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Some Special Smarandache-Ruled Surfaces According to quasi-Frame in Euclidean 3-Space E^3

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Abstract: In our article we define special kinds of Smarandache ruled surfaces according to quasi frame called $T_qN_qB_q$- Smarandache ruled surfaces. We investigated the first and second fundamental forms, the Gaussian curvature and Mean curvature of these surfaces. The necessary and sufficient conditions for such surfaces to be developable surfaces will be introduced. Also, the condition for such surfaces to be minimal surfaces is obtained. Finally the normal curvature, and geodesic curvature for these surfaces are investigated.

Keywords: Smarandache curves, Ruled Surface, Frenet Frame, Bishop Frame, Quasi-Frame, Gaussian curvature, Mean curvature.

1 Introduction

In differential geometry, ruled surfaces hold a special place as a distinct type of surface, simply a ruled surface is defined by selection of a curve and a line along that curve. Garspard Monge discovered and studied these surfaces, and he established the partial differential equation that satisfies all ruled surfaces. A(differential) one-parameter of (straight) lines $\{\alpha(t), w(t)\}$ can be considered as a correspondence that assigns to each $t \in I$, where I is a subset of the real numbers R a point $\alpha(t) \in R^3$ and a vector $w(t) \in R^3$, $w(t) \neq 0$, the line L_t which passes through $\alpha(t)$ and parallel to $w(t)$ is called the line of the family at t, the parametrized surface $\sigma(t,v) = \alpha(t) + vw(t), t \in I, v \in R$ is called the ruled surface generated by the family $\{\alpha(t), w(t)\}$ where $\{\alpha(t), w(t)\}$ a one-parameter family of lines. The lines L_t are called the rulings, and the curve $\alpha(t)$ is called a directrix of the surface $\sigma(t,v)$ [1]. Many properties of ruled surface have been investigated in both Euclidean and non- Euclidean spaces from the past to today as seen in [2,3,4,5,6,7,8,9,10,11,12,13]. Ruled surface have applications in civil engineering, computer programing, architecture and solid modeling for examples one can see the references [14,15].
Ouarab presents the concept of Smarandache-ruled surface according to Frenet-Serret frame of a curve in E^3 in [16]. The author investigates TN-Smarandache-ruled surfaces, TB-Smarandache-ruled surfaces, and NB-Smarandache-ruled surfaces. The author determine the essential conditions for these ruled surfaces to be classified as developable surfaces and minimal surfaces. Ouarab investigates Smarandache-ruled surface according to Darboux frame in E^3 [17]. The author introduces a new approach of constructing special ruled surfaces and definitions of Smarandache-ruled surface according to Darboux frame of a curve lying on an arbitrary regular surface in E^3. Senyurt, Canli, and Hilal, in their work[18], introduce the concept of Smarandache-ruled surface according to Bishop frame in E^3. In [11] the authors investigated the fundamental forms and the corresponding curvatures and they provided several examples. Some special Smarandache-ruled surfaces by Frenet frame in E^3 introduced by Senyurt, Canli, and Con [19,20]. The authors present some new special ruled surfaces with the base TNB- Smarandache curve where the unit-vector of the generator is taken as one of other Frenet vectors and their linear combinations. In [21], Al-dayel and Solouma introduce a special type of ruled surface known as type-2 Smarandache-ruled surface. These surfaces are specifically related to the type-2 Bishop frame in E^3. The authors define and investigate the characteristic properties of type-2 Smarandache-ruled surfaces, providing insights into their unique features and behaviors within the context of the type-2 Bishop frame.

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Ouarab introduces two types of Smarandache-ruled surfaces according to alternative moving frame in E^3, these are NC- Smarandacehe-ruled surface and NW-Smarandache-ruled surface [22].

2 Preliminaries

Let $\alpha = \alpha(s)$ be a curve in three-dimensional Euclidean space E^3 parameterized by arc length s, denoted by $\alpha : I \longrightarrow R^3$. We define the Frenet-Serret frame of the curve α as the orthonormal frame $t(s), n(s), b(s)$, where $t(s)$ represents the unit tangent vector field of $\alpha(s)$, $n(s)$ represents the principal normal vector field of $\alpha(s)$, and $b(s) = t(s) \times n(s)$ represents the binormal vector field of $\alpha(s)$. Throughout this paper, we denote the derivative of $\alpha(s)$ with respect to the arc length parameter s by $\alpha'(s)$, which is equivalent to $t(s)$. The principal normal vector field $n(s)$ is given by $n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$, and the binormal vector field $b(s)$ is obtained by taking the cross product of $t(s)$ and $n(s)$, that is $b(s) = t(s) \times n(s)$. In this article, we use the notation $\kappa(s)$ and $\tau(s)$ to represent the curvature and torsion of the curve $\alpha(s)$, respectively. The curvature $\kappa(s)$ is computed as $\kappa(s) = \| \alpha''(s) \|$, and the torsion $\tau(s)$ is given by $\tau(s) = - < b'(s), n(s) >$. The arc-length derivative of the Frenet-Serret frame can be expressed using the following formula.

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}$$

The Frenet-Serret frame is not defined at inflection points (where the curvature is zero), and it is also not continuously defined for C^1-continuous space curves. To address these issues, Coquillart, Mustafa etal [**?**,**?**] introduced a New frame called a quasi-frame. For a unit speed curve $\alpha(s)$, the quasi-frame along $\alpha(s)$ can be expressed as follows:

$$T_q(s) = t(s),\ N_q(s) = \frac{t(s) \times \zeta}{\| t(s) \times \zeta \|},\ B_q(s) = T_q(s) \times N_q(s),$$

The quasi-frame along a unit speed curve $\alpha(s)$ consists of the projection vector ζ, as well as the unit tangent vector field $T_q(s)$, quasi-normal vector $N_q(s)$, and quasi-binormal vector $B_q(s)$. The relationship between the Frenet-Serret frame and the quasi-frame can be described using the following formula [23, 10, 26, 25]:

$$\begin{pmatrix} T_q(s) \\ N_q(s) \\ B_q(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\varphi & sin\varphi \\ 0 & -sin\varphi & cos\varphi \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \quad (1)$$

Where, as shown in Figure 1, φ the angle between the principal normal $n(s)$ and the quasi-normal N_q.The Frenet-Serret frame is written as $\{t(s), n(s), b(s)\}$ while the quasi-frame is $\{T_q(s), N_q(s), B_q(s)\}$. If $\varphi = 0$ the quasi frame becomes to the Frenet-Serret frame, and if $k_3 = 0$ the quasi becomes to Bishop-frame. The variation equations of the quasi-frame, as stated in equation (1), can be simplified to the following formula:

$$\begin{pmatrix} T_q'(s) \\ N_q'(s) \\ B_q'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{pmatrix} \begin{pmatrix} T_q(s) \\ N_q(s) \\ B_q(s) \end{pmatrix}$$

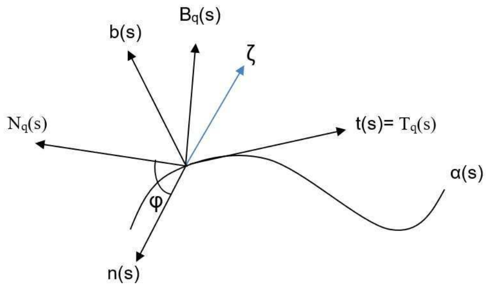

Fig. 1: Relation between Frenet and quasi frames

The quasi curvatures functions of the curve $\alpha(s)$ are represented by the triple $k_1(s)$, $k_2(s)$, and $k_3(s)$. They are given by:

$$k_1 = < T_q'(s), N_q(s) > = \kappa cos\varphi,$$
$$k_2 = < T_q'(s), B_q(s) > = -\kappa sin\varphi,$$
$$k_3 = < N_q'(s), B_q(s) > = \tau(s) + \varphi'(s).$$

The quasi-frame provides many advantages over the Frenet-Serret frame and Bishop frame. For instance, the quasi-frame can be defined along a line where the curvature (κ) is zero. The quasi-frame becomes singular when the torsion (τ) and the projection vector (ζ) are parallel. In this cases, the projection vector ζ can be chosen as $(0,0,1)$, $(0,1,0)$, or $(1,0,0)$. The parametric equation of the ruled surface $\sigma(s,v)$ is defined by:

$$\sigma(s,v) = \alpha(s) + vX(s).$$

Where $\alpha(s)$ is a curve and $X(s)$ is a generator vector. Suppose M is a surface in $\mathbb{E}^3$ defined by the parametrization $\sigma(s,v)$. M is regular if the cross product of the partial derivatives of $\sigma(s,v)$, denoted as σ_s and σ_v, is non-zero for all points on M, i.e., $\sigma_s \times \sigma_v \neq 0$. Where σ_s and σ_v represent the partial derivatives of $\sigma(s,v)$ with respect to s and v. The coefficients of the first fundamental form for the surface M are obtained by [1].

