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Computational Solutions for Fractional Atangana-Baleanu PDEs: An Exploration of Sawi Transform and Homotopy Perturbation Method

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Abstract: In this paper, the Sawi transform and the homotopy perturbation method are combined to present a novel and effective approach to solving fractional Atangana-Baleanu partial differential equations (PDEs). The suggested method is efficient in computing, decreasing time and resources needed to solve complicated and nonlinear equations. The essay introduces an interdisciplinary method for connecting fractional calculus theory and practice. Due to its solid and adaptable solution framework, this method can be used to solve numerous fluid mechanics, biological, and nonlinear wave equation problems. Numerous numerical applications have been examined to demonstrate the method's efficacy. The work introduces an approach that may be applied to various fractional differential equations and operators, paving the way for further research. It greatly advances the field by expanding research and application.

Keywords: Fractional partial differential equations; Sawi transform; Atangana-Baleanu fractional derivative; Homotopy perturbation method.

1 Introduction

Fractional partial differential equations (FPDEs) are used in biology, engineering, and physics to mimic various phenomena [1,2]. Due to the nonlocal and unique structure of fractional derivatives, FPDEs are often challenging to solve numerically. Many analytical and numerical methods have been developed to solve FPDEs, including fixed point, power series, fractional differential transform, and finite difference methods [3,4].

Fractional calculus is widely studied in fields such as physics, biology, and finance due to its practical applications [5,6]. Among fractional derivatives, the Atangana-Baleanu (AB) derivative excels in explaining complex systems with memory and hereditary traits [7,8,9,10,11]. Solving fractional AB PDEs is complex, requiring efficient and accurate approaches [12,13].

One of the most compelling aspects of this article is its efficiency in solving fractional AB Caputo PDEs. Traditional methods often require extensive computational resources and time, especially for complex and high-dimensional problems [14,15]. Our proposed method amalgamates the Sawi Transform and Homotopy Perturbation Method, both of which are known for their computational efficiency [16,17,18]. By synthesizing these methods, we have significantly reduced the computational time and resources required, making it a highly efficient approach for both academic researchers and industry professionals [19,20].

The novelty of this article lies in its interdisciplinary approach. While fractional calculus has been extensively studied, the unique combination of this method in the context of AB Caputo fractional differential equations is unprecedented [21,22]. This innovative approach allows for a more versatile and robust solution framework capable of tackling a wide array of problems in various scientific domains [23,24].

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Another novel contribution is the combination of theoretical and applied aspects. While many existing methods focus either on the theoretical underpinnings or the applied solutions, our approach provides a seamless transition between the two [5,6]. This is particularly beneficial for fields that require quick and accurate solutions to complex problems [19,20].

The efficiency and novelty of this article pave the way for future research that enables the proposed method and extends it to other types of fractional differential equations and operators, thus opening new horizons for research and application [8,16]. In summary, the efficiency and novelty of this article make it a significant contribution to the field of fractional calculus, particularly in solving AB fractional differential equations [7,8]. It not only advances the existing methods but also opens the door for future innovations and applications [25,26,27,28,29].

The article's purpose is to illustrate a general, rapid, and correct technique to solve fractional AB PDEs using the proposed method. The following sections will explore the mathematical formulations, methods of solution, and applications, supported by comparative and numerical studies [14].

The goal of this study is to combine the advantages of the homotopy perturbation approach and the Sawi transform to provide a generalized, accurate, and efficient technique for solving fractional AB partial differential equations. Our methodology synthesizes the body of prior work and introduces new strategies that improve the solutions' computing efficiency and accuracy. By incorporating effective analytical methods into the AB operator, we build on the study of earlier scholars and offer a thorough examination of the solutions' uniqueness and convergence.

The paper is organized as follows: Section 2 provides a review of the AB fractional differential equations, incorporating insights from comparative studies. Section 3 delves into the Sawi transform and homotopy perturbation method, detailing their applications and limitations in solving fractional differential equations. Section 4 presents the proposed method, its mathematical formulation, and its application to various real-world problems. Finally, Section 5 concludes the paper, offering a summary of the contributions and suggesting avenues for future research.

2 Basic Concepts of Sawi Transform

The Sawi transform is presented in this section. We outline a few fundamental characteristics pertaining to this transform's existence conditions, linearity, and inverse. Furthermore, the Sawi transform is applied to elementary basic functions using a few key features and findings. We present the derivative properties as well as the Sawi convolution theory [20].

Definition 1. *if the function $w(t)$ is defined over a positive domain and is a function of t. Next, $\mathbb{S}[w(t)]$, which represents the Sawi transformation of $w(t)$, is provided by*

$$\mathbb{S}[w(t)] = \Psi(v) = \frac{1}{v^2}\int_0^\infty w(t)e^{\frac{-t}{v}}dt, \ \ t \geq 0, \ \ v > 0. \tag{1}$$

The inverse Sawi transformation is provided as

$$\mathbb{S}^{-1}[\Psi(v)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{v^2}e^{\frac{1}{v}t}\Psi(v)dv = w(t), \ \ t > 0, \ c \in \mathbb{R}. \tag{2}$$

Theorem 1. *Let $w(t)$ be a continuous function of exponential order ρ defined for $t > 0$. Afterward, $\mathbb{S}[w(t)]$ exists and fulfills $v > \rho$*

$$|w(t)| \leq Me^{\rho t}, \tag{3}$$

where $M > 0$, then Sawi transformation exists for $v > \rho$.

Suppose that $\mathbb{S}[w(t)] = \Psi(v)$ and $\mathbb{S}[h(t)] = H(v)$ and $i, j \in \mathbb{R}$, then:

- $\mathbb{S}[iw(t) + jh(t)] = i\mathbb{S}[w(t)] + j\mathbb{S}[h(t)]$.
- $\mathbb{S}^{-1}[i\,\Psi(v) + j\,H(v)] = i\,\mathbb{S}^{-1}[\Psi(v)] + j\,\mathbb{S}^{-1}[H(v)]$.
- $\mathbb{S}\left[t^j\right] = v^{j-1}\Gamma(j+1)$.
- $\mathbb{S}\left[e^{jt}\right] = \frac{1}{v(1-jv)}$.
- $\mathbb{S}[\cos(jt)] = \frac{1}{v\left(1+j^2v^2\right)}$.
- $\mathbb{S}[\sin(jt)] = \frac{j}{1+j^2v^2}$.
- $\mathbb{S}[\cosh(jt)] = \frac{1}{v\left(1-j^2v^2\right)}$.
- $\mathbb{S}[\sinh(jt)] = \frac{j}{1-j^2v^2}$.
- $\mathbb{S}\left[\frac{d^jw(t)}{dt^j}\right] = \frac{\Psi(v)}{v^j} - \sum_{i=0}^{j-1}\frac{w^{(i)}(0)}{v^{j-i+1}}$.

