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Vega and Theta of an Interest Rate Derivative

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Abstract: Interest rate derivatives are important financial instruments whose values are influenced by movements in interest rates. A good risk manager has to price and compute sensitivities in order to evade unnecessary risks for the interest rate derivatives. In this paper, we derive expressions for the important greeks, namely, *vega* and *Theta* needed to determine how sensitive a derivative price is to changes in the volatility of its underlying interest rate and how the price changes as time to maturity draws near. This is achieved using integration by parts techniques of Malliavin calculus. The derived expressions will assist a risk manager in order to curtail risks.

Keywords: Interest rate derivatives, Vasicek model, Variance gamma process, Malliavin calculus

1 Introduction

A variance gamma (VG) process established by Madan & Seneta [1], being a class of Lévy processes, has contributed to better pricing and hedging of financial instruments such as interest rate derivatives (Bayazit & Nolder [2], Bavouzet et al [3], Udoye & Ekhaguere [4] and Bavouzet & Messaoud [5]). To minimize risks, an experienced risk manager has to study and understand possible effects of changes in parameters representing certain factors, such as unexpected happenings and abrupt information, that bring about spikes and jumps in the market.

Udoye & Ekhaguere [4] initiated a broadened Vasicek model motivated by a VG process, applied the modified Vasicek short rate model in deriving expression representing the price of a certain interest rate derivative (IRD) named a *zero-coupon bond* (ZCB). In this paper, we extend the work of Udoye & Ekhaguere [4] by deriving expressions for the special greeks that help to understand the effects of changes in the volatility of the interest rates and possible reactions as time to maturity draws near. Conor [6] emphasized that Theta calculates the extent at which an option will decay theoretically in price.

Bayazit & Nolder [2] considered sensitivity study for a stock market motivated by a Lévy process of exponential attribute using the Malliavin calculus. We employ the Malliavin calculus in the derivation of important greeks namely, vega and Theta. The Ornstein-Uhlenbeck operator and differentiation tools of the Malliavin calculus in Bayazit & Nolder [2], Bavouzet et al [3], Bavouzet & Messaoud [5] and Udoye & Ekhaguere [4] are to be employed in obtaining expressions for the desired greeks that deal with the sensitivities of a bond price with respect to alterations in selected parameters. The greeks assist risk managers in hedging so as to minimize risks.

The rest of this paper is arranged in the following order: Section 2 considers important facts needed for the success of the work. Section 3 deals with our results. Then, conclusion follows.

2 Foundational Notion

This section considers important definitions. Some results of Udoye & Ekhaguere [4] required for the realization of this paper are specified.

Definition 1. *The* Vasicek model *[7] is an interest rate model with dynamics given by*

$$dr_t = \alpha(\flat - r_t)dt + \sigma dX_t,$$

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where $\alpha, \flat$ and $\sigma \neq 0$ represent mean-reversion speed, long-term mean rate and volatility for the interest rate, respectively. X_t stands for a given Lévy process.

Definition 2. *An* arithmetic Brownian motion *is indicated by*

$$X_t = \theta t + \breve{\sigma} W_t,$$

where θ stands for drift and $\breve{\sigma} \neq 0$ stands for its volatility, while W_t denotes a Wiener process. Time-changing the arithmetic Brownian motion by a gamma process gives a VG process.

Theorem 1. *[4] Let the price of a ZCB at time t having maturity time T of an extended Vasicek model under a VG process be $P = P(t,T)$. Then, the price satisfies*

$$\begin{aligned} P(t,T) = \exp\Bigg(- \Bigg(\Bigg[-\frac{r_0}{\alpha}\left(e^{-\alpha T} - e^{-\alpha t}\right) + \flat\left(T - t + \frac{1}{\alpha}\left(e^{-\alpha T} - e^{-\alpha t}\right)\right) + \frac{\sigma \widetilde{w}}{\alpha}\Bigg[T - t + \frac{1}{\alpha}(e^{-\alpha T} \\ - e^{-\alpha t})\Bigg] + \sigma \sum_{\upsilon \in [t,T]} \sum_{s \in [0,t]} \left(\theta \triangle G(s) e^{-\alpha(\upsilon - s)} + \breve{\sigma}\Delta (G(s))^{1/2} e^{-\alpha(\upsilon - s)} Z\right)\Bigg] + \sigma\Bigg(\widetilde{w}[T-t] \\ + \sum_{\upsilon \in [t,T]} \left(\breve{\sigma} \triangle (G(\upsilon))^{1/2} Z + \theta \triangle G(\upsilon)\right)\Bigg) - \frac{\sigma^2}{2}\Bigg(\sum_{\upsilon \in [t,T]} \left(\breve{\sigma} \triangle (G(\upsilon))^{1/2} Z + \theta \triangle G(\upsilon)\right)^2\Bigg)\Bigg)\Bigg), \end{aligned} \tag{1}$$