$$E = < \sigma_s, \sigma_s >,\ F = < \sigma_s, \sigma_v >,\ G = < \sigma_v, \sigma_v > .$$

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The unit normal vector field of the regular surface M is defined using the Euclidean inner product denoted by $<,>$:

$$U = \frac{\sigma_s \times \sigma_v}{\| \sigma_s \times \sigma_v \|}.$$

The coefficients of the second fundamental form of M are calculated by the following equations:

$$e =< \sigma_{ss}, U >, \ f =< \sigma_{sv}, U >, \ g =< \sigma_{vv}, U > .$$

The Gaussian curvature and mean curvature of the surface M can be expressed as follows:

$$K = \frac{eg - f^2}{EG - F^2}, \quad and$$
$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}.$$

The distribution parameter, denoted as λ for the ruled surface $\sigma(s,v) = \alpha(s) + vX(s)$, is determined by the following expression:

$$\lambda = \frac{det(\alpha'(s), X(s), X'(s))}{\| X'(s) \|}.$$

The geodesic curvature, the normal curvature and the geodesic torsion which associate with the curve $\alpha(s)$ on the surface $\sigma(s,v)$ can be calculated as follows [21]:

$$k_g =< U \wedge t(s), t'(s) >,$$
$$k_n =< \alpha'', U >,$$
$$\tau_g =< U \wedge U', t'(s) > .$$

For a curve $\alpha(s)$ lying on a surface, it is important to note the following definitions:

I. $\alpha(s)$ is a geodesic if and only if $k_g \equiv 0$.
II. $\alpha(s)$ is a asymptotic line if and only if $k_n \equiv 0$.
III. $\alpha(s)$ is a principal line if and only if $\tau_g \equiv 0$.
IV. A regular surface if flat(developable) if and only if $K \equiv 0$.
V. A regular surface is a minimal surface if $H \equiv 0$.

3 Results and Discussion

In this section, we will define $T_qN_qB_q$- Smarandache ruled surfaces in Euclidean 3-space E^3 according to quasi frame $\{T_q(s), N_q(s), B_q(s)\}$. The following are seven types of $T_qN_qB_q$- Smarandache ruled surfaces.

$$\begin{cases} \Gamma(s,v) = \frac{1}{\sqrt{3}}[(1+\sqrt{3}v)T_q + N_q + B_q] \\ M(s,v) = \frac{1}{\sqrt{3}}[T_q + (1+\sqrt{3}v)N_q + B_q] \\ \omega(s,v) = \frac{1}{\sqrt{3}}[T_q + N_q + (1+\sqrt{3}v)B_q] \\ \phi(s,v) = (\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(T_q + N_q) + \frac{B_q}{\sqrt{3}} \\ \Omega(s,v) = (\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(T_q + B_q) + \frac{N_q}{\sqrt{3}} \\ \sigma(s,v) = \frac{T_q}{\sqrt{3}} + (\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(N_q + B_q) \\ \Upsilon(s,v) = (\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{3}})(T_q + N_q + B_q) \end{cases}$$

3.1 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\mathbf{T}_q$

In this subsection we will define the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector $\mathbf{T}_q$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvature.

Definition 1. *Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\mathbf{T}_q$ defined by the following equation*

$$\Gamma(s,v) = \frac{1}{\sqrt{3}}[(1+\sqrt{3}v)T_q + N_q + B_q]. \tag{2}$$

Theorem 1. *Suppose $\Gamma(s,v)$ be Smarandache Ruled Surface defined as in equation(2). Then the mean curvature $H^{\Gamma(s,v)}$ of $\Gamma(s,v)$ will be given by*

$$H^{\Gamma(s,v)} = \frac{\sqrt{3}}{[(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2]^{\frac{3}{2}}}$$
$$[(1+\sqrt{3}v)^2(k_1'k_2 - k_1k_2' - k_2^2k_3 - k_1^2k_3)$$
$$+ (1+\sqrt{3}v)(k_1'k_3 + k_2'k_3 - k_3'k_2 - k_1k_3' - 2k_3^2k_2$$
$$+ 2k_1k_3^2) + 2k_1k_2k_3 - 2k_3^3 + k_1^2k_3 + k_2^2k_3],$$

and the Gaussian curvature $K^{\Gamma(s,v)}$ of $\Gamma(s,v)$ will be given by

$$K^{\Gamma(s,v)} = \frac{-3k_3^2(k_1+k_2)^2}{[(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2]^2}.$$

Proof. Let $\Gamma(s,v)$ be Smarandache Ruled Surface defined by equation (2), the first and second partial derivatives of $\Gamma(s,v)$are given as follows:

$$\Gamma_s(s,v) = \frac{1}{\sqrt{3}}[-(k_1+k_2)T_q + ((1+\sqrt{3}v)k_1 - k_3)N_q +$$
$$(1+\sqrt{3}v)k_2 + k_3)B_q],$$
$$\Gamma_v(s,v) = T_q, \quad \Gamma_{vs}(s,v) = k_1N_q + k_2B_q, \quad \Gamma_{vv}(s,v) = 0,$$
$$\Gamma_{ss}(s,v) = \frac{1}{\sqrt{3}}[(-(1+\sqrt{3}v)(k_1^2+k_2^2) - k_1' - k_2' + k_1k_3$$
$$- k_2k_3)T_q + ((1+\sqrt{3}v)(k_1' - k_3k_2) - k_3' - k_1^2 - k_3^2$$
$$- k_1k_2)N_q + ((1+\sqrt{3}v)(k_2' + k_1k_3) + k_3' - k_2^2$$
$$- k_3^2 - k_2k_1)B_q].$$

The unit normal of $\Gamma(s,v)$ will be obtained by the following equation

$$U^{\Gamma(s,v)} = \frac{(k_2(1+\sqrt{3}v)+k_3)N_q - (k_1(1+\sqrt{3}v)-k_3)B_q}{\sqrt{(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2}}.$$

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The coefficients of the first and second fundamental forms of $\Gamma(s,v)$ are given by

$$E^{\Gamma(s,v)} = \frac{1}{3}[(k_1+k_2)^2 + (k_1(1+\sqrt{3}v)-k_3)^2 + (k_2(1+\sqrt{3}v)+k_3)^2],$$

$$F^{\Gamma(s,v)} = \frac{-(k_1+k_2)}{\sqrt{3}}, \qquad G^{\Gamma(s,v)} = 1,$$

$$e^{\Gamma(s,v)} = \frac{1}{\sqrt{3}} [(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2]^{\frac{-1}{2}} [(1+\sqrt{3}v)^2(k_1'k_2 - k_1k_2' - k_2^2k_3 - k_1^2k_3) + (1+\sqrt{3}v)(k_1'k_3 + k_2'k_3 - k_3'k_2 - k_1k_3' - 2k_3^2k_2 + 2k_1k_3^2) - 2k_1k_2k_3 - 2k_3^3 - k_1^2k_3 - k_2^2k_3],$$

$$f^{\Gamma(s,v)} = \frac{k_3(k_1+k_2)}{\sqrt{(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2}},$$

$$g^{\Gamma(s,v)} = 0.$$

The mean curvature $H^{\Gamma(s,v)}$ of $\Gamma(s,v)$ and the Gaussian curvature $K^{\Gamma(s,v)}$ of $\Gamma(s,v)$ as follows:

$$H^{\Gamma(s,v)} = \frac{\sqrt{3}}{[(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2]^{\frac{3}{2}}} [(1+\sqrt{3}v)^2(k_1'k_2 - k_1k_2' - k_2^2k_3 - k_1^2k_3) + (1+\sqrt{3}v)(k_1'k_3 + k_2'k_3 - k_3'k_2 - k_1k_3' - 2k_3^2k_2 + 2k_1k_3^2) + 2k_1k_2k_3 - 2k_3^3 + k_1^2k_3 + k_2^2k_3],$$

$$K^{\Gamma(s,v)} = \frac{-3k_3^2(k_1+k_2)^2}{[(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2]^2}.$$

Corollary 1. *$\Gamma(s,v)$-Smarandache-Ruled Surface satisfying equation (2) is developable if and only if $k_3 = 0$ or $k_1 = -k_2$.*

Corollary 2. *$\Gamma(s,v)$-Smarandache-Ruled Surface satisfying equation (2) is minimal surface if and only if $k_3 = k_2 = 0$ and $k_1 \neq 0$.*