Theorem 2. *Let* $\mathbb{S}[w(t)] = \Psi(v)$. *Then,*

$$\mathbb{S}[w(t-j)H(t-j)] = e^{-\frac{1}{v}j}\,\Psi(v), \tag{4}$$

where $H(t)$ *is the unit step function can be found by*

$$H(t-j) = 1, t > j, 0, otherwise.$$

Theorem 3. (Sawi Convolution Theorem). *If* $\mathbb{S}[w(t)] = \Psi(v)$ *and* $\mathbb{S}[h(t)] = H(v)$, *then*

$$\mathbb{S}[(w*h)(t)] = v^2\,\Psi(v)H(v). \tag{5}$$

3 Basic Principles of Fractional Calculus

This section discusses the definitions and properties of fractional calculus that will be utilized in this study.

Definition 2.[7] *The Mittag-Leffler function is defined as*

$$E^{\mu}_{r,\mathscr{K}}(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\mu n}{\Gamma(rn+\mathscr{K})} \quad ,t,\mu,r\in\mathbb{C} \quad , Re(r) > 0. \tag{6}$$

Lemma 1. *Let* $0 < r < 1$ *and* $\rho \in \mathbb{R}$ *such that* $v < \frac{1}{|\rho|^{\frac{1}{r}}}$, *then*

$$\mathbb{S}\left[t^{\mathscr{K}-1}E^{\mu}_{r,\mathscr{K}}(\rho t^r)\right] = \frac{v^{\mathscr{K}-2}}{(1-\rho v^r)^{\mu}}. \tag{7}$$

Proof. The Sawi transform of the function $t^{\mathscr{K}-1}E^{\mu}_{r,\mathscr{K}}(\rho t^r)$ yields

$$\begin{aligned}
\mathbb{S}\left[t^{\mathscr{K}-1}E^{\mu}_{r,\mathscr{K}}(\rho t^r)\right] &= \frac{1}{v^2}\int_0^{\infty}\left[t^{\mathscr{K}-1}E^{\mu}_{r,\mathscr{K}}(\rho t^r)\right]e^{\frac{-t}{v}}dt = \frac{1}{v^2}\int_0^{\infty}t^{\mathscr{K}-1}\sum_{n=0}^{\infty}\frac{\mu n}{\Gamma(nr+\mathscr{K})}\frac{(\rho t^r)^n}{n!}e^{\frac{-t}{v}}dt \\
&= \sum_{n=0}^{\infty}\frac{\mu n}{\Gamma(nr+\mathscr{K})}\frac{\rho^n}{n!}\frac{1}{v^2}\int_0^{\infty}t^{\mathscr{K}-1}t^{nr}e^{\frac{-t}{v}}dt = \sum_{n=0}^{\infty}\frac{\mu n}{\Gamma(nr+\mathscr{K})}\frac{\rho^n}{n!}\mathbb{S}\left[t^{nr+\mathscr{K}-1}\right] \\
&= \sum_{n=0}^{\infty}\frac{\mu n}{\Gamma(nr+\mathscr{K})}\frac{\rho^n}{n!}v^{nr+\mathscr{K}-2}\Gamma(nr+\mathscr{K}) = v^{\mathscr{K}-2}\sum_{n=0}^{\infty}\frac{\mu n}{n!}(\rho v^r)^n = \frac{v^{\mathscr{K}-2}}{(1-\rho v^r)^{\mu}}.
\end{aligned}$$

Corollary 1. *Under the same conditions of Lemma 1, we have*

- $\mathbb{S}\left[t^{\mathscr{K}-1}E_r(\rho t^r)\right] = \frac{v^{\mathscr{K}-2}}{1-\rho v^r}$.
- $\mathbb{S}[E_r(\rho t^r)] = \frac{1}{v(1-\rho v^r)}$.
- $\mathbb{S}\left[E_r\left(\frac{r}{r-1}t^r\right)\right] = \frac{1-r}{v(rv^r-r+1)}$.

Definition 3.[7] *Let* $w \in H^1(0,1)$ *and* $0 < r < 1$.*Then the fractional AB derivative is defined as*

$${}^{ABC}_{0}D^r_t w(t) = \frac{N(r)}{1-r}\int_0^t E_r\left(\frac{r(t-\tau)^r}{r-1}\right)w'(\tau)\,d\tau, \tag{8}$$

where the normalization term $N(r) > 0$ *and satisfies these conditions* $N(1) = N(0) = 1$.

Definition 4.[7] *Let* $w \in H^1(0,1)$ *and* $0 < r < 1$. *Then the fractional AB is represented using the definition of Riemann-Liouville as*

$${}^{ABR}_{0}D^r_t w(t) = \frac{N(r)}{1-r}\frac{d}{dt}\int_0^t E_r\left(\frac{r(t-\tau)^r}{r-1}\right)w(\tau)\,d\tau. \tag{9}$$

Theorem 4. *Let $\Psi(v)$ be Sawi transform of $w(t)$. Then using Caputo's sense, the Sawi transform for the fractional AB derivative is written as*

$$\mathbb{S}\left[{}_0^{ABC}D_t^{\alpha} w(t)\right] = \frac{N(r)}{rv^r - r + 1}\left(\Psi(v) - \frac{1}{v}w(0)\right). \tag{10}$$

Proof. According to the convolution integral definition, we have

$$\int_0^t E_r\left(\frac{r(t-\tau)^r}{r-1}\right) w'(\tau)\, d\tau = E_r\left(\frac{rt^r}{r-1}\right) * w'(t).$$

Thus,

$$\begin{aligned}\mathbb{S}\left[{}_0^{ABC}D_t^{r} w(t)\right] &= \mathbb{S}\left[\frac{N(r)}{1-r}\int_0^t E_r\left(\frac{r(t-\tau)^r}{r-1}\right) w'(\tau)\, d\tau\right] \\ &= \frac{N(r)}{1-r}\mathbb{S}\left[E_r\left(\frac{rt^r}{r-1}\right) * w'(t)\right].\end{aligned}$$

Using Sawi transform and convolution theorem, we get

$$\mathbb{S}\left[{}_0^{ABC}D_t^{r} w(t)\right] = \frac{N(r)}{1-r}\left(v^2\mathbb{S}\left[E_r\left(\frac{rt^r}{r-1}\right)\right]\mathbb{S}\left[w'(t)\right]\right).$$

Using Lemma 1 and applying the result obtained in Corollary 1, and derivative properties of Sawi transform, then we have

$$\mathbb{S}\left[{}_0^{ABC}D_t^{r} w(t)\right] = \frac{N(r)}{1-r}\left(\left(\frac{1-r}{v\,(rv^r - r + 1)}\right)(v\,\Psi(v) - w(0))\right).$$

Therefore,

$$\mathbb{S}\left[{}_0^{ABC}D_t^{r} w(t)\right] = \frac{N(r)}{rv^r - r + 1}\left(\Psi(v) - \frac{1}{v}w(0)\right).$$

Theorem 5. *Let $\Psi(v)$ is Sawi transform of $w(t)$. By adding Eqs. (16) and (17) into Eq. (15), the recursive connection is provided by*

$$\mathbb{S}\left[{}_0^{ABR}D_t^{r} w(t)\right] = \frac{N(r)\Psi(v)}{rv^r - r + 1}. \tag{11}$$

Proof. According to the convolution integral definition, we have

$$\int_0^t E_r\left(\frac{r(t-\tau)^r}{r-1}\right) w(\tau)\, d\tau = E_r\left(\frac{rt^r}{r-1}\right) * w(t).$$