where

$$\widetilde{w} = \frac{1}{\nu}\ln(1 - \theta\nu - 0.5\breve{\sigma}^2\nu);$$

r_0 represents the initial interest rate, ν stands for variance of a gamma process utilized as the subordinator; $\triangle G(t)$ represents $G(t_+) - G(t_-)$; while Z symbolizes Gaussian random variables, G symbolizes gamma random variable.

Definition 3. *A call option price having P as its underlying satisfies*

$$\mathbf{V} := e^{-r_0 T}\mathbb{E}[\max(P - K, 0)] = e^{-r_0 T}\mathbb{E}[\bar{\Phi}(P)],$$

where $\bar{\Phi}(P)$ is the payoff function, K is the strike price, while $\mathbf{V}$ reacts to alterations in several parameters.

Theorem 2. *(Integration by part theorem of Malliavin calculus [2]) Let $Q_\eta = \dfrac{\partial P}{\partial \eta}$ where η stands for selected parameters of the ZCB price. Assume that D stands for Malliavin derivative operator, it follows that $\mathbb{M}(P) = \langle DP, DP\rangle$ denotes the Malliavin covariance matrix, having inverse $\dfrac{1}{\mathbb{M}(P)} = \mathbb{M}(P)^{-1}$ such that $DP \neq 0$. Also, let L stands for the Ornstein-Uhlenbeck operator. Given a smooth function denoted by $\bar{\Phi} : \mathbb{R} \to \mathbb{R}$, the equation below holds:*

$$\mathbb{E}[\partial\bar{\Phi}(P)Q] := \mathbb{E}[\bar{\Phi}(P)H(P,Q)].$$

Moreover, $H(P,Q)$ stands for the Malliavin weight given by

$$H(P,Q) = Q\mathbb{M}(P)^{-1}LP - \langle DP, D\mathbb{M}(P)^{-1}\rangle Q - \langle DP, DQ\rangle \mathbb{M}(P)^{-1}$$

such that $\mathbb{E}[H(P,Q)] < \infty$.

The following theorems from Udoye & Ekhaguere [4] will be useful in the computation of the desired greeks.

Theorem 3. *Let $P = P(t,T)$ be given by equation (1). Then, the Malliavian derivative on P becomes*

$$\begin{aligned} DP = -\Bigg[\sigma\breve{\sigma}\Bigg(\sum_{\upsilon \in [t,T]} \sum_{s \in [0,t]} \triangle (G(s))^{1/2} e^{-\alpha(\upsilon - s)} + \sum_{\upsilon \in [t,T]} \left(\triangle (G(\upsilon))^{1/2}\right)\Bigg) - \sigma^2\breve{\sigma}\Bigg(\sum_{\upsilon \in [t,T]} (\breve{\sigma} \triangle (G(\upsilon))^{1/2} Z \\ + \theta \triangle G(\upsilon)) \triangle (G(\upsilon))^{1/2}\Bigg)\Bigg]P. \end{aligned} \tag{2}$$

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Theorem 4. *Let $P = P(t,T)$ be given by equation (1). Then, the Ornstein-Uhlenbeck operator L acts on P to give*

$$\begin{aligned} LP = -\Bigg[&\sigma^2\breve{\sigma}^2 \sum_{\upsilon\in[t,T]} (\triangle(G(\upsilon))^{1/2})^2 + \Bigg(\sigma\breve{\sigma}\cdot \sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]} \triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} + \sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) \\ &-\sigma^2\breve{\sigma}(\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\Bigg)^2 \\ &+Z\Bigg(\sigma\breve{\sigma}(\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} + \sum_{\upsilon\in[t,T]}\triangle(G(\upsilon))^{1/2}) \\ &-\sigma^2\breve{\sigma}\Bigg(\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\Bigg)\Bigg)\Bigg]P. \end{aligned} \tag{3}$$

