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# Quantifying Risk of Insurance Claims Data Using Various Loss Distributions

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**Abstract:** This research work presents an empirical and theoretical discussion on the area of quantifying risk using parametric loss distributions to model insurance claims data. That is, this paper provides a large-scale comparison of 19 standard parametric distributions for curve-fitting using the South African taxi claims data and the Danish fire loss data. When a few standard loss distributions (to be exact, six, i.e., exponential, gamma, Weibull, lognormal, Pareto, and Burr) were considered for the taxi claims data, the lognormal and Pareto distributions were said to have the best fit for that insurance dataset. In this research work, the list of fitted standard distributions is extended from 6 to 19 (for taxi claims data), and it is observed that there are some standard distributions that provide a better fit than the lognormal and Pareto distributions for both datasets when evaluating the fit using goodness-of-fit measures and then compute their corresponding risk measures (i.e., value-at-risk (VaR) and tail value-at-risk (TVaR)). In general, when fitting the standard loss distributions to both datasets, the transformed beta family of distributions has the best fit, whereas the transformed gamma family of distributions provides the worst fit. Another observation is that the more parameters the distribution has, the more flexible the distribution is, and the better the fit to the data when compared to the other distributions in that parametric family. However, most of the fitted loss distributions tend to overestimate (or underestimate) the risk metrics which may lead to over-reserving (under-reserving), respectively.

**Keywords:** loss distributions; claims data; risk measures; fire insurance losses; taxi claims losses; heavy tailed; skewness.

## 1 Introduction

Modelling insurance losses by fitting statistical distributions and estimating risk is an important practice in risk analysis. Statistical distributions such as the exponential, gamma, Weibull, lognormal, Pareto and Burr distributions have been widely used for modelling claims data, see [1] and [2]. Klugman et al. [3] summarizes parametric families which are obtained when certain parameters of a distribution are set to one or equal to each other. A summary and distributional properties of the latter parametric families is given in the Appendix. It is important to note that in this paper the terms ‘model’ and ‘distribution’ are used interchangeably.

First, we discuss the transformed beta family which consists of the four-parameter transformed beta distribution, the three-parameter Burr distribution, the three-parameter inverse Burr distribution, the three-parameter generalized Pareto distribution, the two-parameter Pareto distribution, the two-parameter inverse Pareto distribution, the two-parameter paralogistic distribution, the two-parameter inverse paralogistic distribution and the two-parameter loglogistic distribution. The transformed beta (or generalized beta of the second kind) distribution has shape parameters $\alpha$, $\gamma$ and $\tau$ and scale parameter $\theta$. The Burr (or Singh-Maddala) distribution with shape parameters $\alpha$ and $\gamma$, and scale parameter $\theta$, is a special case of the transformed beta distribution when $\tau = 1$. The distribution is said to be heavy tailed for $\alpha < 2$ and very heavy tailed for $\alpha < 1$, see [1]. The inverse Burr (or Dagum) distribution with shape parameters $\tau$ and $\gamma$, and scale parameter $\theta$, is special case of the transformed beta distribution when $\alpha = 1$. The generalized Pareto (or beta of the second kind) distribution with shape parameters $\alpha$ and $\tau$, and scale parameter $\theta$, is a special case of the transformed beta distribution when $\gamma = 1$. The Pareto (or Pareto Type II or Lomax) distribution with shape parameter $\alpha$ and scale parameter $\theta$, is a special case of the transformed beta distribution when $\gamma = \tau = 1$. The distribution is said to be very heavy tailed for $\alpha < 1$, see [1]. The inverse Pareto distribution with shape parameter $\tau$ and scale parameter $\theta$, is a special case of the transformed distribution when $\gamma = \alpha = 1$. The paralogistic distribution with shape parameter $\alpha$ and scale parameter $\theta$, is a special case of the transformed beta distribution when $\alpha = \gamma, \tau = 1$. The inverse paralogistic distribution with shape parameter $\tau$ and scale parameter $\theta$, is a special case of the transformed beta distribution when $\tau = \gamma, \alpha = 1$. The loglogistic (or Fisk) distribution with shape parameter $\gamma$ and scale parameter $\theta$, is a special case of the transformed beta distribution when $\alpha = \tau = 1$.