Thus,

$$\begin{aligned}\mathbb{S}\left[{}_0^{ABR}D_t^{r} w(t)\right] &= \mathbb{S}\left[\frac{N(r)}{1-r}\frac{d}{dt}\int_0^t E_r\left(\frac{r(t-\tau)^r}{r-1}\right) w(\tau)\, d\tau\right] \\ &= \frac{N(r)}{1-r}\mathbb{S}\left[\frac{d}{dt}\left(E_r\left(\frac{rt^r}{r-1}\right) * w(t)\right)\right].\end{aligned}$$

Using derivative properties of Sawi transform, we get

$$\mathbb{S}\left[{}_0^{ABR}D_t^{r} w(t)\right] = \frac{N(r)}{1-r}\left(\frac{1}{v}\mathbb{S}\left[E_r\left(\frac{rt^r}{r-1}\right) * w(t)\right] - \frac{1}{v^2}E_r(0) * w(0)\right).$$

Using convolution theorem of Sawi transform and applying the result obtained in Corollary 1, then we have

$$\mathbb{S}\left[{}_0^{ABR}D_t^{r} w(t)\right] = \frac{N(r)\Psi(v)}{rv^r - r + 1}.$$

4 Investigation of the Sawi Transform Homotopy Perturbation Method

This section of the study discusses the fundamental principle of the Sawi transform homotopy perturbation method for FPDEs. We examine the following general partial differential equations to demonstrate the basic structure of the Sawi transform homotopy perturbation method:

$$D_t^r w(u,t) = k(u,t) + L(w(u,t)) + N(w(u,t)), \quad (u,t) \in [0,1] \times [0,T], \; n-1 < r < n, \tag{12}$$

subject to the conditions

$$\frac{d^s w(u,0)}{dt^s} = k_s(u) \quad , \quad s = 0,1,\ldots,n-1, \tag{13}$$

where $k(u,t)$ is a given function, $w(u,t)$ is the unknown function, L is linear differential operator, N is nonlinear differential operators, and D_t^r is the AB fractional derivative.

Applying the Sawi transform for Eq. (12), we obtain

$$W(u,v) = v^r \left(K(u,v) + \mathbb{S}[L(w(u,t)) + N(w(u,t))]\right) + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}} \left(\frac{\partial^i w(u,0)}{\partial t^i}\right). \tag{14}$$

The fractional AB derivative is given by

$$W(u,v) = \frac{rv^r - r + 1}{N(r)} \left(K(u,v) + \mathbb{S}[L(w(u,t)) + N(w(u,t))]\right) + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}} \left(\frac{\partial^i w(u,0)}{\partial t^i}\right). \tag{15}$$

Consequently, the definition of the homotopy parameter q is

$$w(u,t) = \sum_{z=0}^{\infty} q^z w_z(u,t), \tag{16}$$

and Eq. (12)'s nonlinear terms can be expressed as

$$N(w(u,t)) = \sum_{z=0}^{\infty} q^z \mathscr{H}_z, \tag{17}$$

where

$$\mathscr{H}_z = \frac{1}{z!} \frac{\partial^z}{\partial q^z} \left(N \left(\sum_{i=0}^{\infty} q^i w_i(u,t) \right) \right)_{q=0} , \quad z = 0,1,2,\cdots. \tag{18}$$

$\mathscr{H}_z$ is He's polynomials. The recursive connection that is the solution of the AB derivative, by putting Eqs. (16) and (17) into Eq. (15), is given by

$$\sum_{z=0}^{\infty} q^z W_z(u,v) = \left(\frac{rv^r - r + 1}{N(r)}\right) K(u,v) + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}} \left(\frac{\partial^i w(u,0)}{\partial t^i}\right) + q \left(\frac{rv^r - r + 1}{N(r)}\right) \mathbb{S}\left[L\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right) + \sum_{z=0}^{\infty} q^z \mathscr{H}_z\right]. \tag{19}$$

Eq. (19) can be transformed using the Sawi transform, giving us

$$\sum_{z=0}^{\infty} q^z w_z(u,t) = \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right) K(u,v) + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}} \left(\frac{\partial^i w(u,0)}{\partial t^i}\right)\right] + q\,\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right) \mathbb{S}\left[L\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right) + \sum_{z=0}^{\infty} q^z \mathscr{H}_z\right]\right]. \tag{20}$$

Therefore, When solved for q, Eq. (20) is defined as

$$
\begin{aligned}
q^0 : w_0(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right) K(u,v) + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}} \left(\frac{\partial^i w(u,0)}{\partial t^i}\right)\right],\\
q^1 : w_1(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right) \mathbb{S}[L(w_0(u,t)) + \mathscr{H}_0]\right],\\
q^2 : w_2(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right) \mathbb{S}[L(w_1(u,t)) + \mathscr{H}_1]\right],\\
&\vdots\\
q^{z+1} : w_{z+1}(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right) \mathbb{S}[L(w_z(u,t)) + \mathscr{H}_z]\right], \qquad z \geq 0.
\end{aligned}
\tag{21}
$$

Assume that Eq. (21) is the approximate solution to Eq. (12) when $q \to 1$ is applied, and the solution is

$$
w = w_0 + w_1 + w_2 + w_3 + \cdots .
$$

5 Numerical Examples

This section of the research analyzes the efficacy of the novel approach for AB fractional derivatives in addressing an initial value problem.

Example 1. Find the solution of the IVP

$$
\frac{\partial^r w(u,t)}{\partial t^r} = \frac{\partial^2 w(u,t)}{\partial u^2} + w(u,t) - \frac{\partial w(u,t)}{\partial u} + w(u,t)\frac{\partial w(u,t)}{\partial u} - w^2(u,t), \tag{22}
$$

where $0 < r \leq 1$, and subject to the conditions

$$
w(u,0) = e^u. \tag{23}
$$

Solution. For Eq. (22), we apply the Sawi transform AB operator homotopy perturbation, and we get

$$
\begin{aligned}
\sum_{z=0}^{\infty} q^z W_z(u,v) = \frac{1}{v} w(u,0) + q\left(\frac{rv^r - r + 1}{N(r)}\right)\left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_{uu}\right.\right.\\
\left.\left. + \sum_{z=0}^{\infty} q^z w_z(u,t) - \left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_u\right]\right) + q\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[\sum_{z=0}^{\infty} \mathscr{H}_z\right].
\end{aligned}
\tag{24}
$$

Applying the Sawi transform in reverse to Eq. (24), we obtain

$$
\begin{aligned}
\sum_{z=0}^{\infty} q^z w_z(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v} w(u,0)\right] + q\,\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_{uu}\right.\right.\right.\\
\left.\left.\left. + \sum_{z=0}^{\infty} q^z w_z(u,t) - \left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_u\right]\right)\right] + q\,\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[\sum_{z=0}^{\infty} \mathscr{H}_z\right]\right].
\end{aligned}
\tag{25}
$$