Theorem 5. *Let $P = P(t,T)$ be given by equation (1). Then, the Malliavin covariance matrix of P satisfies*

$$\begin{aligned} \mathbb{M}(P) = &\sigma^2\breve{\sigma}^2\cdot\Bigg[\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} + \sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) \\ &-\sigma\sum_{\upsilon\in[t,T]}(\theta\triangle G(\upsilon)+\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z)\triangle(G(\upsilon))^{1/2}\Bigg]^2 P^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{M}(P)^{-1} = &(\sigma\breve{\sigma})^{-2}\cdot\Bigg(\Bigg[\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} + \sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) \\ &-\sigma\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\Bigg]P\Bigg)^{-2}. \end{aligned} \tag{4}$$

Theorem 6. *Let $P = P(t,T)$ be given by equation (1). Assume that $\mathbb{M}(P)^{-1}$ is the inverse Malliavin covariance matrix of $P(t,T)$. Then,*

$$\begin{aligned} D\mathbb{M}(P)^{-1} = \Bigg[\Bigg(&\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} + \sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) - \sigma^2\breve{\sigma}\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z \\ &+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\Bigg)^{-3}\Bigg]2P^{-2}\times\Bigg[\sigma^2\breve{\sigma}^2\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2 + \Bigg[\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} \\ &+\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) - \sigma^2\breve{\sigma}\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\Bigg]^2\Bigg]. \end{aligned} \tag{5}$$

3 Results

The expressions for the greeks are obtained in what follows.

3.1 Computation of Vega *for the VG-driven interest rate derivatives*

In this subsection, the greek *vega* for VG-driven IRD is obtained.

$$\mathscr{V} = \frac{\partial}{\partial\sigma}e^{-r_0T}\mathbb{E}[\bar{\Phi}(P)] = e^{-r_0T}\mathbb{E}\left[\bar{\Phi}'(P)\frac{\partial P}{\partial\sigma}\right] = e^{-r_0T}\mathbb{E}\left[\bar{\Phi}(P)H\left(P,\frac{\partial P}{\partial\sigma}\right)\right].$$

We state Lemmas 1-4 needed for Theorem 7.

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Lemma 1. *Let $P = P(t,T)$ be given by equation (1) and $Q_\sigma = \dfrac{\partial P}{\partial \sigma}$. Suppose that DQ_σ is the Malliavin derivative of Q_σ. Then,*

$$\begin{aligned} Q_\sigma =& -\Bigg[\frac{\widetilde{w}}{\alpha}\big[T-t+\frac{1}{\alpha}(e^{-\alpha T}-e^{-\alpha t})\big]+\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(\theta\cdot\triangle G(s)e^{-\alpha(\upsilon-s)}+\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}Z\big) \\ &+\widetilde{w}[T-t]+\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))-\sigma\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))^2\Bigg]P. \end{aligned} \tag{6}$$

Also,

$$\begin{aligned} DQ_\sigma =& -\Bigg[\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}(\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)})+\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) \\ &-2\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\theta\triangle G(\upsilon)+\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z)\triangle(G(\upsilon))^{1/2}\Bigg]P \\ &+\Bigg[\frac{\widetilde{w}}{\alpha}\big[T-t+\frac{1}{\alpha}(e^{-\alpha T}-e^{-\alpha t})\big]+\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(\theta\triangle G(s)e^{-\alpha(\upsilon-s)}+\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}Z\big) \\ &+\widetilde{w}[T-t]+\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))-\sigma\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))^2\Bigg]\mathscr{K}P \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathscr{K} =& \sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}+\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})-\sigma^2\breve{\sigma}\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z \\ &+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}. \end{aligned} \tag{8}$$