Second, we discuss the transformed gamma family which consists of the three-parameter transformed gamma distribution,

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the two-parameter gamma distribution, the two-parameter Weibull distribution, and the one-parameter exponential distribution. The transformed gamma (or generalized gamma) distribution has shape parameters $\alpha$ and $\tau$, and scale parameter $\theta$. The gamma distribution with shape parameter $\alpha$ and scale parameter $\theta$, is a special case of the transformed gamma distribution when $\tau = 1$. If $\alpha$ is a positive integer, then the distribution is an Erlang distribution with shape parameter $\alpha$ and scale parameter $\theta$, see [1]. Also, if $\alpha = \frac{\upsilon}{2}$ and $\theta = \frac{1}{2}$, then the distribution is a chi-squared distribution with $\upsilon$ degrees of freedom, i.e., $\chi^2_\upsilon$. The Weibull distribution with shape parameter $\tau$ and scale parameter $\theta$, is a special case of the transformed gamma when $\alpha = 1$. The shape parameter $\tau$, affects the tail weight of the distribution. For $\tau < 1$, [1] stated that the distribution is heavy tailed. For $\tau \approx 3.6$, the distribution is approximately symmetrical and for smaller values of $\tau$, it is skewed to the right and for larger values of $\tau$, it is skewed to the left, see [4]. The exponential distribution with scale parameter $\theta$ (or rate parameter $\frac{1}{\theta}$), is a special case of the three-parameter transformed gamma distribution when $\alpha = \tau = 1$.

Finally, we discuss the inverse transformed gamma family which consists of the three-parameter inverse transformed gamma distribution, the two-parameter inverse gamma distribution, the two-parameter inverse Weibull distribution, and the one-parameter inverse exponential distribution, see [3] for more details. The inverse transformed gamma (or inverse generalized gamma) distribution has shape parameters $\alpha$ and $\tau$ and scale parameter $\theta$. The inverse gamma distribution with shape parameter $\alpha$ and scale parameter $\theta$, is a special case of the inverse transformed gamma distribution when $\tau = 1$. The inverse Weibull (or log-Gompertz) distribution with shape parameter $\tau$ and scale parameter $\theta$, is a special case of the inverse transformed gamma when $\alpha = 1$. The inverse exponential distribution with scale parameter $\theta$, is a special case of the three-parameter inverse transformed gamma distribution when $\alpha = \tau = 1$.

Two more distributions which are not part of the parametric families that will be studied here are the two-parameter lognormal distribution and the two-parameter inverse Gaussian distribution. The lognormal (or Cobbs-Douglas) distribution has location parameter $\mu$ and scale parameter $\sigma$. The lognormal distribution is similar to a normal distribution when the scale parameter $\sigma$, is small, which is an undesirable for skewed, heavy tailed loss data; see [4]. The inverse Gaussian (or Wald) distribution has mean parameter $\mu$ and shape parameter $\theta$.

About half of this paper is an extension of the recent work by [2] where six standard loss distributions were considered for the taxi claims data, and it was observed that the lognormal distribution and the Pareto distribution were the most efficient. The other half is based on the Danish fire loss data. We undertake a large-scale comparison of 19 standard distributions for curve-fitting of which most of the distributions emerge from three parametric distribution families described above. This evaluation of standard distributions has the following objectives:

- To discover other standard distributions that have not been studied previously;
- To assess the implications of the different statistical distributions on risk measures such as Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR).

This paper is structured as follows: The description of the distributions is outlined in the Appendix. Section 2 provides the risk measures and model selection criteria. Section 3 provides the analysis for which all the results for the distributions fit to the Danish fire loss data and the South African taxi claims data are discussed. The goodness-of-fit statistics, information criteria and the risk measures are evaluated for all the models studied in this paper. Finally, Section 4 provides the concluding remarks.

## 2 Methodology

### 2.1 Risk measures

A risk measure, which can also be referred to as a key risk indicator [3], is defined as a mathematical function of the probability of an event and the consequences of that event. It is important to realize that decision-making regarding risks is very complex and risk measures are essential for actuaries, investors, and financial institutions to make informed decisions about investments and risk management strategies. Two main risk measures are considered, i.e., VaR and TVaR. Let $F(\cdot)$ and $F^{-1}(\cdot)$ denote the cdf and inverse cdf of a continuous random variable $X$, respectively. Then, the VaR of $X$ at a $100p\%$ security level denoted by $\mathrm{VaR_p}(X)$, is the $100p\%$ quantile of $F$ such that

$$P\left(X < \mathrm{VaR_p}(X)\right) = p,\ F^{-1}(p) = \mathrm{VaR_p}(X). \tag{1}$$

The TVaR of $X$ at a $100p\%$ security level denoted by $\mathrm{TVaR_p}(X)$, represents the average of all VaR values exceeding security level, $p$, such that

$$\mathrm{TVaR_p}(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR_u}(X)du = \mathbb{E}[X|X > \mathrm{VaR_p}(X)]. \quad (2)$$

Finite values of Equation (2) can only be obtained if the first moment of the distribution of $X$ exists (see [5]).