The initial terms of $\mathscr{H}_z$ are provided by

$$
\begin{aligned}
\mathscr{H}_0 &= w_0 w_{0u} - (w_0)^2,\\
\mathscr{H}_1 &= w_0 w_{1u} + w_1 w_{0u} - 2w_0 w_1,\\
\mathscr{H}_2 &= w_0 w_{2u} + w_1 w_{1u} + w_2 w_{0u} - 2w_0 w_2 - (w_2)^2,\\
&\vdots
\end{aligned}
\tag{26}
$$

To determine the function of the AB derivative result, the powers of:

$$q^0 : w_0(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v}w(u,0)\right] = \mathbb{S}^{-1}\left[\frac{1}{v}e^u\right] = e^u. \tag{27}$$

$$q^{z+1} : w_{z+1}(u,t) = \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{z_{uu}}(u,t) + w_z(u,t) - w_{z_u}(u,t) + \mathscr{H}_z\right]\right], z \geq 0. \tag{28}$$

Putting $z = 0$ into Eq. (28), we get

$$\begin{aligned}
q^1 : w_1(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{0_{uu}}(u,t) + w_0(u,t) - w_{0_u}(u,t) + \mathscr{H}_0\right]\right] \\
&= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[e^{\,u} + e^{\,u} - e^{\,u} + e^{\,2u} - e^{\,2u}\right]\right] \\
&= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\frac{1}{v}e^{\,u}\right] = \frac{1}{N(r)}\mathbb{S}^{-1}\left[\left(rv^{r-1} - \frac{r}{v} + \frac{1}{v}\right)e^{\,u}\right] \\
&= \frac{e^{\,u}}{N(r)}\left(\frac{t^{\,r}}{\Gamma(r)} - r + 1\right).
\end{aligned}$$

Putting $z = 1$ into Eq. (28), we get

$$\begin{aligned}
q^2 : w_2(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{1_{uu}}(u,t) + w_1(u,t) - w_{1_u}(u,t) + \mathscr{H}_1\right]\right] \\
&= \frac{1}{N(r)}\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[\frac{e^u}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right)\right]\right] \\
&= \frac{e^u}{(N(r))^2}\mathbb{S}^{-1}\left[(rv^r - r + 1)\mathbb{S}\left[\left(\frac{t^r}{\Gamma(r)} - r + 1\right)\right]\right] \\
&= \frac{e^u}{(N(r))^2}\mathbb{S}^{-1}\left[\left(r^2v^{2r-1} + 2r(1-r)v^{r-1} + v^{-1}(1-r)^2\right)\right] \\
&= \frac{e^u}{(N(r))^2}\left(r^2\frac{t^{2r}}{\Gamma(2r+1)} + 2r(1-r)\frac{t^r}{\Gamma(r+1)} + (1-r)^2\right).
\end{aligned}$$

Consequently, Eq. (22)'s solution is provided by

$$\begin{aligned}
w(u,t) = w_0 + w_1 + w_2 + \ldots &= e^u + \frac{e^u}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right) + \frac{e^u}{(N(r))^2}\left(\frac{r^2t^{2r}}{\Gamma(2r+1)} + \frac{2r(1-r)t^r}{\Gamma(r+1)} + (1-r)^2\right) + \ldots \\
&= e^u\left(1 + \frac{1}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right) + \frac{1}{(N(r))^2}\left(\frac{r^2t^{2r}}{\Gamma(2r+1)} + \frac{2r(1-r)t^r}{\Gamma(r+1)} + (1-r)^2\right) + \ldots\right).
\end{aligned}$$

Thus, the exact solution is $w(u,t) = e^{u+t}$.

Below, in Figure 1 and 2 we plot the exact and the approximate solutions. We sketch the graph of the approximate solutions for different values of $r = 1, 0.9, 0.8$ and 0.6.

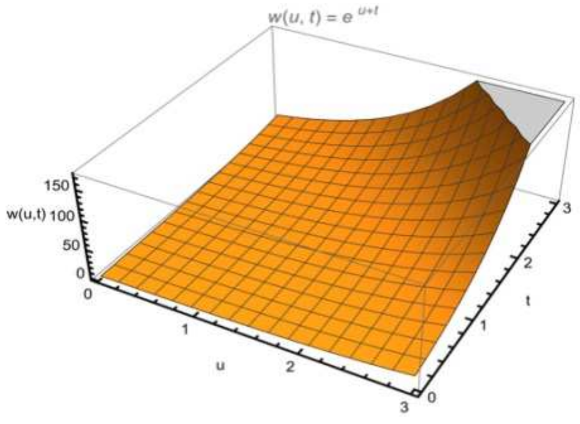

Fig. 1: The exact solution of Example 1.