Proof. Since $Q_\sigma = \dfrac{\partial P}{\partial \sigma}$, it follows from equation (1) that

$$\begin{aligned} Q_\sigma =& -\Bigg[\frac{\widetilde{w}}{\alpha}\big[T-t+\frac{1}{\alpha}(e^{-\alpha T}-e^{-\alpha t})\big]+\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(\theta\triangle G(s))\cdot e^{-\alpha(\upsilon-s)}+\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}Z\big)+\widetilde{w}[T-t] \\ &+\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))-\frac{2\sigma}{2}\Big(\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))^2\Big)\Bigg]P \end{aligned}$$

which gives equation (6).
Furthermore, the Malliavin derivative

$$\begin{aligned} DQ_\sigma =\;& P\times\Bigg(-\Bigg[\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}(\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)})+\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2})-2\sigma\Big(\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z \\ &+\theta\triangle G(\upsilon))\breve{\sigma}\triangle(G(\upsilon))^{1/2}\Big)\Bigg]\Bigg)+\Bigg(-\Bigg[\frac{\widetilde{w}}{\alpha}\big[T-t+\frac{1}{\alpha}(e^{-\alpha T}-e^{-\alpha t})\big]+\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(\theta\triangle G(s)e^{-\alpha(\upsilon-s)} \\ &+\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}Z\big)+\widetilde{w}[T-t] \\ &+\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))-\sigma\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))^2\Bigg]\Bigg)DP \end{aligned}$$

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$$=-\Bigg[\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}(\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)})+\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})-2\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z$$
$$+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\Bigg]P+-\Bigg[\frac{\widetilde{w}}{\alpha}[T-t+\frac{1}{\alpha}(e^{-\alpha T}-e^{-\alpha t})]+\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(\theta\triangle G(s)e^{-\alpha(\upsilon-s)}$$
$$+\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}Z)\Bigg]+\widetilde{w}[T-t]+\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))$$
$$-\sigma\sum_{\upsilon\in[t,T]}(\theta\triangle G(\upsilon)+\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z)^2\Bigg](-\mathscr{K}P)$$

where $\mathscr{K}$ follows from equation (8).

Lemma 2. *Let $P=P(t,T)$ be given by equation (1). Then,*

$$Q_\sigma\mathbb{M}(P)^{-1}LP=\Lambda\left[\mathscr{K}^{-2}\sigma^2\breve{\sigma}^2\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2+1+\frac{Z}{\mathscr{K}}\right] \tag{9}$$

where $\mathscr{K}$ is specified in equation (8) and

$$\Lambda=\frac{\widetilde{w}}{\alpha}[T-t+\alpha^{-1}(e^{-\alpha T}-e^{-\alpha t})]+\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(\theta\triangle G(s)\quad\cdot e^{-\alpha(\upsilon-s)+\breve{\sigma}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}Z}\big)+\widetilde{w}[T-t]$$
$$+\sum_{\upsilon\in[t,T]}(\theta\triangle G(\upsilon)+\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z)-\sigma\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))^2. \tag{10}$$

Proof. Let Λ be as given above; from equations (6), (4) and (3), it follows that

$$Q_\sigma\mathbb{M}(P)^{-1}LP=-\Lambda P\cdot\mathscr{K}^{-2}P^{-2}\left(-\left[\sigma^2\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2})^2+\mathscr{K}^2+Z\mathscr{K}\right]\right)P$$
$$=\Lambda\mathscr{K}^{-2}\sigma^2\breve{\sigma}^2\big(\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2\big)+\Lambda+\frac{\Lambda Z}{\mathscr{K}}.$$

Lemma 3. *Let $P=P(t,T)$ be given by equation (1). It implies that*

$$\mathbb{M}(P)^{-1}\langle DP,DQ_\sigma\rangle=-\Lambda+\Bigg[\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}(\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)}+\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})-2\sigma\breve{\sigma}$$
$$\cdot\left(\sum_{\upsilon\in[t,T]}(\breve{\sigma}\triangle(G(\upsilon))^{1/2}Z+\theta\triangle G(\upsilon))\triangle(G(\upsilon))^{1/2}\right)\Bigg]\mathscr{K}^{-1} \tag{11}$$

where $\mathscr{K}$ follows from equation (8) and Λ is specified in equation (10).

Proof.
Expression in equation (11) is obtained by replacing and simplifying equation (4) in place of $\mathbb{M}(P)^{-1}$, (2) in place of DP, while equation (7) is in place of DQ_σ.