The main difference between the two risk measures is understood through their interpretations. VaR can be interpreted as the lower bound for the capital required to avoid insolvency, whereas TVaR can be interpreted as the expected value of total loss, given that the loss exceeds VaR. Another key difference between the two risk measures is that TVaR is a coherent risk measure (see [3]), unlike VaR - which makes TVaR more attractive for an organisation with many business lines.

In this research work, the fit of the theoretical model is also assessed by comparing the empirical risk estimates to the theoretical risk estimates. Underestimating the risk measures may result in under-reserving - which may lead to insolvency, i.e., not enough capital to cover future claims. Overestimating the risk measures may result in over-reserving - which may affect the profitability of the insurer due to fewer funds available for investment purpose.

## 2.2 Model selection

This section discusses some commonly used model selection criteria that appear in the area of loss distributions. The first three are goodness-of-fit statistics: Kolmogorov-Smirnov, Cramer-von Mises, and the Anderson-Darling, which are popularly known as KS, CvM, and AD statistics, respectively. The next three are the negative log-likelihood function, Akaike Information Criterion [6] and Bayesian Information Criterion (or Schwarz Bayesian Criterion) [7], which are popularly known as NLL, AIC and BIC (or SBC), respectively.

The values for the KS, CvM and AD test statistics are computed for each distribution using the following equations:

$$\text{KS: } \max_x |F_n(x) - F(x)| \quad (3)$$

$$\text{CvM: } n\int \left(F_n(x) - F(x)\right)^2 f(x)dx \quad (4)$$

$$\text{AD: } n\int \frac{\left(F_n(x)-F(x)\right)^2}{F(x)\left(1-F(x)\right)} f(x)dx \quad (5)$$

where $n$ is the number of observations, $F_n(x)$ is the empirical cdf, $F(x)$ is the theoretical (fitted) cdf and $f(x)$ is the corresponding pdf.

The KS statistic computes the maximum absolute vertical differences between the empirical cdf and the theoretical cdf. Chernobai et al. [1] described it as a statistic which captures differences between the middle of the data and the proposed model. The CvM statistic considers the integral of the squared differences between the empirical cdf and the theoretical cdf rather than just considering differences between points. The AD statistic places emphasis on the tails of the distribution, i.e., where $F(x)$ or $1 - F(x)$ are small.

Let $\ell(\theta)$ denote the maximised log-likelihood function of a model, then the NLL is defined as

$$\mathrm{NLL} = -\ell(\theta). \quad (6)$$

The AIC is defined as

$$\mathrm{AIC} = 2\mathrm{NLL} + 2\mathrm{p}, \quad (7)$$

and the BIC is defined as

$$\mathrm{BIC} = 2\mathrm{NLL} + \mathrm{p}\log(\mathrm{n}). \quad (8)$$

where p is the number of parameters or degrees of freedom and n is the number of observations.

The classical likelihood ratio test [8] can be used to assess the goodness-of-fit of two models, where one is a subset (or is nested). That is, the null model is a special case of the alternative model. The test statistic is defined as

$$\mathrm{D} = -2[\ell(\theta_0) - \ell(\theta_1)] = 2[NLL_0 - NLL_1], \quad (9)$$

where $\ell(\theta_0)$ and $\ell(\theta_1)$ are the maximised log-likelihood values of the non-nested model and the nested model, respectively and $NLL_0$ and $NLL_1$ are the corresponding negative log-likelihood values. Wilks's theorem states that D can be approximated by a chi-square random variable with degrees of freedom equal to the difference in the number of parameters (dimensionality) of the two models. The null hypothesis is rejected if the test statistic is greater than the critical value. An analysis of the results is given with a special focus on the BIC.

# 3 Analysis

### 3.1 Some descriptives

In this section, we illustrate the proposed methodology for the taxi claims data and the popular Danish fire loss data and discuss our findings. Statistical computations were performed in R – see [9]. This study also made use of R add-on packages **actuar** and **fitdistrplus** developed by [10] and [11], respectively.

The South African taxi claims data consists of 48043 claims, which are reported in hundreds of South African Rands (ZARs), see [2]. The taxi claims data is skewed to the right with a skewness coefficient of 6.474 and leptokurtic with a kurtosis coefficient of 63.64. The Danish fire loss data consists of 2492 claims, which are reported in millions of Danish Krones (DKKs) and are adjusted for inflation to reflect 1985 values. The Danish data is skewed to the right with a skewness coefficient of 19.896 and leptokurtic with a kurtosis coefficient of 549.57. The Danish dataset is available in the add-on R package **SMPracticals** built by [12]. Table 1 presents the summary of the descriptive statistics for both datasets.

