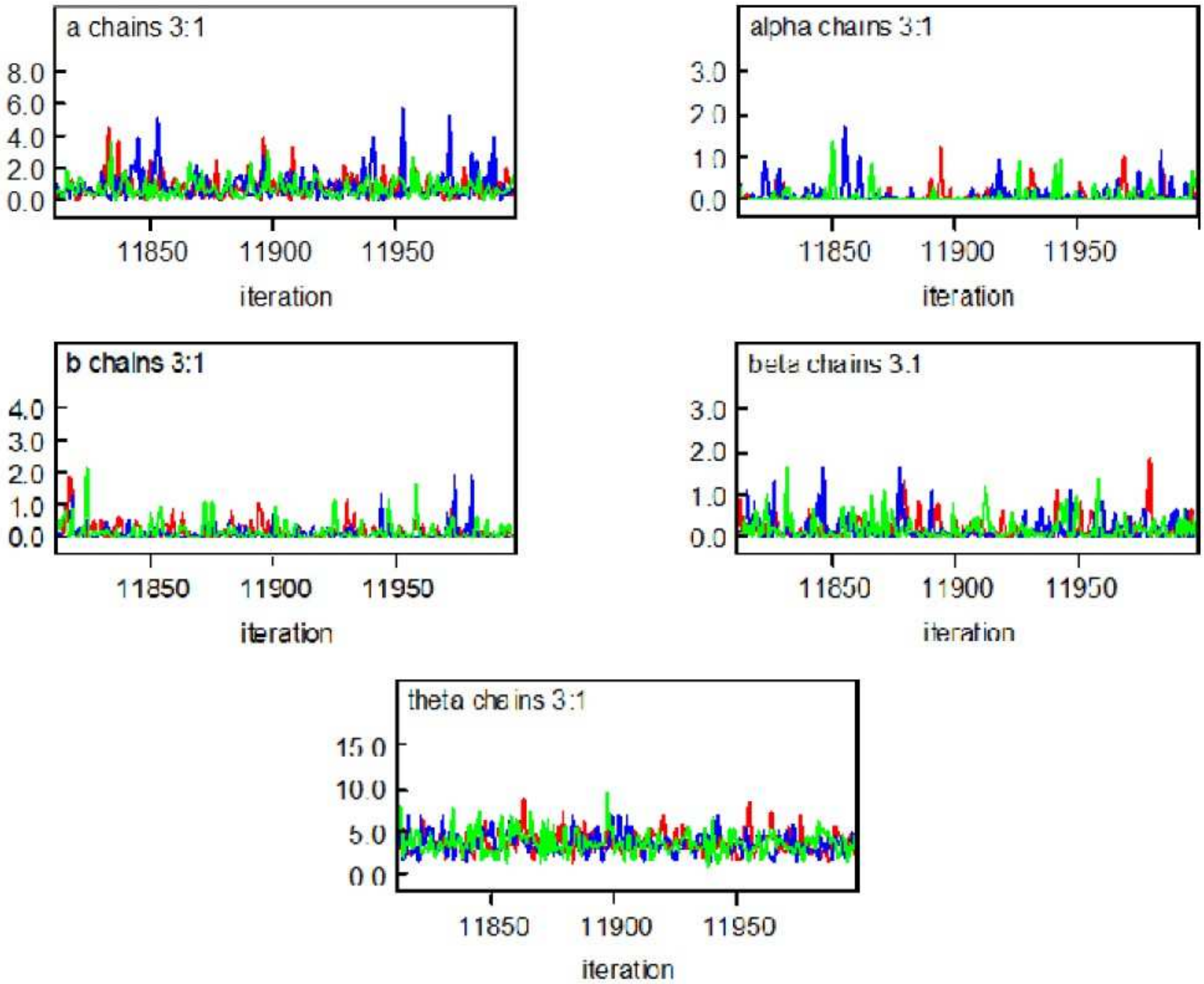

Figure 8. Trace plots for glass fiber data.

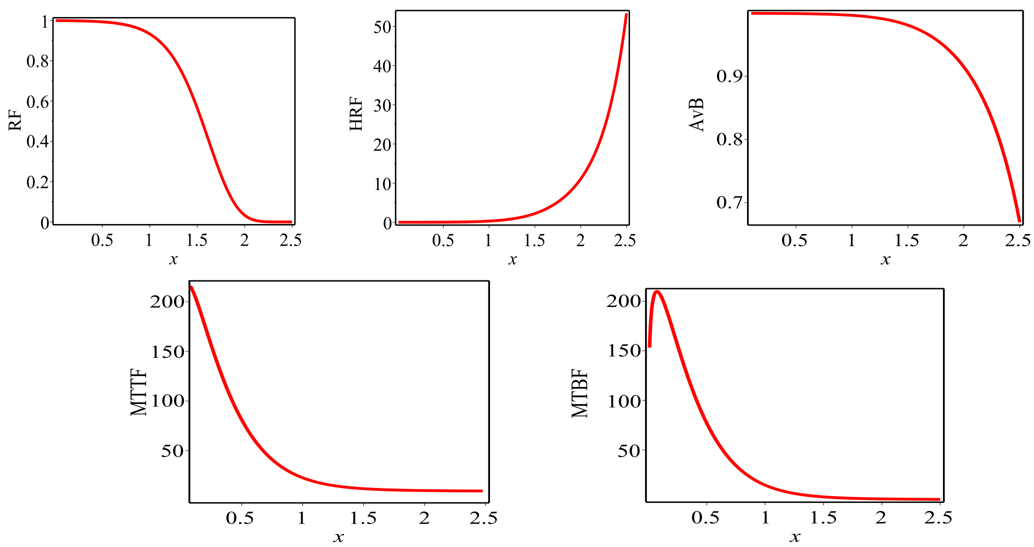

Figure 9. Some reliability plots for glass fiber data.

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