Lemma 4. *Let $P=P(t,T)$ be given by equation (1). This implies that*

$$Q_\sigma\langle DP,D\mathbb{M}(P)^{-1}\rangle=2\Lambda\left[\mathscr{K}^{-2}\sigma^2\breve{\sigma}^2\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2+1\right] \tag{12}$$

where $\mathscr{K}$ follows from equation (8) and Λ is specified in equation (10).

Proof. Let $\mathscr{K}$ and Λ be as specified in equations (8) and (10), respectively. Then, the result of equation (12) is derived by replacement of equation (6) for Q_σ, equation (2) in place of DP, while equation (5) in place of $D\mathbb{M}(P)^{-1}$, and further simplification.

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Theorem 7. *Let $P = P(t,T)$ be given by equation (1). Then, the greek* vega *satisfies*

$$\mathscr{V} = e^{-r_0 T}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\Phi}(\wp(t,T,g,z)) \cdot H\left(\wp, \frac{\partial \wp}{\partial \sigma}\right)(2\pi)^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}z^2}\left(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\left(\frac{t}{\nu}\right)} g^{\frac{t}{\nu}-1} e^{-\frac{1}{\nu}g}\right) dz dg\right),$$

where

$$H\left(\wp, \frac{\partial \wp}{\partial \sigma}\right) = \frac{\bar{\Lambda} z}{\hat{\mathscr{K}}} - \left[\breve{\sigma} \sum_{\upsilon\in[t,T]} \sum_{s\in[0,t]} (\triangle(g(s))^{1/2} \cdot e^{-\alpha(\upsilon-s)} + \breve{\sigma} \sum_{\upsilon\in[t,T]} (\triangle(g(\upsilon))^{1/2}) - 2\sigma\breve{\sigma} \left(\sum_{\upsilon\in[t,T]} (\breve{\sigma} \triangle (g(\upsilon))^{1/2} z + \theta \triangle g(\upsilon)) \triangle (g(\upsilon))^{1/2}\right)\right] \hat{\mathscr{K}}^{-1} - \frac{\bar{\Lambda}}{\hat{\mathscr{K}}^2}\left[\sigma^2 \breve{\sigma}^2 \sum_{\upsilon\in[t,T]} (\triangle(g(\upsilon))^{1/2})^2\right],$$

$$\hat{\mathscr{K}} = \sigma\breve{\sigma}\left[\sum_{\upsilon\in[t,T]} \sum_{s\in[0,t]} \triangle(g(s))^{1/2} e^{-\alpha(\upsilon-s)} + \sum_{\upsilon\in[t,T]} (\triangle(g(\upsilon))^{1/2}) - \sigma \sum_{\upsilon\in[t,T]} (\breve{\sigma} \triangle (g(\upsilon))^{1/2} z + \theta \triangle g(\upsilon)) \triangle (g(\upsilon))^{1/2}\right]. \tag{13}$$

and

$$\bar{\Lambda} = \frac{\widetilde{w}}{\alpha}[T - t + \alpha^{-1}(e^{-\alpha T} - e^{-\alpha t})] + \sum_{\upsilon\in[t,T]} \sum_{s\in[0,t]} \left(\theta \triangle g(s) e^{-\alpha(\upsilon-s)} + \breve{\sigma} \triangle (g(s))^{1/2} e^{-\alpha(\upsilon-s)} z\right) + \widetilde{w}[T-t] + \sum_{\upsilon\in[t,T]} (\theta \triangle g(\upsilon) + \breve{\sigma} \triangle (g(\upsilon))^{1/2} z) - \sigma \sum_{\upsilon\in[t,T]} (\breve{\sigma} \triangle (g(\upsilon))^{1/2} z + \theta \triangle g(\upsilon))^2. \tag{14}$$

Proof.

$$\mathscr{V} = \frac{\partial \mathbb{V}}{\partial \sigma} = e^{-r_0 T}\mathbb{E}\left[\bar{\Phi}(P) H(P, Q_\sigma)\right].$$

Also, from Theorem 2,

$$H(P, Q_\sigma) = Q_\sigma \mathbb{M}(P)^{-1} LP - \langle DP, DQ_\sigma \rangle \mathbb{M}(P)^{-1} - \langle DP, D\mathbb{M}(P)^{-1} \rangle Q_\sigma.$$