**Table 1:** Descriptive statistics for South African taxi claims data and Danish fire insurance loss data

| | **South African taxi claims** | **Danish fire loss** |
|---|---|---|
| **Country (Currency)** | South Africa (in 100s of ZARs) | Denmark (in mil DKKs) |
| **Insurance type** | Motor commercial insurance | Fire insurance |
| **Number of observations** | 48043 | 2492 |
| **Minimum** | 0.1 | 0.3134 |
| **1st quartile** | 20.8 | 1.1572 |
| **Median** | 45 | 1.6339 |
| **3rd quartile** | 120.8 | 2.6455 |
| **Mean** | 132.3 | 3.0627 |
| **Maximum** | 4803.3 | 263.2504 |
| **Sum** | 6 357 247 | 7 623.246 |
| **Standard deviation** | 284.1563 | 7.976703 |
| **Coefficient of variation** | 2.147426 | 2.6045 |
| **Skewness** | 6.474064 | 19.89612 |
| **Kurtosis** | 63.63799 | 549.5736 |

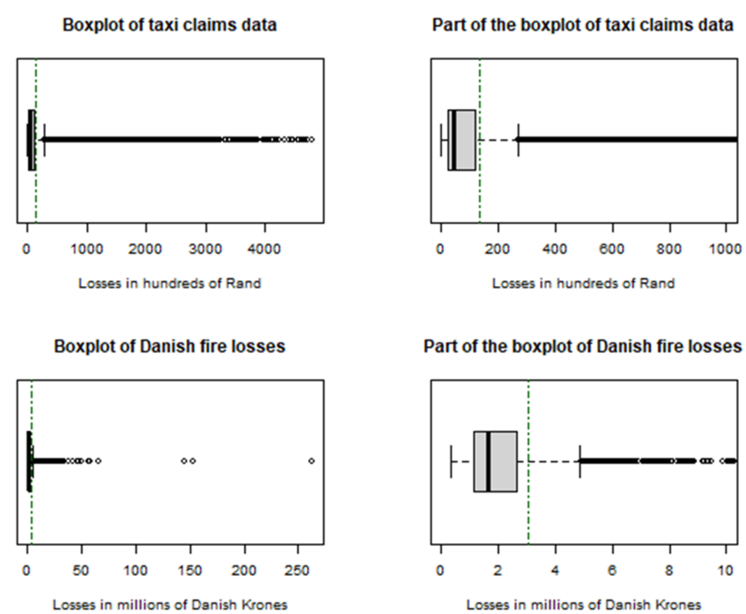


**Fig. 1:** Boxplots of taxi claims and Danish loss data

The boxplots of the complete datasets are given on the left-hand side (LHS) of Figure 1, while the zoomed in version for clearer visualization of the box are given on the right-hand side (RHS) of Figure 1. The mean value for each dataset is represented by the dotted vertical line. Overall, from Figure 1, it is clear that each dataset is skewed to the right.

Figure 2 shows the histograms of the taxi claims and the Danish fire loss data. By visual inspection of the histograms, the claims data are skewed to the right with long upper tails. Another observation from the histograms is that the claims data are positive (or at least nonnegative) and unimodal. Lastly, the smaller claims are more frequent, and the larger claims are less frequent.