Corollary 3. *The geodesic curvature, and the normal curvature for the Smarandache curve lying on the surface $\Gamma(s,v)$ satisfying equation (2) is given by*

$$k_g^{\Gamma(s,v)} = \frac{k_3(k_1-k_2) - (k_1^2+k_2^2)(1+\sqrt{3}v)}{\sqrt{(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2}},$$

$$k_n^{\Gamma(s,v)} = \frac{1}{\sqrt{3}\sqrt{(k_2(1+\sqrt{3}v)+k_3)^2 + (k_1(1+\sqrt{3}v)-k_3)^2}} [(k_2(1+\sqrt{3}v)+k_3)(k_1' - k_3' - k_1^2 - k_3^2 - k_1k_2 - k_2k_3) - (k_1(1+\sqrt{3}v)-k_3)(k_2' + k_3' - k_2^2 - k_3^2 - k_1k_2 + k_1k_3)].$$

Corollary 4. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (2) is a geodesic if and only if $k_1 = k_2 = 0$.*

Corollary 5. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (2) is asymptotic line if and only if $k_1 = k_3 = 0$.*

Corollary 6. *The distribution parameter for the $\Gamma(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{\Gamma(s,v)} = \frac{k_3(k_1+k_2)}{\sqrt{3}\sqrt{k_1^2+k_2^2}}.$$

3.2 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\mathbf{N}_q$

In this subsection we will be introduce the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector $\mathbf{N}_q$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvatures.

Definition 2. *Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\mathbf{N}_q$ defined by equation*

$$M(s,v) = \frac{1}{\sqrt{3}}[T_q + (1+\sqrt{3}v)N_q + B_q]. \tag{3}$$

Theorem 2. *Suppose $M(s,v)$ be Smarandache Ruled Surface defined by equation (3). Then the mean curvature $H^{M(s,v)}$ of $M(s,v)$ will be given by*

$$H^{M(s,v)} = \frac{\sqrt{3}}{2[(k_3(1+\sqrt{3}v)+k_2)^2 + (k_1(1+\sqrt{3}v)+k_2)^2]^{\frac{3}{2}}} [(1+\sqrt{3}v)^2(k_1'k_3 - k_3'k_1 + k_1^2k_2 + k_3^2k_2) + (1+\sqrt{3}v)(2k_2^2k_3 + 2k_2^2k_1 + k_1'k_2 + k_2'k_3 - k_3'k_2 - k_2'k_1) + 2k_2^3 + 2k_1k_2k_3 - k_2k_1^2 - k_3^2k_2],$$

and the Gaussian curvature $K^{M(s,v)}$ of $M(s,v)$ will be given by

$$K^{M(s,v)} = \frac{-3k_2^2(k_1-k_3)^2}{[(k_3(1+\sqrt{3}v)+k_2)^2 + (k_1(1+\sqrt{3}v)+k_2)^2]^2}.$$

Proof. Let $M(s,v)$ be Smarandache Ruled Surface defined by equation (3), the first and second partial derivatives of

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$M(s,v)$are given as follows:

$$M_s(s,v) = \frac{1}{\sqrt{3}}[(-k_1(1+\sqrt{3}v)-k_2)T_q+(k_1-k_3)N_q + (k_3(1+\sqrt{3}v)+k_2)B_q],$$
$$M_v(s,v)=N_q,\ \ M_{vs}(s,v)=-k_1T_q+k_3B_q,\ M_{vv}(s,v)=0,$$
$$M_{ss}(s,v)=\frac{1}{\sqrt{3}}[(-(1+\sqrt{3}v)(k_2k_3+k_1')-k_2'-k_1^2+k_3k_1 - k_2^2)T_q+(-(1+\sqrt{3}v)(k_1^2+k_3^2)+k_1'-k_3'-k_1k_2 - k_2k_3)N_q+((1+\sqrt{3}v)(k_3'-k_1k_2)+k_2'-k_2^2 - k_3^2+k_1k_3)B_q].$$

The unit normal of $M(s,v)$ will be obtained by the following equation

$$U^{M(s,v)}=\frac{-(k_3(1+\sqrt{3}v)+k_2)T_q-(k_1(1+\sqrt{3}v)+k_2)B_q}{\sqrt{(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2}}$$

The coefficients of the first and second fundamental forms of $M(s,v)$ are given by

$$E^{M(s,v)}=\frac{1}{3}[(k_1(1+\sqrt{3}v)+k_2)^2+(k_1-k_3)^2+ (k_3(1+\sqrt{3}v)+k_2)^2],$$
$$F^{M(s,v)}=\frac{k_1-k_3}{\sqrt{3}},\qquad G^{M(s,v)}=1,$$
$$e^{M(s,v)}=\frac{1}{\sqrt{3}} [(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2]^{\frac{-1}{2}} [(1+\sqrt{3}v)^2(k_1'k_3-k_3'k_1+k_1^2k_2+k_3^2k_2) + (1+\sqrt{3}v)(2k_2^2k_3+2k_2^2k_1+k_1'k_2+k_2'k_3-k_3'k_2 - k_2'k_1)+2k_2^3-2k_1k_2k_3+k_2k_1^2+k_3^2k_2],$$
$$f^{M(s,v)}=\frac{k_2(k_1-k_3)}{\sqrt{(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2}},$$
$$g^{M(s,v)}=0.$$

The mean curvature $H^{M(s,v)}$ of $M(s,v)$ and the Gaussian curvature $K^{M(s,v)}$ of $M(s,v)$ will be given by:

$$H^{M(s,v)}=\frac{\sqrt{3}}{2[(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2]^{\frac{3}{2}}} [(1+\sqrt{3}v)^2(k_1'k_3-k_3'k_1+k_1^2k_2+k_3^2k_2) + (1+\sqrt{3}v)(2k_2^2k_3+2k_2^2k_1+k_1'k_2+k_2'k_3 - k_3'k_2-k_2'k_1)+2k_2^3+2k_1k_2k_3-k_2k_1^2-k_3^2k_2],$$

$$K^{M(s,v)}=\frac{-3k_2^2(k_1-k_3)^2}{[(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2]^2}.$$

Corollary 7.*$M(s,v)$-Smarandache-Ruled Surface satisfying equation (3) is developable if and only if $k_2=0$ or $k_1=k_3$.*

Corollary 8.*$M(s,v)$-Smarandache-Ruled Surface satisfying equation (3) is minimal surface if and only if $k_1=k_2=0$ and $k_3\neq 0$.*

Corollary 9.*The geodesic curvature, and the normal curvature for the Smarandache curve lying on the surface $M(s,v)$ satisfying equation (3) is given by*

$$k_g^{M(s,v)}=\frac{-k_1(k_1(1+\sqrt{3}v)+k_2)}{\sqrt{(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2}},$$

$$k_n^{M(s,v)}=\frac{1}{\sqrt{3}\sqrt{(k_3(1+\sqrt{3}v)+k_2)^2+(k_1(1+\sqrt{3}v)+k_2)^2}} [(-k_3(1+\sqrt{3}v)-k_2)(-k_1'-k_2'-k_1^2-k_2^2+k_1k_3 - k_2k_3)-(k_1(1+\sqrt{3}v)+k_2)(k_2'+k_3'-k_2^2-k_3^2 +k_1k_3-k_1k_2)].$$

Corollary 10.*$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (3) is a geodesic if and only if $k_1=0$.*

Corollary 11.*$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (3) is asymptotic line if and only if $k_1=k_2=0$.*

Corollary 12.*The distribution parameter for the $M(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{M(s,v)}=\frac{k_2(k_1-k_3)}{\sqrt{3}\sqrt{k_1^2+k_3^2}}.$$

3.3 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\mathbf{B}_q$

In this subsection we will define the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector$\mathbf{B}_q$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvature.