Substituting equations (9), (11) and (12) into the above equation yields the desired weight function given by

$$H(P, Q_\sigma) = \frac{\Lambda Z}{\mathscr{K}} - \left[\breve{\sigma} \sum_{\upsilon\in[t,T]} \sum_{s\in[0,t]} (\triangle(G(s))^{1/2} e^{-\alpha(\upsilon-s)}) + \breve{\sigma} \sum_{\upsilon\in[t,T]} (\triangle(G(\upsilon))^{1/2}) - 2\sigma\breve{\sigma}\left(\sum_{\upsilon\in[t,T]} (\breve{\sigma} \triangle (G(\upsilon))^{1/2} Z + \theta \triangle G(\upsilon) \triangle (G(\upsilon))^{1/2}\right)\right] \mathscr{K}^{-1} - \frac{\Lambda}{\mathscr{K}^2}\left[\sigma^2 \breve{\sigma}^2 \sum_{\upsilon\in[t,T]} (\triangle(G(\upsilon))^{1/2})^2\right].$$

Moreover,

$$\begin{aligned}
\mathscr{V} &= e^{-r_0 T} \cdot \mathbb{E}\left[\bar{\Phi}(P) \cdot H\left(P, \frac{\partial P}{\partial \sigma}\right)\right] \\
&= e^{-r_0 T} \cdot \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\Phi}(\wp(t,T,g,z) \cdot H\left(\wp, \frac{\partial \wp}{\partial \sigma}\right) f_{\mathscr{N}}(z;0,1) \cdot f_G(g; \frac{t}{\nu}, \frac{1}{\nu}) dz dg\right) \\
&= e^{-r_0 T}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\Phi}(\wp(t,T,g,z) H\left(\wp, \frac{\partial \wp}{\partial \sigma}\right)(2\pi)^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}z^2}\left(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\left(\frac{t}{\nu}\right)} g^{\frac{t}{\nu}-1} e^{-\frac{1}{\nu}g}\right) dz dg\right)
\end{aligned}$$

where $f_{\mathscr{N}}(z;0,1)$ stands for the probability density function of a Gaussian random variable, while $f_G(g; \frac{t}{\nu}, \frac{1}{\nu})$ stands for the probability density function of a gamma random variable. Consequently, the result follows.

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3.2 Computation of Theta *for the VG-driven interest rate derivatives*

We compute the greek *Theta* Θ as follows:

$$\begin{aligned}\Theta &= \frac{\partial}{\partial T}e^{-r_0T}\mathbb{E}[\bar{\Phi}(P)] = -r_0e^{-r_0T}\mathbb{E}[\bar{\Phi}(P)] + e^{-r_0T}\mathbb{E}\left[\bar{\Phi}(P)H\left(P, \frac{\partial P}{\partial T}\right)\right] \\ &= -r_0e^{-r_0T}\mathbb{E}[\bar{\Phi}(P)] + e^{-r_0T}\mathbb{E}\left[\bar{\Phi}(P)H(P, \mathscr{Q}_T)\right].\end{aligned}$$

Lemma 5. *Let P be a VG-driven ZCB price. Then,*

$$\mathscr{Q}_T = -\big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)P \tag{15}$$

and

$$D\mathscr{Q}_T = \big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)\mathscr{K}P, \tag{16}$$

where $\mathscr{K}$ *is as specified in equation (8).*

Proof. Applying partial derivative to equation (1) with respect to maturity time T and simplifying gives equation (15). Furthermore, the Malliavin derivative

$$D\mathscr{Q}_T = -\big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)DP.$$

Substituting DP from equation (2) into the above equation gives

$$\begin{aligned}D\mathscr{Q}_T =& -\big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big) \times -\Bigg[\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} \\ &+ \sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) - \sigma^2\Bigg(\sum_{\upsilon\in[t,T]}(\theta\triangle G(\upsilon) + \breve{\sigma}\triangle(G(\upsilon))^{1/2}Z)\breve{\sigma}\triangle(G(\upsilon))^{1/2}\Bigg)\Bigg]P \\ =& \big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)\cdot\Bigg[\sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\triangle(G(s))^{1/2}e^{-\alpha(\upsilon-s)} \\ &+ \sigma\breve{\sigma}\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2}) - \sigma^2\breve{\sigma}\Bigg(\sum_{\upsilon\in[t,T]}(\theta\triangle G(\upsilon) + \breve{\sigma}\triangle(G(\upsilon))^{1/2}Z)\triangle(G(\upsilon))^{1/2}\Bigg)\Bigg]P\end{aligned}$$

where $\mathscr{K}$ is specified in equation (8).