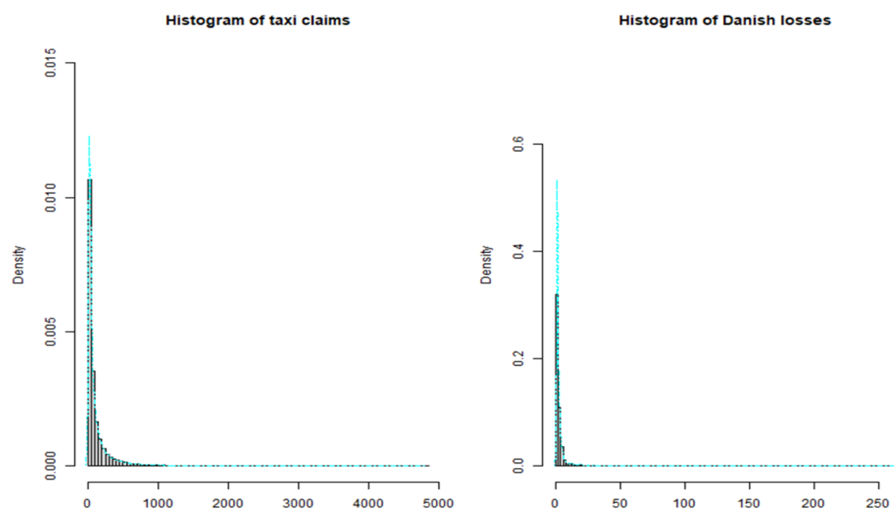


**Fig. 2:** Histograms of taxi claims and Danish loss data

Next, researchers often visually inspect the mean excess plot to determine the heaviness of the tail of the dataset. Stated differently, the mean excess function, denoted by $e(k)$, plotted against various threshold levels yields the mean excess plot [1]. Note that $e(k)$ is the mean of all differences between the data values and the threshold value, given that data values exceed the threshold, where $k$ denotes a threshold variable, i.e., $e(k) = \mathbb{E}[X - k|X > k]$. An ultimately increasing (decreasing) mean excess plot suggests that the underlying distribution is heavy- (light-) tailed. Figure 3 shows the mean excess plots for taxi claims on the left and the Danish fire losses data on the right. The mean excess plot for the taxi claims data is initially ultimately increasing, then constant and then ultimately decreasing. Therefore, the underlying distribution of the taxi claims data can be interpreted as initially heavy-tailed and then light-tailed in the upper tail. The mean excess plot for the Danish losses is ultimately increasing (with the two observations being an exception). Therefore, the underlying distribution of the Danish data is heavy-tailed throughout.

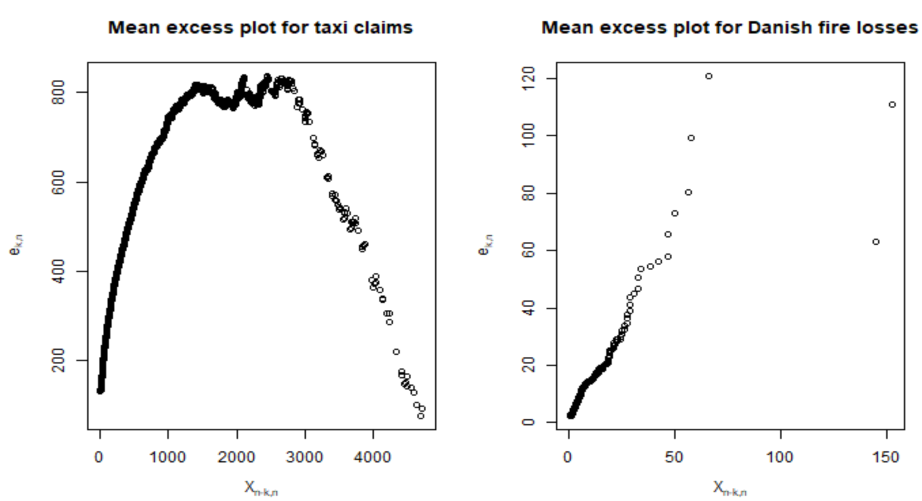


**Fig. 3:** Mean excess plots for taxi claims and Danish fire losses data

### 3.2 South African taxi claims

Based on the NLL, AIC, and BIC, the four-parameter transformed beta distribution provides an optimal fit for the taxi claims data, see Table 2. The three-parameter generalized Pareto, inverse Burr, inverse transformed gamma, and the two-parameter inverse paralogistic distributions also have a better performance than the two-parameter lognormal distribution proposed in [2] for the taxi claims data. The inverse paralogistic distribution is the only two-parameter distribution which performs better than the two-parameter lognormal distribution. Of the distributions which outperform the lognormal distribution, it is only the inverse transformed gamma distribution which comes from the class of inverse transformed gamma distributions, the rest are from the class of transformed beta distributions.

Furthermore, it is observed from Table 2 that based on the KS and CvM test statistics, the three-parameter Burr, two-parameter inverse Pareto, loglogistic and paralogistic distributions also perform better than the lognormal distribution. Based on the AD test statistic, the three-parameter Burr, two-parameter inverse Pareto and loglogistic distributions also perform better than the lognormal distribution. If the model selection were based on the KS, CvM and/or AD goodness-of-fit statistics (such as in [2]), then the three-parameter inverse Burr would have been the ideal model. In general, the more flexibility (parameters) the distribution has, the better the fit to the taxi claims data. Note that the boldfaced values in Table 2 denote the best model with respect to that specific model selection criteria on each respective column.