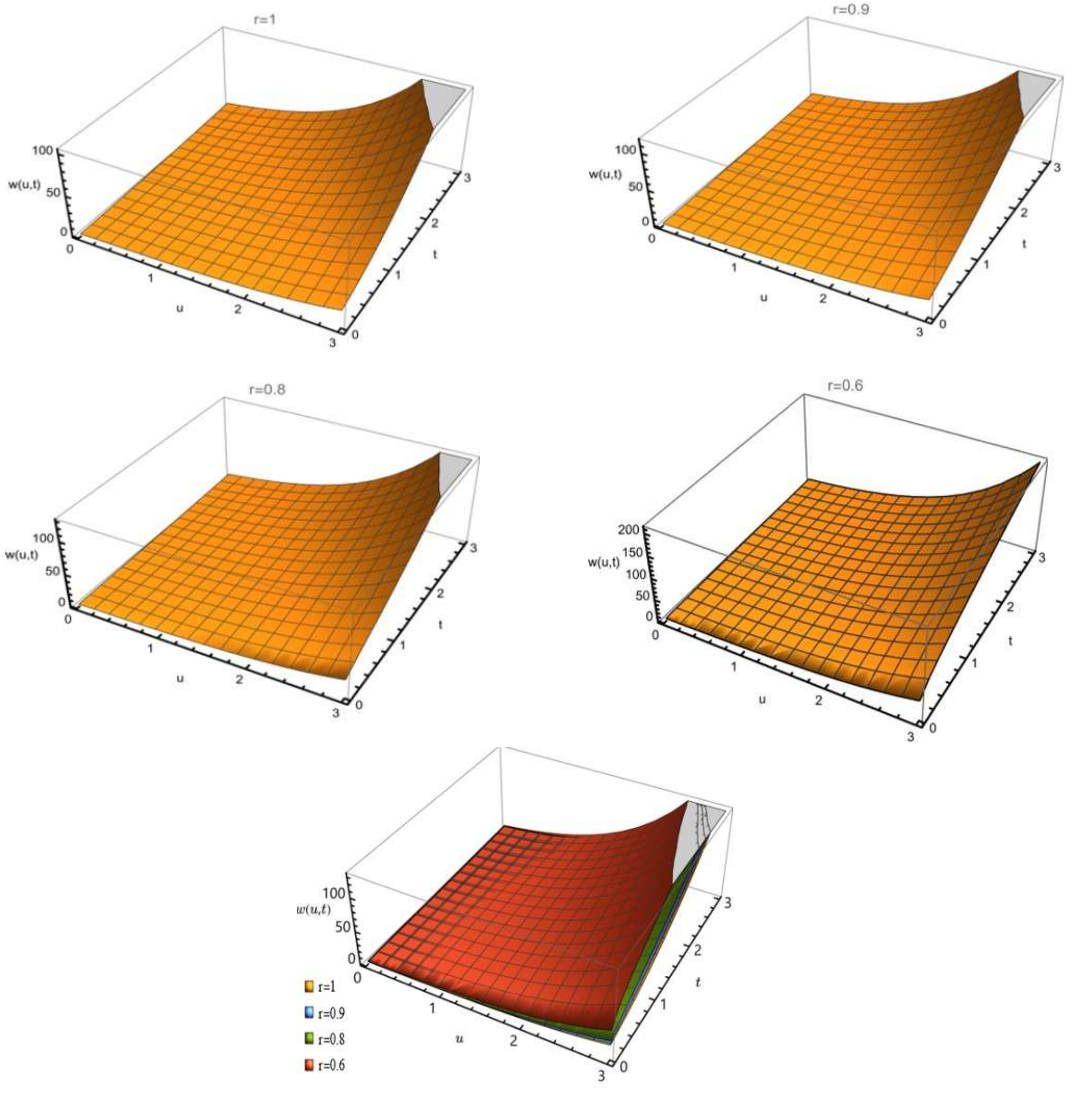

Fig. 2: Approximate solutions of Example 1 for different values of r.

In Figure 3, we sketch the absolute error for different values of $r = 1, 0.9, 0.8$ and 0.6.

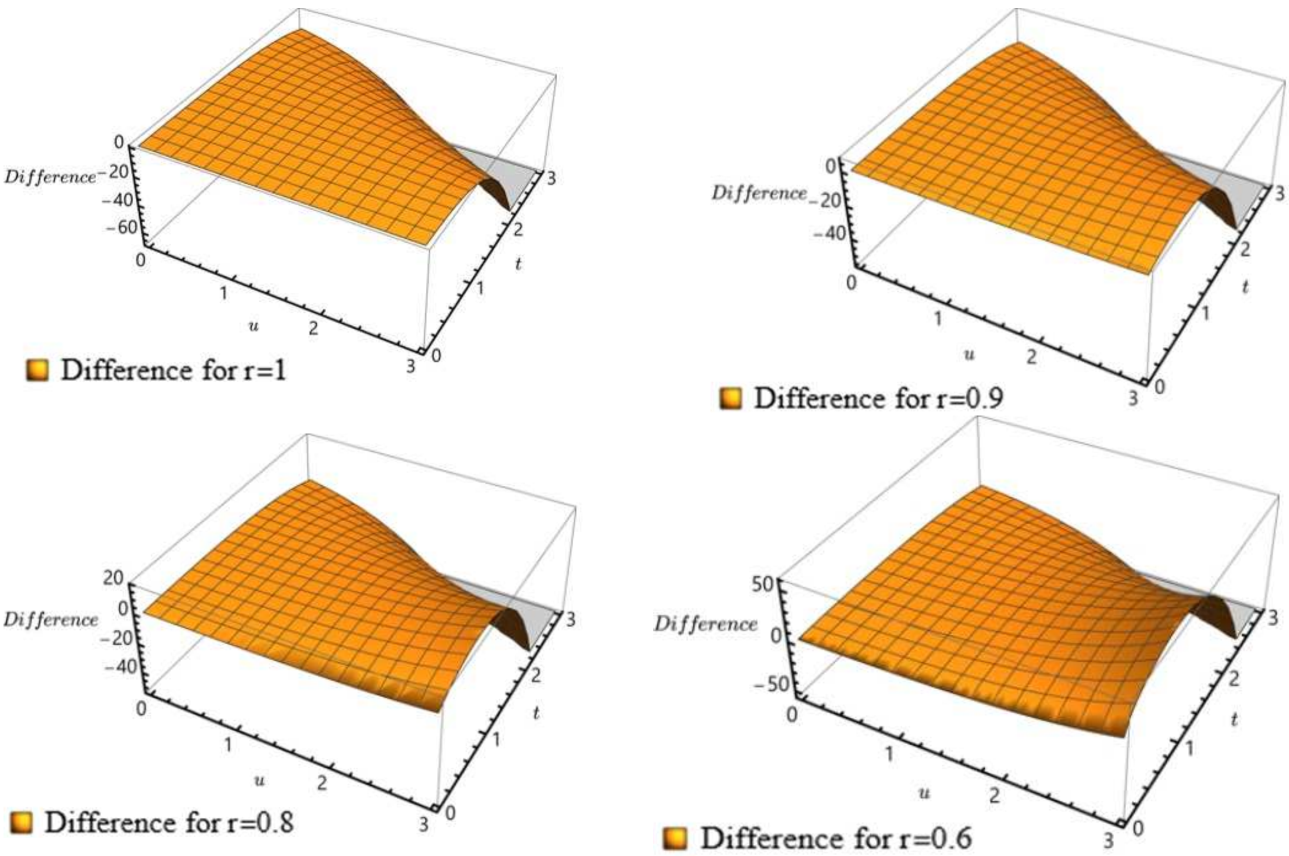

Fig. 3: The absolute error for various values of r in Example 1.

Plots of the approximate solution and e^{u+t} for various values of r are displayed. The complex function's departure from the simple exponential function e^{u+t} can be understood as this difference. It is possible to infer from the figures that:

- The difference is relatively small, indicating that the complex function closely approximates e^{u+t} when $r = 1$.
- It is more noticeable when r falls, particularly when u and t are larger. According to this, the complex function deviates from e^{u+t} more as r falls.
- The function's sensitivity to changes in u and t increases as r decreases.

Example 2. Take a look at the convection-reaction-diffusion equation that follows

$$\frac{\partial^r w(u,t)}{\partial t^r} = \frac{\partial^2 w(u,t)}{\partial u^2} - (1+4u^2)w(u,t), \tag{29}$$

according to the conditions

$$w(u,0) = e^{u^2}. \tag{30}$$

Solution. For Eq. (29), we use the Sawi transform homotopy perturbation method, and we get

$$\begin{aligned}\sum_{z=0}^{\infty} q^z W_z(u,v) &= \frac{1}{v} w(u,0) \\ &+ q\left(\frac{rv^r - r + 1}{N(r)}\right)\left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_{uu} - (1+4u^2)\sum_{z=0}^{\infty} q^z w_z(u,t)\right]\right).\end{aligned} \tag{31}$$

The inverse Sawi transform applied to Eqs. (31) yields

$$\sum_{z=0}^{\infty} q^z w_z(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v}w(u,0)\right] + q\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_{uu} - (1+4u^2)\sum_{z=0}^{\infty} q^z w_z(u,t)\right]\right]. \tag{32}$$

To determine the function of the AB derivative result, the powers of:

$$q^0 : w_0(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v}w(u,0)\right] = \mathbb{S}^{-1}\left[\frac{1}{v}e^{u^2}\right] = e^{u^2}. \tag{33}$$

$$q^{z+1} : w_{z+1}(u,t) = \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{z_{uu}}(u,t) - (1+4u^2)w_z(u,t)\right]\right]. \tag{34}$$