Lemma 6. *Let P be a VG-driven ZCB price. Then,*

$$\mathscr{Q}_T\mathbb{M}(P)^{-1}LP = \big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)\cdot\left[\frac{\sigma^2\breve{\sigma}^2\big(\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2\big)}{\mathscr{K}^2} + 1 + \frac{Z}{\mathscr{K}}\right]. \tag{17}$$

Proof. By equations (15), (4) and (3), the expression in equation (17) follows by the replacement of equation (15) for $\mathscr{Q}_T$, equation (4) in place of $\mathbb{M}(P)^{-1}$ while equation (3) is in place of LP, and further simplification.

Lemma 7. *Let P be a ZCB price under a VG process. It follows that*

$$\mathbb{M}(P)^{-1}\langle DP, D\mathscr{Q}_T\rangle = -\big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big). \tag{18}$$

Proof. The result in equation (18) is derived by replacing equations (4), (2) including (16) for $\mathbb{M}(P)^{-1}$, DP and $D\mathscr{Q}_T$, respectively, with appropriate simplification.

Lemma 8. *Let P be a VG-driven ZCB price. Then,*

$$\mathscr{Q}_T\langle DP, D\mathbb{M}(P)^{-1}\rangle = 2\big(r_0e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)\cdot\left[\frac{\sigma^2\breve{\sigma}^2\big(\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2\big)}{\mathscr{K}^2} + 1\right]. \tag{19}$$

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Proof. The result follows by substituting and simplifying equations (15), (2) and (5) into $\mathscr{Q}_T$, DP with $D\mathbb{M}(P)^{-1}$, respectively.

Theorem 8. *Let P denote a VG-driven ZCB price. Then,*

$$\Theta = -r_0 e^{-r_0 T}\mathbb{E}[\bar{\Phi}(P)] + e^{-r_0 T}\cdot\mathbb{E}\big[\bar{\Phi}(P)H(P,\mathscr{Q}_T)\big] = e^{-r_0 T}\Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{\Phi}(\wp(t,T,g,z))\cdot H\Big(\wp,\frac{\partial\wp}{\partial T}\Big)(2\pi)^{-\frac{1}{2}}$$
$$\cdot e^{-\frac{1}{2}z^2}\Bigg(\frac{\nu^{-\frac{t}{\nu}}}{\Gamma\big(\frac{t}{\nu}\big)}g^{\frac{t}{\nu}-1}e^{-\frac{1}{\nu}g}\Bigg)dzdg\Bigg),$$

where

$$H\Big(\wp,\frac{\partial\wp}{\partial T}\Big) = \big(r_0 e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)\times\Bigg[\frac{z}{\hat{\mathscr{K}}} - \frac{\sigma^2\breve{\sigma}^2\big(\sum_{\upsilon\in[t,T]}(\triangle(g(\upsilon))^{1/2})^2\big)}{\hat{\mathscr{K}}^2}\Bigg]$$

and $\hat{\mathscr{K}}$ *is specified in equation (13).*

Proof. $\Theta = \dfrac{\partial\mathbb{V}}{\partial T} = -r_0 e^{-r_0 T}\mathbb{E}[\bar{\Phi}(P)] + e^{-r_0 T}\mathbb{E}\big[\bar{\Phi}(P)H(P,\mathscr{Q}_T)\big]$. From Theorem 2,

$$H(P,\mathscr{Q}_T) = \mathscr{Q}_T\mathbb{M}(P)^{-1}LP - \langle DP, D\mathscr{Q}_T\rangle\mathbb{M}(P)^{-1} - \mathscr{Q}_T\langle DP, D\mathbb{M}(P)^{-1}\rangle.$$