Definition 3.*Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s),N_q(s),B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\mathbf{B}_q$ defined by equation*

$$\omega(s,v)=\frac{1}{\sqrt{3}}[T_q+N_q+(1+\sqrt{3}v)B_q]. \tag{4}$$

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Theorem 3. *Suppose $\omega(s,v)$ be Smarandache Ruled Surface defined by equation(4). Then the mean curvature $H^{\omega(s,v)}$ of $\omega(s,v)$ will be given by*

$$H^{\omega(s,v)} = \frac{\sqrt{3}}{2[(k_1+k_2(1+\sqrt{3}v))^2+(k_1-k_3(1+\sqrt{3}v))^2]^{\frac{3}{2}}} [(1+\sqrt{3}v)^2(k_2'k_3-k_3'k_2-k_2^2k_1-k_3^2k_1) + (1+\sqrt{3}v)(k_1'k_3-k_2'k_1-k_3'k_1+k_1'k_2+2k_1^2k_3 - 2k_1^2k_2)-2k_1^3+k_1k_2^2+k_1k_3^2+2k_1k_2k_3],$$

and the Gaussian curvature $K^{\omega(s,v)}$ of $\omega(s,v)$ will be given by

$$K^{\omega(s,v)} = \frac{-3k_1^2(k_2+k_3)^2}{[(k_1+k_2(1+\sqrt{3}v))^2+(k_1-k_3(1+\sqrt{3}v))^2]^2}.$$

Proof. Let $\omega(s,v)$ be Smarandache Ruled Surface defined by equation (4), the first and second partial derivatives of $\omega(s,v)$are given as follows:

$$\omega_s(s,v) = \frac{1}{\sqrt{3}}[(-k_1-k_2(1+\sqrt{3}v))T_q + (k_1-k_3(1+\sqrt{3}v))N_q+(k_2+k_3)B_q],$$

$$\omega_v(s,v) = B_q, \quad \omega_{vs}(s,v) = -k_2T_q-k_3N_q, \quad \omega_{vv}(s,v)=0,$$

$$\omega_{ss}(s,v) = \frac{1}{\sqrt{3}}[((1+\sqrt{3}v)(k_1k_3-k_2')-k_1'-k_1^2-k_2^2 - k_2k_3)T_q+((1+\sqrt{3}v)(-k_3'-k_1k_2)+k_1'-k_1^2 - k_3^2-k_2k_3)N_q+((1+\sqrt{3}v)(-k_2^2-k_3^2)+k_2' + k_3'-k_1k_2+k_1k_3)B_q].$$

The unit normal of $\omega(s,v)$ will be given by the following equation

$$U^{\omega(s,v)} = \frac{(k_1-k_3(1+\sqrt{3}v))T_q+(k_1+k_2(1+\sqrt{3}v))N_q}{\sqrt{(k_1-k_3(1+\sqrt{3}v))^2+(k_1+k_2(1+\sqrt{3}v)^2}}.$$

The coefficients of the first and second fundamental forms of $\omega(s,v)$ are given by

$$E^{\omega(s,v)} = \frac{1}{3}[(k_1+k_2(1+\sqrt{3}v))^2+((k_1-k_3(1+\sqrt{3}v)))^2 + (k_2+k_3)^2],$$

$$F^{\omega(s,v)} = \frac{k_2+k_3}{\sqrt{3}}, \qquad G^{\omega(s,v)} = 1,$$

$$e^{\omega(s,v)} = \frac{1}{\sqrt{3}\sqrt{(k_1-k_3(1+\sqrt{3}v))^2+(k_1+k_2(1+\sqrt{3}v)^2}} [(1+\sqrt{3}v)^2(k_2'k_3-k_3'k_2-k_2^2k_1-k_3^2k_1) + (1+\sqrt{3}v)(k_1'k_3-k_2'k_1-k_3'k_1+k_1'k_2+2k_1^2k_3 - 2k_1^2k_2)-2k_1^3-k_1k_2^2-k_1k_3^2-2k_1k_2k_3],$$

$$f^{\omega(s,v)} = \frac{-k_1(k_2+k_3)}{\sqrt{(k_1-k_3(1+\sqrt{3}v))^2+(k_1+k_2(1+\sqrt{3}v)^2}},$$

$$g^{\omega(s,v)} = 0.$$

The mean curvature $H^{\omega(s,v)}$ of $\omega(s,v)$ and the Gaussian curvature $K^{\omega(s,v)}$ of $\omega(s,v)$ will be given by:

$$H^{\omega(s,v)} = \frac{\sqrt{3}}{2[(k_1+k_2(1+\sqrt{3}v))^2+(k_1-k_3(1+\sqrt{3}v))^2]^{\frac{3}{2}}} [(1+\sqrt{3}v)^2(k_2'k_3-k_3'k_2-k_2^2k_1-k_3^2k_1) + (1+\sqrt{3}v)(k_1'k_3-k_2'k_1-k_3'k_1+k_1'k_2+2k_1^2k_3 - 2k_1^2k_2)-2k_1^3+k_1k_2^2+k_1k_3^2+2k_1k_2k_3],$$

$$K^{\omega(s,v)} = \frac{-3k_1^2(k_2+k_3)^2}{[(k_1+k_2(1+\sqrt{3}v))^2+(k_1-k_3(1+\sqrt{3}v))^2]^2}.$$

Corollary 13. *$\omega(s,v)$-Smarandache-Ruled Surface satisfying equation (4) is developable if and only if $k_1 = 0$ or $k_2 = -k_3$.*

Corollary 14. *$\omega(s,v)$-Smarandache-Ruled Surface satisfying equation (4) is minimal surface if and only if $k_1 = k_2 = 0$ and $k_3 \neq 0$.*

Corollary 15. *The geodesic curvature, and the normal curvature for the $T_qN_qB_q$-Smarandache curve satisfying equation (4) is given by*

$$k_g^{\omega(s,v)} = \frac{-k_2(k_1+k_2(1+\sqrt{3}v))}{\sqrt{(k_1-k_3(1+\sqrt{3}v))^2+(k_1+k_2(1+\sqrt{3}v)^2}},$$

$$k_n^{\omega(s,v)} = \frac{1}{\sqrt{3}\sqrt{(k_1-k_3(1+\sqrt{3}v))^2+(k_1+k_2(1+\sqrt{3}v)^2}} [(k_1-k_3(1+\sqrt{3}v))(-k_1'-k_2'-k_1^2-k_2^2+k_1k_3 - k_2k_3)+(k_1+k_2(1+\sqrt{3}v))(k_1'-k_3'-k_1^2-k_3^2 - k_1k_2-k_2k_3)].$$

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Corollary 16. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (4) is a geodesic if and only if $k_2 = 0$.*

Corollary 17. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (4) is asymptotic line if and only if $k_1 = k_2 = 0$.*

Corollary 18. *The distribution parameter for the $\omega(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{\omega(s,v)} = \frac{-k_1(k_2+k_3)}{\sqrt{3}\sqrt{k_2^2+k_3^2}}.$$

3.4 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\frac{\mathbf{T_q}+\mathbf{N_q}}{\sqrt{2}}$

In this subsection we will define the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector $\frac{\mathbf{T_q}+\mathbf{N_q}}{\sqrt{2}}$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvature.

Definition 4. *Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\frac{\mathbf{T_q}+\mathbf{N_q}}{\sqrt{2}}$ defined by equation*

$$\phi(s,v) = (\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(T_q + N_q) + \frac{B_q}{\sqrt{3}}. \qquad (5)$$

Theorem 4. *Suppose $\phi(s,v)$ be Smarandache Ruled Surface defined by equation(5).Then the mean curvature $H^{\phi(s,v)}$ of $\phi(s,v)$ will be given by*

$$H^{\phi(s,v)} = \frac{1}{6f_1(s,v)\mu_1(s,v)}[(k_2+k_3)(k_3-k_2)^2 + 3(\nu_1(s) (\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})^2 + \nu_2(s)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) + \nu_3(s))],$$

and the Gaussian curvature $K^{\phi(s,v)}$ of $\phi(s,v)$ will be given by

$$K^{\phi(s,v)} = \frac{-(k_3^2-k_2^2)^2}{6f_1(s,v)\mu_1^2(s,v)},$$

where

$$f_1(s,v) = (k_1(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) + \frac{k_2}{\sqrt{3}})^2 + (k_1(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) - \frac{k_3}{\sqrt{3}})^2 + ((k_2+k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}))^2 - \frac{(k_2+k_3)^2}{6},$$

$$\mu_1(s,v) = [2((k_2+k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}))^2 + (-2k_1(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) + \frac{k_3-k_2}{\sqrt{3}})^2]^{\frac{1}{2}},$$

$$\nu_1(s) = (k_2+k_3)(k_2^2 - k_3^2 + 2k_1') - 2k_1(k_2'+k_3') + 2k_1^2(k_2-k_3),$$

$$\nu_2(s) = \frac{1}{\sqrt{3}}(2k_1(k_2^2+k_3^2) - 4k_1k_2k_3 - 2k_3'k_2 + 2k_2'k_3),$$

$$\nu_3(s) = \frac{-(k_2^2+k_3^2)(k_3-k_2)}{3}.$$

Proof. Let $\phi(s,v)$ be Smarandache Ruled Surface defined by equation (5), the first and second partial derivatives of $\phi(s,v)$ are given as follows:

$$\phi_s(s,v) = -(k_1(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) + \frac{k_2}{\sqrt{3}})T_q + (k_1(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) - \frac{k_3}{\sqrt{3}})N_q + ((k_2+k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}))B_q,$$

$$\phi_v(s,v) = \frac{T_q+N_q}{\sqrt{2}},$$

$$\phi_{vs}(s,v) = \frac{1}{\sqrt{2}}[-k_1T_q + k_1N_q + (k_2+k_3)B_q],$$

$$\phi_{vv}(s,v) = 0,$$

$$\phi_{ss}(s,v) = (-(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(k_1' + k_1^2 + k_2^2 + k_2k_3) - \frac{k_2'}{\sqrt{3}} + \frac{k_1k_3}{\sqrt{3}})T_q + ((\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(k_1' - k_1^2 - k_3^2 - k_2k_3) - \frac{1}{\sqrt{3}}(k_3' + k_1k_2))N_q + ((\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})(k_2' + k_3' - k_1k_2 + k_1k_3) - \frac{1}{\sqrt{3}}(k_2^2+k_3^2))B_q.$$