**Table 2:** Summary of the goodness-fit statistics and information criteria for the standard distributions for taxi claims data

| Distribution | p | KS | CvM | AD | NLL | AIC | BIC |
|---|---|---|---|---|---|---|---|
| Transformed beta | 4 | 0.0224 | 6.2957 | 37.2347 | **270499.5** | **541007.1** | **541042.2** |
| Generalized Pareto | 3 | 0.0199 | 5.2892 | 34.0960 | 270561.6 | 541129.2 | 541155.5 |
| Inverse transformed gamma | 3 | 0.0242 | 4.8895 | 36.1285 | 270603.2 | 541212.5 | 541238.8 |
| Inverse Burr | 3 | **0.0182** | **4.6666** | **32.1616** | 270606.3 | 541218.7 | 541245.0 |
| Inverse paralogistic | 2 | 0.0268 | 10.9516 | 70.2140 | 270743.9 | 541491.8 | 541509.3 |
| Lognormal | 2 | 0.0410 | 21.4363 | 115.313 | 270766.9 | 541537.9 | 541555.4 |
| Burr | 3 | 0.0235 | 7.26634 | 49.0775 | 270768.8 | 541543.6 | 541570.0 |
| Inverse Pareto | 2 | 0.0198 | 4.52623 | 54.4015 | 270833.8 | 541671.5 | 541689.1 |
| Loglogistic | 2 | 0.0321 | 15.8827 | 109.9894 | 271020.8 | 542045.7 | 542063.2 |
| Paralogistic | 2 | 0.0361 | 20.9142 | 152.6501 | 271298.0 | 542600.0 | 542617.5 |
| Transformed gamma | 3 | 0.0565 | 44.1059 | 242.1923 | 271447.7 | 542901.4 | 542927.7 |
| Pareto | 2 | 0.0614 | 43.4782 | 338.3163 | 272260.0 | 544524.0 | 544541.5 |
| Inverse Gaussian | 2 | 0.0631 | 55.5081 | 316.0335 | 272436.8 | 544877.5 | 544895.1 |
| Inverse Weibull | 2 | 0.0518 | 43.9268 | 354.7448 | 273135.0 | 546274.0 | 546291.6 |
| Inverse gamma | 2 | 0.0747 | 103.5090 | 619.1477 | 274740.3 | 549484.6 | 549502.2 |
| Weibull | 2 | 0.0885 | 167.8688 | 1040.437 | 276332.4 | 552668.8 | 552686.4 |
| Inverse exponential | 1 | 0.1100 | 300.3772 | 1573.600 | 276412.7 | 552827.4 | 552836.2 |
| Gamma | 2 | 0.1354 | 331.6400 | 1723.5081 | 279102.6 | 558209.2 | 558226.8 |
| Exponential | 1 | 0.2197 | 916.3690 | 4496.479 | 282745.3 | 565492.5 | 565501.3 |

In Table 3, the shape parameter, $\alpha$ for the Burr distribution is less than 1, which suggests a *very* heavy tailed distribution as discussed previously. Similarly, for the Weibull distribution, the shape parameter $\tau$ is less than 1, which suggests a heavy tailed distribution.

**Table 3:** Parameter estimates and standard errors (in parenthesis) for the standard distributions for the taxi claims data

| Distribution | Parameters |
|---|---|
| Transformed beta | $\alpha = 3.0293(0.3222)$, $\gamma = 0.5626\ (0.0369)$,<br>$\tau = 6.5787\ (0.94975)$, $\theta = 10.9048(1.8338)$ |
| Generalized Pareto | $\alpha = 1.244\ (0.0122)$, $\tau = 2.1670\ (0.0346)$, $\theta = 24.012\ (0.7281)$ |
| Inverse Burr | $\tau = 1.76200\ (0.0388)$, $\gamma = 1.14515\ (0.00733)$, $\theta = 24.90825\ (0.71485)$ |
| Inverse transformed gamma | $\alpha = 7.666$, $\tau = 0.27974$, $\theta = 58753.79$ |
| Inverse paralogistic | $\tau = 1.2593(0.00365)$, $\theta = 37.6879(0.2497)$ |
| Lognormal | $\mu = 3.9385(0.006)$, $\sigma = 1.32115(0.0043)$ |
| Burr | $\alpha = 0.6650(0.0113)$, $\gamma = 1.5833(0.0137)$, $\theta = 31.8763\ (0.5572)$ |
| Inverse Pareto | $\tau = 2.6679\ (0.0426)$, $\theta = 13.9203\ (0.29275)$ |
| Loglogistic | $\gamma = 1.3335\ (0.0051)$, $\theta = 49.2026\ (0.2938)$ |