Putting $z = 0$ into Eq. (34), we get

$$\begin{aligned}
q^1 : w_1(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{0_{uu}}(u,t) - (1+4u^2)w_0(u,t)\right]\right] \\
&= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[(2+4u^2)e^{u^2} - (1+4u^2)e^{u^2}\right]\right] \\
&= \frac{1}{N(r)}\mathbb{S}^{-1}\left[(rv^r - r + 1)v^{-1}e^{u^2}\right] \\
&= \frac{e^{u^2}}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right).
\end{aligned}$$

Putting $z = 1$ into Eq. (34), we get

$$\begin{aligned}
q^2 : w_2(u,t) &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{1_{uu}}(u,t) - \left(1+4u^2\right)w_1(u,t)\right]\right] \\
&= \frac{e^{u^2}}{(N(r))^2}\mathbb{S}^{-1}\left[(rv^r - r + 1)\mathbb{S}\left[\frac{t^r}{\Gamma(r)} - r + 1\right]\right] \\
&= \frac{e^{u^2}}{(N(r))^2}\mathbb{S}^{-1}\left[r^2\frac{t^{2r}}{\Gamma(2r+1)} + 2r(1-r)\frac{t^r}{\Gamma(r+1)} + (1-r)^2\right] \\
&= \frac{e^{u^2}}{(N(r))^2}\left(r^2\frac{t^{2r}}{\Gamma(2r+1)} + 2r(1-r)\frac{t^r}{\Gamma(r+1)} + (1-r)^2\right).
\end{aligned}$$

Consequently, Eq.(29) solution is provided by

$$\begin{aligned}
w(u,t) &= w_0 + w_1 + w_2 + \ldots \\
&= e^{u^2} + \frac{e^{\,u^2}}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right) + \frac{e^u}{(N(r))^2}\left(\frac{r^2t^{2r}}{\Gamma(2r+1)} + \frac{2r(1-r)t^r}{\Gamma(r+1)} + (1-r)^2\right) + \ldots \\
&= e^{u^2}\left(1 + \frac{1}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right) + \frac{1}{(N(r))^2}\left(\frac{r^2t^{2r}}{\Gamma(2r+1)} + \frac{2r(1-r)t^r}{\Gamma(r+1)} + (1-r)^2\right) + \ldots\right).
\end{aligned}$$

Thus, the exact solution is $w(u,t) = e^{u^2+t}$.

Below, in Figure 4 and Figure 5 we present the graph of the exact and approximate solutions. We sketch the graph of the approximate solutions for different values of $r = 1, 0.9, 0.8$ and 0.6.

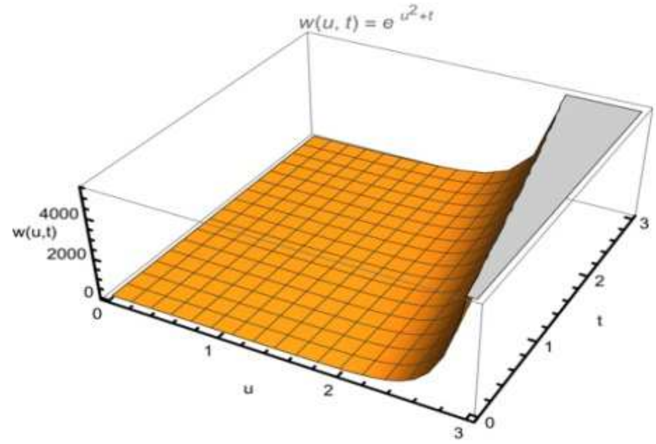

Fig. 4: The exact solution e^{u^2+t}.

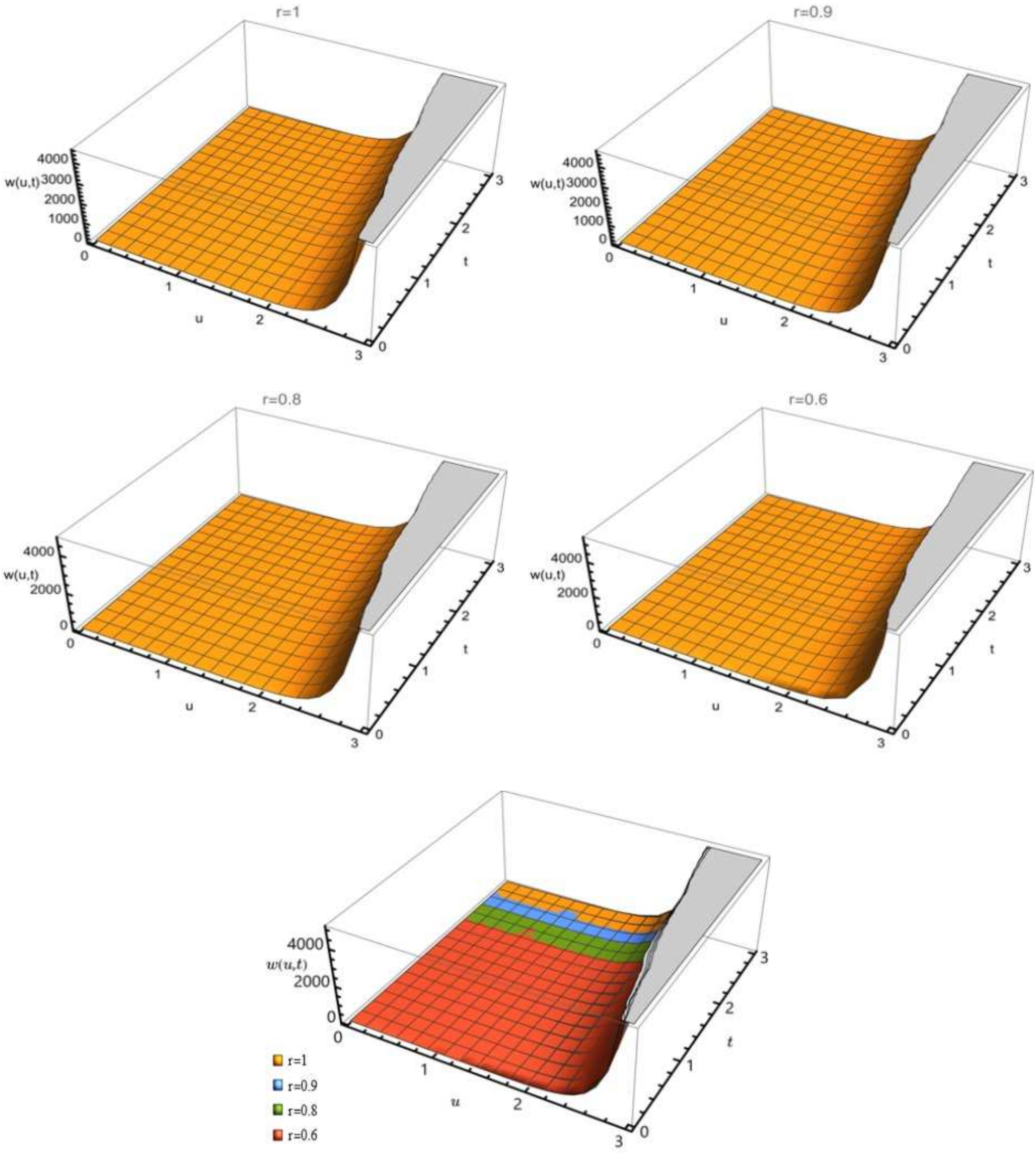

Fig. 5: Estimated solutions for various values of r.