The unit normal of $\phi(s,v)$ will be given by the following equation

$$U^{\phi(s,v)} = \frac{1}{\mu_1(s,v)}[-(k_2+k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})T_q + (k_2+k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})N_q + (-2k_1(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}) + \frac{1}{\sqrt{3}}(k_3-k_2))B_q].$$

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The coefficients of the first and second fundamental forms of $\phi(s,v)$ are given by

$$E^{\phi(s,v)}=(k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_2}{\sqrt{3}})^2+(k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})-\frac{k_3}{\sqrt{3}})^2+((k_2+k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2$$

$$F^{\phi(s,v)}=\frac{-(k_2+k_3)}{\sqrt{6}},\qquad G^{\phi(s,v)}=1,$$

$$e^{\phi(s,v)}=\frac{\nu_1(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})^2+\nu_2(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\nu_3(s)}{\mu_1(s,v)},$$

$$f^{\phi(s,v)}=\frac{(k_3-k_2)^2}{\mu_1(s,v)\sqrt{6}},\qquad g^{\phi(s,v)}=0.$$

where

$$f_1(s,v)=(k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_2}{\sqrt{3}})^2+(k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})-\frac{k_3}{\sqrt{3}})^2+((k_2+k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2-\frac{(k_2+k_3)^2}{6},$$

$$\mu_1(s,v)=[2((k_2+k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2+(-2k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_3-k_2}{\sqrt{3}})^2]^{\frac{1}{2}},$$

$$\nu_1(s)=(k_2+k_3)(k_2^2-k_3^2+2k_1')-2k_1(k_2'+k_3')+2k_1^2(k_2-k_3),$$

$$\nu_2(s)=\frac{1}{\sqrt{3}}(2k_1(k_2^2+k_3^2)-4k_1k_2k_3-2k_3'k_2+2k_2'k_3),$$

$$\nu_3(s)=\frac{-(k_2^2+k_3^2)(k_3-k_2)}{3}.$$

The mean curvature $H^{\phi(s,v)}$ of $\phi(s,v)$ and the Gaussian curvature $K^{\phi(s,v)}$ of $\phi(s,v)$ will be given by:

$$H^{\phi(s,v)}=\frac{1}{6f_1(s,v)\mu_1(s,v)}[(k_2+k_3)(k_3-k_2)^2+3(\nu_1(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})^2+\nu_2(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\nu_3(s))],$$

$$K^{\phi(s,v)}=\frac{-(k_3^2-k_2^2)^2}{6f_1(s,v)\mu_1^2(s,v)}.$$

Corollary 19. *$\phi(s,v)$-Smarandache-Ruled Surface satisfying equation (5) is developable if and only if $k_2=k_3$.*

Corollary 20. *$\phi(s,v)$-Smarandache-Ruled Surface satisfying equation (5) is minimal surface if and only if $k_2=k_3=0$ and $k_1\neq 0$.*

Corollary 21. *The geodesic curvature, and the normal curvature for the $T_qN_qB_q$-Smarandache curve satisfying equation (5) is given by*

$$k_g^{\phi(s,v)}=\frac{1}{\mu_1(s,v)}[k_1(-2k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{1}{\sqrt{3}}(k_3-k_2))-k_2((k_2+k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))],$$

$$k_n^{\phi(s,v)}=\frac{1}{\mu_1(s,v)}[(k_2+k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_1(k_1-k_3)+k_2(k_2+k_3)+k_1'+k_2')+(k_2+k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(-k_1(k_1+k_2)-k_3(k_2+k_3)+k_1'-k_3')+(-2k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{1}{\sqrt{3}}(k_3-k_2))(k_3(k_1-k_3)-k_2(k_1+k_2)+k_2'+k_3')].$$

Corollary 22. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (5) is a geodesic if and only if $k_1=k_2=0$ and $k_3\neq 0$.*

Corollary 23. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (5) is asymptotic line if and only if $k_2=k_3=0$ and $k_1\neq 0$.*

Corollary 24. *The distribution parameter for the $\phi(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{\phi(s,v)}=\frac{k_3^2-k_2^2}{\sqrt{6}\sqrt{2k_1^2+(k_1-k_3)^2}}.$$

3.5 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\frac{\mathbf{T_q}+\mathbf{B_q}}{\sqrt{2}}$

In this subsection we will define the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector $\frac{\mathbf{T_q}+\mathbf{B_q}}{\sqrt{2}}$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvature.

Definition 5. *Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s),N_q(s),B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\frac{\mathbf{T_q}+\mathbf{B_q}}{\sqrt{2}}$ defined by equation*

$$\Omega(s,v)=(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(T_q+B_q)+\frac{N_q}{\sqrt{3}}. \qquad (6)$$

Theorem 5. *Suppose $\Omega(s,v)$ be Smarandache Ruled Surface defined by equation(6). Then the mean curvature $H^{\Omega(s,v)}$ of $\Omega(s,v)$ will be given by*

$$H^{\Omega(s,v)}=\frac{1}{6\mu_2(s,v)f_2(s,v)}[-(k_1^2-k_3^2)(k_3-k_1)+3(\eta_1(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})^2+\eta_2(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\eta_3(s))]$$

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and the Gaussian curvature $K^{\Omega(s,v)}$ of $\Omega(s,v)$ will be given by

$$K^{\Omega(s,v)} = \frac{-(k_1^2-k_3^2)^2}{6f_2(s,v)\mu_2(s,v)^2}.$$

where

$$\mu_2(s,v) = [2((k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2+(2k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}) + \frac{k_1+k_3}{\sqrt{3}})^2]^{\frac{1}{2}},$$

$$f_2(s,v) = (k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1}{\sqrt{3}})^2+((k_1-k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}))^2+(k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_3}{\sqrt{3}})^2-\frac{(k_3-k_1)^2}{6},$$

$$\eta_1(s) = (k_1-k_3)(k_3^2-k_1^2-2k_2')+2k_2(k_1'-k_3') - 2k_2^2(k_1+k_3),$$

$$\eta_2(s) = \frac{1}{\sqrt{3}}(2k_1'k_3-2k_3'k_1-2k_2(k_1^2+k_3^2)-4k_1k_2k_3)$$

$$\eta_3(s) = \frac{-(k_1+k_3)(k_3^2+k_1^2)}{3}.$$

Proof. Let $\Omega(s,v)$ be Smarandache Ruled Surface defined by equation (6), the first and second partial derivatives of $\Omega(s,v)$are given as follows:

$$\Omega_s(s,v) = -(k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1}{\sqrt{3}})T_q+(k_1-k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})N_q+(k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_3}{\sqrt{3}})B_q,$$

$$\Omega_v(s,v) = \frac{T_q+B_q}{\sqrt{2}},$$

$$\Omega_{vs}(s,v) = \frac{1}{\sqrt{2}}[-k_2T_q+(k_1-k_3)N_q+k_2B_q],$$

$$\Omega_{vv}(s,v) = 0,$$

$$\Omega_{ss}(s,v) = (-(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_2'+k_1^2+k_2^2-k_1k_3)-\frac{k_1'}{\sqrt{3}} - \frac{k_2k_3}{\sqrt{3}})T_q+((\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_1'-k_3'-k_1k_2 - k_2k_3)-\frac{1}{\sqrt{3}}(k_3^2+k_1^2))N_q+((\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_2' + k_1k_3-k_3^2-k_2^2)+\frac{1}{\sqrt{3}}(k_3'-k_1k_2))B_q.$$

The unit normal of $\Omega(s,v)$ will be given by the following equation

$$U^{\Omega(s,v)} = \frac{1}{\mu_2(s,v)}[(k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})T_q + (2k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1+k_3}{\sqrt{3}})N_q- (k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})B_q].$$