| | |
|---|---|
| Paralogistic | $\alpha = 1.2192\ (0.00338),\ \theta = 62.60275\ (0.4225)$ |
| Transformed gamma | $\alpha = 18.3243,\ \tau = 0.1759,\ \theta = 0.00000398$ |
| Pareto | $\alpha = 1.75885\ (0.02095),\ \theta = 106.0148\ (1.8297)$ |
| Inverse Gaussian | $\mu = 132.2698\ (1.3240),\ \theta = 27.44295\ (0.1771)$ |
| Inverse Weibull | $\tau = 0.7875\ (0.0025),\ \theta = 26.93325\ (0.1652)$ |
| Inverse gamma | $\alpha = 0.7364\ (0.0041),\ \theta = 16.7328\ (0.1283)$ |
| Weibull | $\tau = 0.7228\ (0.0023),\ \theta = 100.9417564\ (0.67684)$ |
| Inverse exponential | $\theta = 22.7286\ (0.10369)$ |
| Gamma | $\alpha = 0.6458\ (0.0035),\ \theta = 204.6238\ (1.606995)$ |
| Exponential | $\frac{1}{\theta} = 0.00756\ (0.000034)$ |

**Error! Reference source not found.** 4 reports the empirical risk estimates, the estimated risk measures for the standard distributions for taxi claims and the percentage deviation in parenthesis of each estimated risk measure with respect to the empirical risk estimates. For the distributions where there is a '-', it means that the corresponding metric is undefined. Of the 19 parametric distributions considered for risk estimation, the inverse Gaussian distribution provides close estimates to the empirical risk estimate. The transformed gamma family distributions tend to underestimate the VaR and the TVaR at both the 95% and 99% security levels. For the inverse Pareto, the inverse Weibull, the inverse gamma, and the inverse exponential, the TVaR goes to infinity. Most of the other distributions in Table 4 overestimate the TVaR estimates at both the 95% and 99% security levels which indicates that the distributions do not adequately capture the tail area.

**Table 4:** Summary of the empirical risk estimates, estimated risk measures for the standard distributions for taxi claims data and the percentage deviation of the with respect to the empirical risk estimates in parenthesis

| | $\mathbf{VaR_{0.95}}$ | $\mathbf{VaR_{0.99}}$ | $\mathbf{TVaR_{0.95}}$ | $\mathbf{TVaR_{0.99}}$ |
|---|---|---|---|---|
| **Empirical estimates** | 525.1509 | 1396.901 | 1085.583 | 2206.203 |
| **Parametric** | | | | |
| Transformed beta | 495.65 (-5.6%) | 1575.42 (12.8%) | 1506.83 (38.8%) | 4286.07 (94.3%) |
| Generalized Pareto | 514.87 (-2.0%) | 1975.14 (41.4%) | 2776.24 (155.7%) | 10220.95 (363.3%) |
| Inverse Burr | 539.60 (2.8%) | 2262.96 (62.0%) | 4391.92 (304.6%) | 17962.8 (714.2%) |
| Inverse transformed gamma | 523.10 (-0.4%) | 1662.99 (19.0%) | 1517.36 (39.8%) | 4147.5 (88.0%) |
| Inverse paralogistic | 470.96 (-10.3%) | 1740.87 (24.6%) | 2357.64 (117.2%) | 8504.43 (285.5%) |
| Lognormal | 451.05 (-14.1%) | 1109.81 (-20.6%) | 916.88 (-15.5%) | 1934.12 (-12.3%) |
| Burr | 544.57 (3.7%) | 2527.34 (80.9%) | 10907.29 (904.7%) | 50308.46 (2180.3%) |
| Inverse Pareto | 717.09 (36.5%) | 3688.23 (164.0%) | - | - |
| Loglogistic | 447.63 (-14.8%) | 1543.51 (10.5%) | 1845.98 (70.0%) | 6208.86 (181.4%) |
| Paralogistic | 436.50 (-16.9%) | 1361.12 (-2.6%) | 1406.74 (29.6%) | 4216.58 (91.1%) |
| Transformed gamma | 427.8 (-18.5%) | 938.3 (-32.8%) | 774.06 (-28.7%) | 1465.37 (-33.6%) |
| Pareto | 476.18 (-9.3%) | 1347.68 (-3.5%) | 1243.40 (14.5%) | 3263.34 (47.9%) |
| Inverse Gaussian | 562.57 (7.1%) | 1428.39 (2.3%) | 1117.52 (2.9%) | 2174.35 (-1.4%) |
| Inverse Weibull | 1170.42 (122.9%) | 9723.76 (596.1%) | - | - |
| Inverse gamma | 1091.86 (107.9%) | 9788.55 (600.7%) | - | - |
| Weibull | 460.6 (-12.3%) | 835.01 (-40.2%) | 696.69 (-35.8%) | 1104.62 (-49.9%) |
| Inverse exponential | 443.11 (-15.6%) | 2261.47 (61.9%) | - | - |
| Gamma | 463.06 (-11.8%) | 763.65 (-45.3%) | 650.16 (-40.1%) | 955.59 (-56.7%) |
| Exponential | 396.41 (-24.5%) | 609.38 (-56.4%) | 528.73 (-51.3%) | 741.70 (-66.4%) |