Figure 6 presents the graphs depicting the absolute error for the values: $r = 1, 0.9, 0.8$, and 0.6.

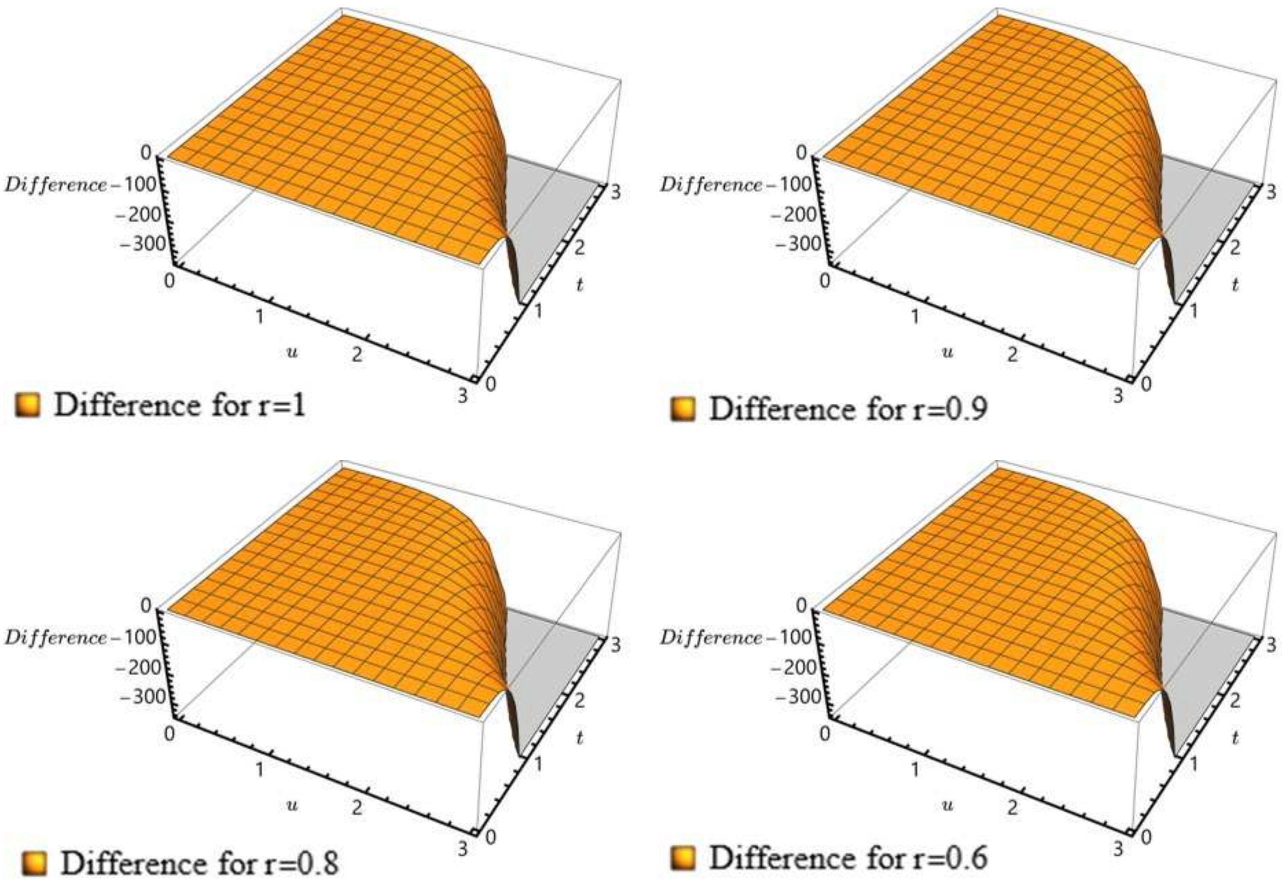

Fig. 6: The absolute error for $r = 1, 0.9, 0.8$.

Example 3. Find the solution of the IVP for convection-reaction-diffusion.

$$\frac{\partial^r w(u,t)}{\partial t^r} = \frac{\partial^2 w(u,t)}{\partial u^2} + w(u,t) + w(u,t)\frac{\partial w(u,t)}{\partial u} - w^2(u,t). \tag{35}$$

Depending on conditions

$$w(u,0) = 1 + e^u. \tag{36}$$

Solution. Using the Sawi transform AB operator homotopy perturbation for Eq. (35), we arrive to

$$\sum_{z=0}^{\infty} q^z W_z(u,v) = \frac{1}{v} w(u,0) + q\left(\frac{rv^r - r + 1}{N(r)}\right)\left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_{uu} + \sum_{z=0}^{\infty} q^z w_z(u,t)\right]\right) + q\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[\sum_{z=0}^{\infty} \mathscr{H}_z\right]. \tag{37}$$

Applying the Sawi transform in reverse to Eqs. (37), we obtain

$$\sum_{z=0}^{\infty} q^z w_z(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v} w(u,0)\right] + q\,\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right)_{uu} + \sum_{z=0}^{\infty} q^z w_z(u,t)\right]\right)\right] + q\,\mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[\sum_{z=0}^{\infty} \mathscr{H}_z\right]\right]. \tag{38}$$

The initial terms of $\mathscr{H}_z$ are provided by

$$\begin{aligned} \mathscr{H}_0 &= w_0 w_{0u} - (w_0)^2, \\ \mathscr{H}_1 &= w_0 w_{1u} + w_1 w_{0u} - 2w_0 w_1, \\ \mathscr{H}_2 &= w_0 w_{2u} + w_1 w_{1u} + w_2 w_{0u} - 2w_0 w_2 - (w_2)^2, \\ &\vdots \end{aligned} \tag{39}$$

To determine the function of the Caputo derivative result, one must compute the powers of:

$$q^0 : w_0(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v} w(u,0)\right] = \mathbb{S}^{-1}\left[\frac{1}{v}(1+e^{u})\right] = 1+e^{u}, \tag{40}$$

$$q^{z+1} : w_{z+1}(u,t) = \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[w_{zuu}(u,t) + w_{zu}(u,t) + \mathscr{H}_z\right]\right], z \geq 0. \tag{41}$$

Putting $z=0$ into Eq.(41), we get

$$\begin{aligned} q^1 : w_1(u,t) &= \mathbb{S}^{-1}\left[\frac{1}{v^r}\mathbb{S}\left[w_{0uu}(u,t) + w_{1u}(u,t) + \mathscr{H}_0\right]\right] \\ &= \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)\mathbb{S}\left[e^{u} + (1+e^{u}) + (1+e^u)e^u - (1+e^u)^2\right]\right] = \mathbb{S}^{-1}\left[\left(\frac{rv^r - r + 1}{N(r)}\right)v^{-1}e^u\right] \\ &= \frac{e^u}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right), \end{aligned}$$

in the same way, we get

$$q^2 : w_2(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v^r}\mathbb{S}\left[w_{1uu}(u,t) + w_{1u}(u,t) + \mathscr{H}_1\right]\right] = \frac{e^u}{(N(r))^2}\left(r^2\frac{t^{2r}}{\Gamma(2r+1)} + 2r(1-r)\frac{t^{r}}{\Gamma(r+1)} + (1-r)^2\right).$$