The coefficients of the first and second fundamental forms of $\Omega(s,v)$ are given by

$$E^{\Omega(s,v)} = (k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1}{\sqrt{3}})^2+((k_1-k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}))^2+(k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_3}{\sqrt{3}})^2,$$

$$F^{\Omega(s,v)} = \frac{k_3-k_1}{\sqrt{6}}, \qquad G^{\Omega(s,v)} = 1,$$

$$e^{\Omega(s,v)} = \frac{\eta_1(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})^2+\eta_2(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\eta_3(s)}{\mu_2(s,v)},$$

$$f^{\Omega(s,v)} = \frac{k_1^2-k_3^2}{\mu_2(s,v)}, \qquad g^{\Omega(s,v)} = 0.$$

where

$$\mu_2(s,v) = [2((k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2+(2k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}) + \frac{k_1+k_3}{\sqrt{3}})^2]^{\frac{1}{2}},$$

$$f_2(s,v) = (k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1}{\sqrt{3}})^2+((k_1-k_3)(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}}))^2+(k_2(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_3}{\sqrt{3}})^2-\frac{(k_3-k_1)^2}{6},$$

$$\eta_1(s) = (k_1-k_3)(k_3^2-k_1^2-2k_2')+2k_2(k_1'-k_3') - 2k_2^2(k_1+k_3),$$

$$\eta_2(s) = \frac{1}{\sqrt{3}}(2k_1'k_3-2k_3'k_1-2k_2(k_1^2+k_3^2)-4k_1k_2k_3)$$

$$\eta_3(s) = \frac{-(k_1+k_3)(k_3^2+k_1^2)}{3}.$$

The mean curvature $H^{\Omega(s,v)}$ of $\Omega(s,v)$ and the Gaussian curvature $K^{\Omega(s,v)}$ of $\Omega(s,v)$ will be given by:

$$H^{\Omega(s,v)} = \frac{1}{6\mu_2(s,v)f_2(s,v)}[-(k_1^2-k_3^2)(k_3-k_1)\ +3(\eta_1(s) (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})^2+\eta_2(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\eta_3(s))],$$

$$K^{\Omega(s,v)} = \frac{-(k_1^2-k_3^2)^2}{6f_2(s,v)\mu_2(s,v)^2}.$$

Corollary 25.$\Omega(s,v)$*-Smarandache-Ruled Surface satisfying equation* (6) *is developable if and only if* $k_1 = k_3$.

Corollary 26.$\Omega(s,v)$*-Smarandache-Ruled Surface satisfying equation* (6) *is minimal surface if and only if* $k_1 = k_3 = 0$ *and* $k_2 \neq 0$.

Corollary 27.*The geodesic curvature, and the normal curvature for the* $T_qN_qB_q$*-Smarandache curve satisfying*

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equation (6) is given by

$$k_g^{\Omega(s,v)} = \frac{1}{\mu_2(s,v)}[-k_1(k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})-k_2(2k_2 (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1+k_3}{\sqrt{3}}]$$

$$k_n^{\Omega(s,v)} = \frac{1}{\mu_2(s,v)}[-(k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_1(k_1-k_3) +k_2(k_2+k_3)+k_1'+k_2')+(-2k_1(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}) +\frac{1}{\sqrt{3}}(k_3-k_2))(-k_1(k_1+k_2)-k_3(k_2+k_3) +k_1'-k_3')-(k_1-k_3)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_3(k_1-k_3) -k_2(k_1+k_2)+k_2'+k_3')].$$

Corollary 28. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (6) is a geodesic if and only if $k_1=k_2=0$ and $k_3\neq 0$.*

Corollary 29. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (6) is asymptotic line if and only if $k_2=k_3=0$ and $k_1\neq 0$.*

Corollary 30. *The distribution parameter for the $\Omega(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{\Omega(s,v)} = \frac{k_1^2-k_3^2}{\sqrt{6}\sqrt{2k_2^2+(k_1-k_3)^2}}.$$

3.6 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\frac{\mathbf{N_q+B_q}}{\sqrt{2}}$

In this subsection we will define the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector $\frac{\mathbf{N_q+B_q}}{\sqrt{2}}$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvature.

Definition 6. *Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s),N_q(s),B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\frac{\mathbf{N_q+B_q}}{\sqrt{2}}$ defined by equation*

$$\sigma(s,v) = \frac{T_q}{\sqrt{3}}+(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(N_q+B_q). \tag{7}$$

Theorem 6. *Suppose $\sigma(s,v)$ be Smarandache Ruled Surface defined by equation(7). Then the mean curvature $H^{\sigma(s,v)}$ of $\sigma(s,v)$ will be given by*

$$H^{\sigma(s,v)} = \frac{1}{6f_3(s,v)\mu_3(s,v)}[(k_2^2-k_1^2)(k_1+k_2)+3((\frac{1}{\sqrt{3}}+ \frac{v}{\sqrt{2}})^2\varsigma_1(s)+(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})\varsigma_2(s)+\varsigma_3(s))],$$

and the Gaussian curvature $K^{\sigma(s,v)}$ of $\sigma(s,v)$ will be given by

$$K^{\Omega(s,v)} = \frac{-(k_2^2-k_1^2)^2}{6f_3(s,v)\mu_3^2(s,v)}.$$

where

$$\mu_3(s,v) =[(-2k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1-k_2}{\sqrt{3}})^2+2((k_1+k_2) (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2]^{\frac{1}{2}},$$

$$f_3(s,v) =((k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2+(-k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}) +\frac{k_1}{\sqrt{3}})^2+(k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_2}{\sqrt{3}})^2 -\frac{(k_1+k_2)^2}{6},$$

$$\varsigma_1(s) =2k_3(k_1'+k_2')-2k_3'(k_1+k_2)+(k_2^2-k_1^2-2k_3^2) (k_1+k_2),$$

$$\varsigma_2(s) =\frac{1}{\sqrt{3}}(2k_3(k_1^2+k_2^2)+2k_1'k_2-2k_2'k_1-4k_1k_2k_3),$$

$$\varsigma_3(s) =\frac{-(k_1^2+k_2^2)(k_1-k_2)}{3}.$$

Proof. Let $\sigma(s,v)$ be Smarandache Ruled Surface defined by equation (7), the first and second partial derivatives of $\sigma(s,v)$are given as follows:

$$\sigma_s(s,v) = (-(k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))T_q+(-k_3(\frac{1}{\sqrt{3}}+ \frac{v}{\sqrt{2}})+\frac{k_1}{\sqrt{3}})N_q+(k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_2}{\sqrt{3}})B_q),$$

$$\sigma_v(s,v) = \frac{N_q+B_q}{\sqrt{2}},$$

$$\sigma_{vs}(s,v) = \frac{1}{\sqrt{2}}(-(k_1+k_2)Tq-k_3N_q+k_3B_q),$$

$$\sigma_{vv}(s,v) = 0,$$

$$\sigma_{ss}(s,v) = ((\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})(-k_1'-k_2'+k_1k_3-k_2k_3) -\frac{k_1^2+k_2^2}{\sqrt{3}})T_q+((\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})(-k_3'-k_1^2-k_1k_2 -k_3^2)+\frac{k_1'-k_2k_3}{\sqrt{3}})N_q+((\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})(k_3'-k_2k_1 -k_2^2-k_3^2)+\frac{k_1k_3+k_2'}{\sqrt{3}})B_q.$$

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The unit normal of $\sigma(s,v)$ will be given by the following equation

$$U^{\sigma(s,v)} = \frac{1}{\mu_3(s,v)}[(-2k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1-k_2}{\sqrt{3}})T_q + (k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})N_q - (k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})B_q].$$