Substituting equations (17), (18) and (19), and simplifying gives

$$H(P,\mathscr{Q}_T) = Z\mathscr{K}^{-1}\big(r_0 e^{-\alpha T} - \flat(e^{-\alpha T}-1) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big) - \frac{\sigma^2\breve{\sigma}^2\big(\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2\big)}{\mathscr{K}^2}$$
$$`.0\cdot\big(r_0 e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)$$
$$= \big(r_0 e^{-\alpha T} + \flat(1-e^{-\alpha T}) + \frac{\sigma\widetilde{w}}{\alpha}(1-e^{-\alpha T}) + \widetilde{w}\sigma\big)\times\Bigg[\frac{Z}{\mathscr{K}} - \frac{\sigma^2\breve{\sigma}^2\big(\sum_{\upsilon\in[t,T]}(\triangle(G(\upsilon))^{1/2})^2\big)}{\mathscr{K}^2}\Bigg].$$

Hence, the result follows.

4 Application

The greeks give information on how sensitive the worth of the zero-coupon bond option price is to alterations in its independent variables and parameters. Vega provides a measure of the extent that 1% change in the independent variable *volatility* affects the dependent variable *option price* on the interest rate derivative. A portfolio manager has to understand the greeks to minimize risk. The risk associated with the volatility of interest rate is minimized by understanding vega. A vega $\mathscr{V} = 4$ implies that for a 1% increase in volatility, the value of the option on the interest rate derivative will increase by 0.04. Theta measures the rate at which the option price on the interest rate derivative changes as time to maturity draws near, thus, assists in minimizing of risks.

Data of Bank of Ghana Interbank monthly average rate from January 2002 to March 2021 was used to estimate the parameters of the interest rate derivatives using Least-square regression method and VarianceGamma package version 0.4-0 from R software version 3.2.2. The parameter values of the interest rate derivatives are obtained as follows:
$\alpha = 0.027018$, $\flat = 0.153394$, $\sigma = 0.028108$, $r_0 = 0.262$, $\breve{\sigma} = 0.03318$, $\theta = 0.11673$, $\kappa = 0.11840$, and $n = 225$.
Thus, the price of a zero-coupon price using Bank of Ghana data gives

$$P(t,T) = \exp\Bigg(-\Bigg(\Bigg[-\frac{0.262}{0.02702}\big(e^{-0.02702T} - e^{-0.02702t}\big) + 0.153394\big(T - t + \frac{1}{0.02702}(e^{-0.02702T} - e^{-0.02702t})\big)$$
$$+\frac{0.028108\widetilde{w}}{0.02702}\Bigg[T - t + \frac{1}{0.02702}(e^{-0.02702T} - e^{-0.02702t})\Bigg] + 0.028108\sum_{\upsilon\in[t,T]}\sum_{s\in[0,t]}\big(0.11673\triangle G(s)e^{-0.02702(\upsilon-s)}$$
$$+0.03318\Delta(G(s))^{1/2}e^{-0.02702(\upsilon-s)}Z\big)\Bigg] + 0.028108\Bigg(\widetilde{w}[T-t] + \sum_{\upsilon\in[t,T]}(0.03318\triangle(G(\upsilon))^{1/2}Z$$
$$+0.11673\triangle G(\upsilon))\Bigg) - \frac{0.028108^2}{2}\Bigg(\sum_{\upsilon\in[t,T]}(0.03318\triangle(G(\upsilon))^{1/2}Z + 0.11673\triangle G(\upsilon))^2\Bigg)\Bigg)\Bigg),$$

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where

$$\widetilde{w} = \frac{1}{\nu}\ln(1 - 0.11673\nu - 0.50.03318^2\nu);\ \nu = 0.11840.$$

5 Conclusion

We have derived expressions for the two important greeks, namely, vega and Theta. The greek *vega* gives the rate at which the option price changes as a result of change in volatility, and hence, assists in taking the right position to avoid unnecessary risks. The greek *Theta* gives the rate at which bond option price changes as time to maturity draws near, and thus, assists a risk manager in minimizing risks. Furthermore, data from monthly average internank rate of Bank of Ghana was used to estimate parameters of the interest rate derivative and thus, the zero-coupon bond price driven by a variance gamma process was obtained.

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