Consequently, Eq.(35)'s solution is provided by

$$\begin{aligned} w(u,t) &= w_0 + w_1 + w_2 + \ldots = 1 + e^u + \frac{e^u}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right) + \frac{e^u}{(N(r))^2}\left(\frac{r^2t^{2r}}{\Gamma(2r+1)} + \frac{2r(1-r)t^r}{\Gamma(r+1)} + (1-r)^2\right) + \ldots \\ &= 1 + e^{u}\left(1 + \frac{1}{N(r)}\left(\frac{t^r}{\Gamma(r)} - r + 1\right) + \frac{1}{(N(r))^2}\left(\frac{r^2t^{2r}}{\Gamma(2r+1)} + \frac{2r(1-r)t^{r}}{\Gamma(r+1)} + (1-r)^2\right) + \ldots\right). \end{aligned}$$

Thus, the exact solution is $w(u,t) = 1 + e^{u+t}$. In Figure 7, we sketch the graph of Example 3.

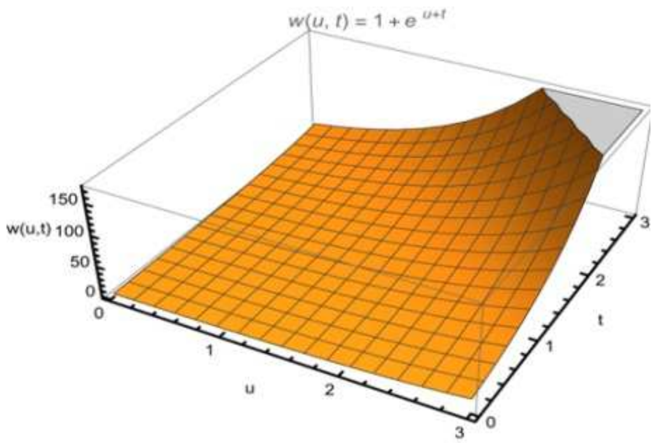

Fig. 7: The graph of the solution $1+e^{u+t}$.

For the variance value of $r = 1, 0.9, 0.8, 0.6$, we present the 3D graphs that demonstrate the disparity between the exact solution $1+e^{u+t}$ and approximate solution.

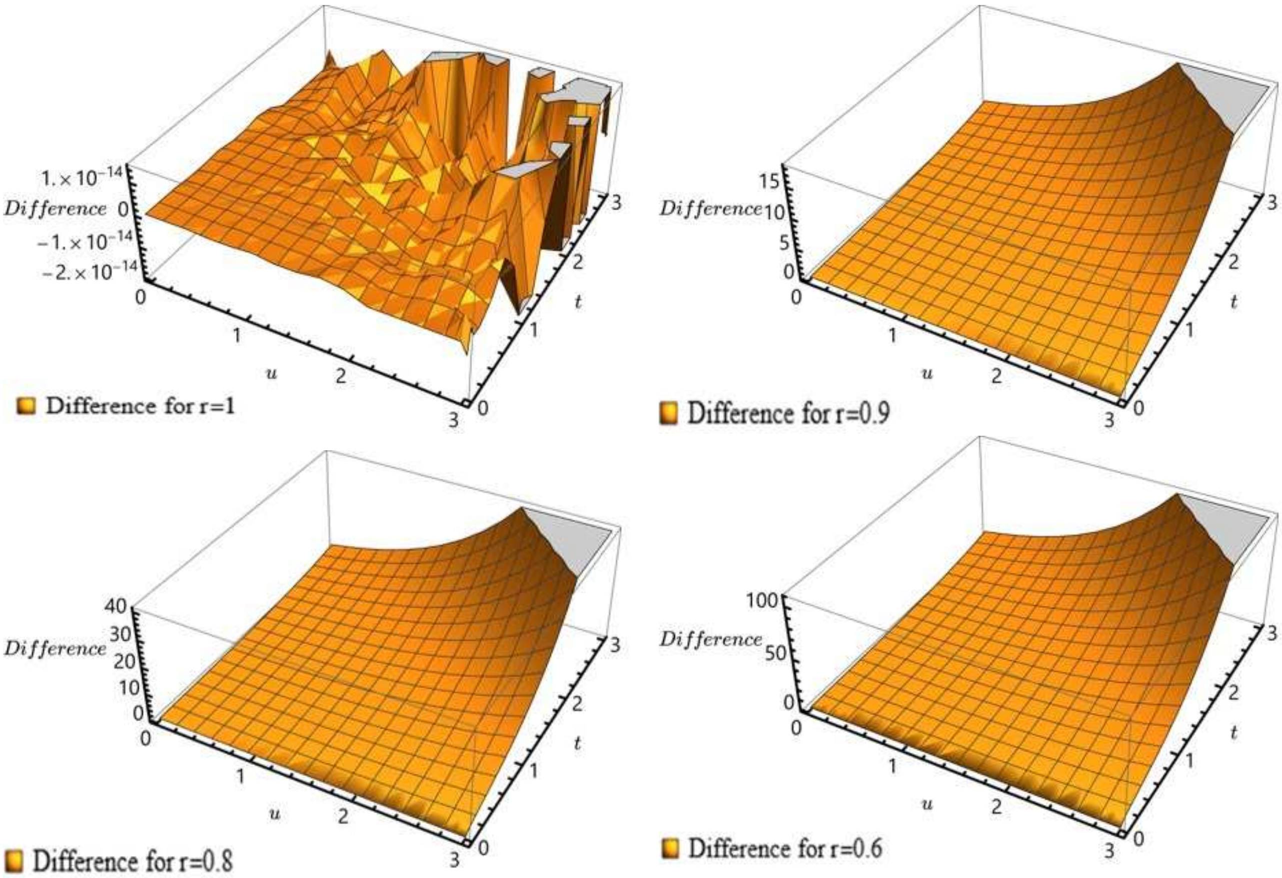

Fig. 8: The absolute error for different values of r.

6 Conclusions

This article presents a new methodology that combines the Sawi transform and the homotopy perturbation method using the definition of fractional AB. The integrated approach demonstrates notable computational efficiency, leading to a reduction in both time and resource expenditure, thereby enhancing its applicability for addressing intricate, high-dimensional challenges. The multidisciplinary nature connects fractional calculus theory with its practical applications in areas such as nonlinear wave equations, biological systems, and fluid mechanics. The method demonstrates versatility in its application to various fractional differential equations, thereby facilitating future research and advancements in the efficient resolution of FPDEs. The integrated approach demonstrates notable computational efficiency, leading to a reduction in both time and resource expenditure, which is particularly advantageous for addressing intricate, high-dimensional challenges. The multidisciplinary nature effectively connects fractional calculus theory with its practical applications across various fields. The method demonstrates a significant level of adaptability to various fractional differential equations, thereby facilitating future research and advancements in the efficient resolution of FPDEs.

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