The coefficients of the first and second fundamental forms of $\sigma(s,v)$ are given by

$$E^{\sigma(s,v)} = ((k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2 + (-k_3(\frac{1}{\sqrt{3}} + \frac{v}{\sqrt{2}})+\frac{k_1}{\sqrt{3}})^2 + (k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_2}{\sqrt{3}})^2,$$

$$F^{\sigma(s,v)} = \frac{k_1+k_2}{\sqrt{6}}, \qquad G^{\sigma(s,v)} = 1,$$

$$e^{\sigma(s,v)} = \frac{(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})^2\varsigma_1(s)+(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})\varsigma_2(s)+\varsigma_3(s)}{\mu_3(s,v)},$$

$$f^{\sigma(s,v)} = \frac{k_2^2-k_1^2}{\sqrt{6}\mu_3}, \qquad g^{\sigma(s,v)} = 0.$$

where

$$\mu_3(s,v) = [(-2k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1-k_2}{\sqrt{3}})^2 + 2((k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2]^{\frac{1}{2}},$$

$$f_3(s,v) = ((k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}))^2 + (-k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}}) + \frac{k_1}{\sqrt{3}})^2 + (k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_2}{\sqrt{3}})^2 - \frac{(k_1+k_2)^2}{6},$$

$$\varsigma_1(s) = 2k_3(k_1'+k_2') - 2k_3'(k_1+k_2) + (k_2^2-k_1^2-2k_3^2)(k_1+k_2),$$

$$\varsigma_2(s) = \frac{1}{\sqrt{3}}(2k_3(k_1^2+k_2^2)+2k_1'k_2-2k_2'k_1-4k_1k_2k_3),$$

$$\varsigma_3(s) = \frac{-(k_1^2+k_2^2)(k_1-k_2)}{3}.$$

The mean curvature $H^{\sigma(s,v)}$ of $\sigma(s,v)$ and the Gaussian curvature $K^{\sigma(s,v)}$ of $\sigma(s,v)$ will be given by:

$$H^{\sigma(s,v)} = \frac{1}{6f_3(s,v)\mu_3(s,v)}[(k_2^2-k_1^2)(k_1+k_2)+3((\frac{1}{\sqrt{3}}+ \frac{v}{\sqrt{2}})^2\varsigma_1(s)+(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})\varsigma_2(s)+\varsigma_3(s))],$$

$$K^{\Omega(s,v)} = \frac{-(k_2^2-k_1^2)^2}{6f_3(s,v)\mu_3^2(s,v)}.$$

Corollary 31. *$\sigma(s,v)$-Smarandache-Ruled Surface satisfying equation (7) is developable if and only if $k_1 = k_2 = 0$ and $k_3 \neq 0$.*

Corollary 32. *$\sigma(s,v)$-Smarandache-Ruled Surface satisfying equation (7) is minimal surface if and only if $k_1 = k_2 = 0$ and $k_3 \neq 0$.*

Corollary 33. *The geodesic curvature, and the normal curvature for the $T_qN_qB_q$-Smarandache curve satisfying equation (7) is given by*

$$k_g^{\sigma(s,v)} = \frac{-k_1(k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})-k_2(k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})}{\mu_3(s,v)},$$

$$k_n^{\sigma(s,v)} = \frac{1}{\mu_3(s,v)}[(-2k_3(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})+\frac{k_1-k_2}{\sqrt{3}})(-k_1'-k_2' - k_1^2-k_2^2+k_1k_3-k_2k_3)+(k_1+k_2)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_1' - k_3'-k_1^2-k_3^2-k_1k_2-k_2k_3)-(k_1+k_2) (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{2}})(k_2'+k_3'-k_2^2-k_3^2-k_1k_2+k_1k_3)].$$

Corollary 34. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (7) is a geodesic if and only if $k_1 = k_2 = 0$ and $k_3 \neq 0$.*

Corollary 35. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (7) is asymptotic line if and only if $k_1 = k_2 = 0$ and $k_3 \neq 0$.*

Corollary 36. *The distribution parameter for the $\sigma(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{\sigma(s,v)} = \frac{k_2^2-k_1^2}{\sqrt{6}\sqrt{2k_3^2+(k_1+k_2)^2}}.$$

3.7 $T_qN_qB_q$- Smarandache ruled surface with unit vector $\frac{\mathbf{T_q+N_q+B_q}}{\sqrt{3}}$

In this subsection we will define the $T_qN_qB_q$-Smarandache ruled surface according to quasi frame with unit vector $\frac{\mathbf{T_q+N_q+B_q}}{\sqrt{3}}$. We will compute the first and the second fundamental forms of such surface. We will also investigate the mean curvature and the Gaussian curvature.

Definition 7. *Let $\alpha(s)$ be a regular curve in E^3 with a quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the $T_qN_qB_q$-Smarandache Ruled Surface with unit vector $\frac{\mathbf{T_q+N_q+B_q}}{\sqrt{3}}$ defined by the following equation*

$$\Upsilon(s,v) = (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})(T_q+N_q+B_q). \tag{8}$$

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Theorem 7. *Suppose $\Upsilon(s,v)$ be Smarandache Ruled Surface defined by equation(8). Then the mean curvature $H^{\Upsilon(s,v)}$ of $\Upsilon(s,v)$ will be given by*

$$H^{\Upsilon(s,v)} = \frac{o_1(s)}{2\mu_4(s,v)f_4(s,v)},$$

and the Gaussian curvature $K^{\Upsilon(s,v)}$ of $\Upsilon(s,v)$ will be given by

$$K^{\Upsilon(s,v)} = 0.$$

where

$$\begin{aligned}
\mu_4(s,v) &= [(k_1-2k_3-k_2)^2+(k_1+2k_2+k_3)^2 \\
&\quad +(k_3-2k_1-k_2)^2]^{\frac{1}{2}} \\
f_4(s,v) &= (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})[(k_1+k_2)^2+(k_1-k_3)^2 \\
&\quad +(k_2+k_3)^2], \\
o_1(s) &= 3k_2'(-k_1+k_3)+3k_1'(k_2+k_3)-3k_3'(k_1+k_2) \\
&\quad -6k_1k_2k_3+2(-k_1^3+k_2^3-k_3^3).
\end{aligned}$$

Proof. Let $\Upsilon(s,v)$ be Smarandache Ruled Surface defined by equation (8), the first and second partial derivatives of $\Upsilon(s,v)$are given as follows:

$$\begin{aligned}
\Upsilon_s(s,v) &= (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})(-(k_1+k_2)Tq+(k_1-k_3)N_q \\
&\quad +(k_2+k_3)B_q), \\
\Upsilon_v(s,v) &= \frac{T_q+N_q+B_q}{\sqrt{3}}, \\
\Upsilon_{vs}(s,v) &= \frac{1}{\sqrt{3}}(-(k_1+k_2)Tq+(k_1-k_3)N_q \\
&\quad +(k_2+k_3)B_q), \\
\Upsilon_{vv}(s,v) &= 0, \\
\Upsilon_{ss}(s,v) &= (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})[(-k_1'-k_2'-k_1(k_1-k_3) \\
&\quad -k_2(k_2+k_3))T_q+(-k_1(k_1+k_2)+k_1'-k_3' \\
&\quad -k_3(k_2+k_3))N_q+(k_2'+k_3'+k_3(k_1-k_3) \\
&\quad -k_2(k_2+k_1))B_q].
\end{aligned}$$

The unit normal of $\Upsilon(s,v)$ will be given by the following equation

$$\begin{aligned}
U^{\Upsilon(s,v)} &= \frac{1}{\mu_4(s,v)}(k_1-2k_3-k_2)T_q \\
&\quad +(k_1+2k_2+k_3)N_q \\
&\quad +(k_3-2k_1-k_2)B_q.
\end{aligned}$$

The coefficients of the first and second fundamental forms of $\Upsilon(s,v)$ are given by

$$\begin{aligned}
E^{\Upsilon(s,v)} &= (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})^2[(k_1+k_2)^2+(k_1-k_3)^2 \\
&\quad +(k_2+k_3)^2], \\
F^{\Upsilon(s,v)} &= 0, \qquad G^{\Upsilon(s,v)} = 1, \\
e^{\Upsilon(s,v)} &= \frac{o_1(s)(\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})}{\mu_4(s,v)}, \\
f^{\Upsilon(s,v)} &= 0, \qquad g^{\Upsilon(s,v)} = 0.
\end{aligned}$$

where

$$\begin{aligned}
\mu_4(s,v) &= [(k_1-2k_3-k_2)^2+(k_1+2k_2+k_3)^2 \\
&\quad +(k_3-2k_1-k_2)^2]^{\frac{1}{2}} \\
f_4(s,v) &= (\frac{1}{\sqrt{3}}+\frac{v}{\sqrt{3}})[(k_1+k_2)^2+(k_1-k_3)^2 \\
&\quad +(k_2+k_3)^2], \\
o_1(s) &= 3k_2'(-k_1+k_3)+3k_1'(k_2+k_3)-3k_3'(k_1+k_2) \\
&\quad -6k_1k_2k_3+2(-k_1^3+k_2^3-k_3^3).
\end{aligned}$$

The mean curvature $H^{\Upsilon(s,v)}$ of $\Upsilon(s,v)$ and the Gaussian curvature $K^{\Upsilon(s,v)}$ of $\Upsilon(s,v)$ will be given by:

$$H^{\Upsilon(s,v)} = \frac{o_1(s)}{2\mu_4(s,v)f_4(s,v)},$$

$$K^{\Upsilon(s,v)} = 0.$$

Corollary 37. *$\Upsilon(s,v)$-Smarandache-Ruled Surface satisfying equation (8) is developable surface.*

Corollary 38. *$\Upsilon(s,v)$-Smarandache-Ruled Surface satisfying equation (8) can not be a minimal surface.*

Corollary 39. *The geodesic curvature, and the normal curvature for the $T_qN_qB_q$-Smarandache curve satisfying equation (6) is given by*

$$k_g^{\Upsilon(s,v)} = \frac{k_1(k_3-k_2-2k_1)-k_2(k_1+2k_2+k_3)}{\mu_4(s,v)},$$

$$\begin{aligned}
k_n^{\Upsilon(s,v)} &= \frac{1}{\sqrt{3}\mu_4(s,v)}[-(k_1-k_2-2k_3)(k_1(k_1-k_3) \\
&\quad +k_2(k_2+k_3)+k_1'+k_2')+(k_1+2k_2+k_3)(-k_1(k_1 \\
&\quad +k_2)-k_3(k_2+k_3)+k_1'-k_3')+(k_3-k_2-2k_1) \\
&\quad (k_3(k_1-k_3)-k_2(k_1+k_2)+k_2'+k_3')].
\end{aligned}$$

Corollary 40. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation (8) is a geodesic if and only if $k_1 = k_2 = 0$ and $k_3 \neq 0$.*

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Corollary 41. *$T_qN_qB_q$-Smarandache curve lying on the surface given by equation 8 can not be asymptotic line.*

Corollary 42. *The distribution parameter for the $\Upsilon(s,v)$-Smarandache Ruled surface is given by*

$$\lambda^{\Upsilon(s,v)} = 0.$$

Example 1. Consider the unit speed helix curve $\alpha(s) = (\frac{3}{5}sins, \frac{3}{5}coss, \frac{4}{5}s)$.
The quasi-frame vectors of $\alpha(s)$ are given as

$$T_q = (\frac{3}{5}coss, -\frac{3}{5}sins, \frac{4}{5}),$$
$$N_q = (-sins, -coss, 0),$$
$$B_q = (\frac{4}{5}coss, -\frac{4}{5}sins, -\frac{3}{5}).$$

Where $\zeta = (0,0,1)$ and the quasi curvature as follows

$$k_1 = \frac{3}{5}, \;\; k_2 = 0, \;\; k_3 = -\frac{4}{5}.$$

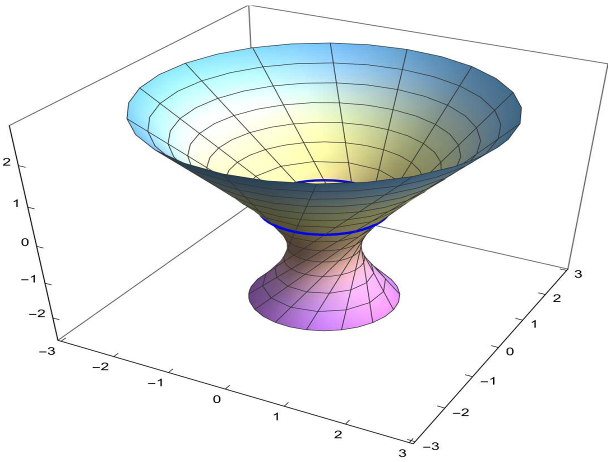

Fig. 2: generated by **T**

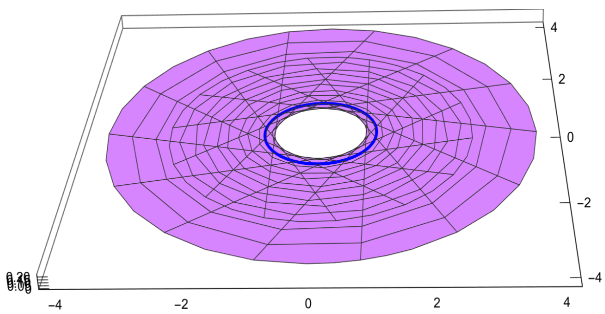

Fig. 3: generated by **N**

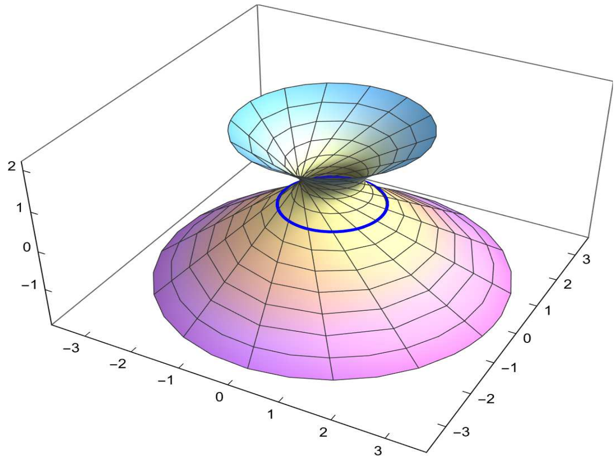

Fig. 4: generated by **B**

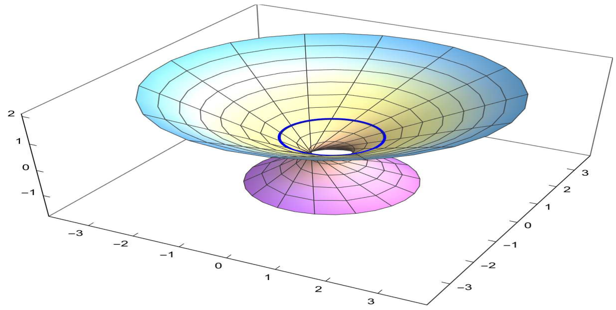

Fig. 5: Generated by $\frac{\mathbf{T_q+N_q}}{\sqrt{2}}$

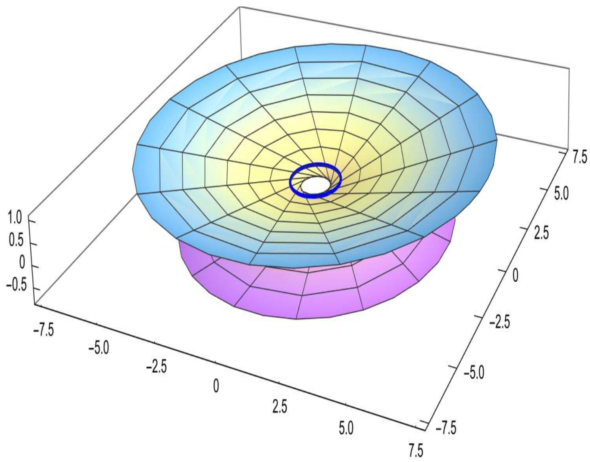

Fig. 6: Generated by $\frac{\mathbf{T_q+B_q}}{\sqrt{2}}$

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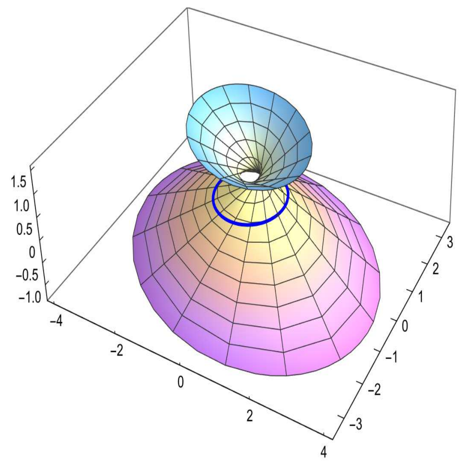

Fig. 7: Generated by $\frac{\mathbf{N_q}+\mathbf{B_q}}{\sqrt{2}}$

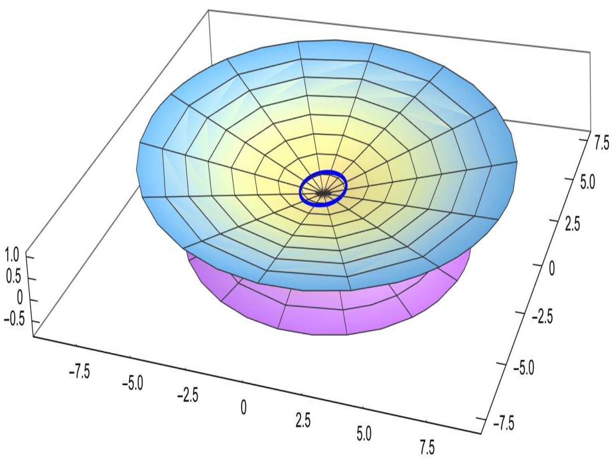

Fig. 8: Generated by $\frac{\mathbf{T_q}+\mathbf{N_q}+\mathbf{B_q}}{\sqrt{3}}$

4 Conclusion

In our article, $T_qN_qB_q$-Smarandache ruled surface are investigated according to quasi frame in Euclidean 3-space. The Gaussian curvature and Mean curvature for these surfaces are calculated. Also the necessary and sufficient conditions for such surfaces to be developable surfaces are obtained. We also introduce some other important geometric properties for these surfaces. Also an example is introduced.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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