Based on the PP plots in Fig. 4, the transformed beta distribution provides a significant improvement to the fit in the middle of the data when compared to the lognormal and the Pareto distributions. Based on the QQ plots in Fig. 4, the underlying distribution of the data is heavier than the lognormal distribution where the plotted points are above the reference line and lighter than the lognormal distribution where the plotted points are below the reference line. The underlying distribution of the data in the upper tail of the data is lighter than the Pareto distribution and the transformed beta distribution.

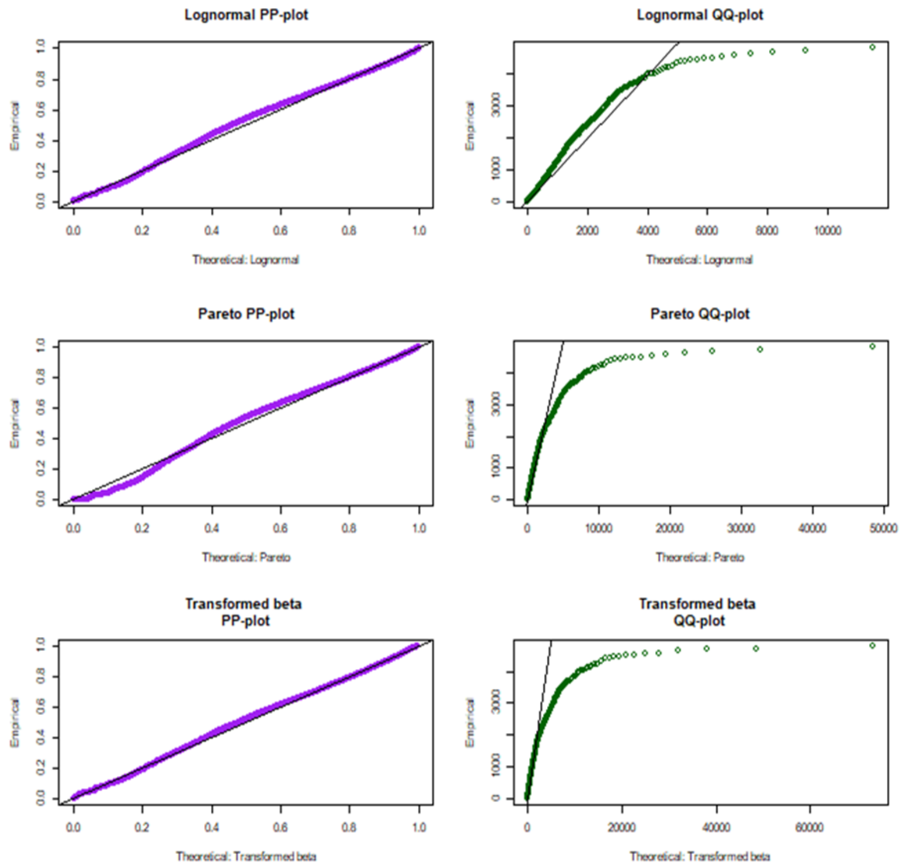


**Fig. 4:** PP plots and QQ plots for the lognormal, Pareto and transformed beta distributions for taxi claims data

### 3.2 Danish fire loss data

Based on Table 5 results, the three-parameter Burr distribution has the best performance out of the entire class transformed beta distributions according to two goodness-of-fit statistics (CvM and AD) and the two information criteria (AIC and BIC). The four-parameter transformed beta is only superior when the negative log-likelihood is the decision factor and the three-parameter inverse transformed gamma distribution only when using the KS statistic. The AD statistics for the inverse Gaussian, gamma, Weibull, and exponential distributions are infinity. A smaller value for the AD statistic suggests a good fit to the data especially in the tail area. We can conclude that these distributions perform poorly in the right tail. Note that the boldfaced values in Table 5 denote the best model with respect to that specific model selection criteria on each respective column.

Using the likelihood ratio test, the null hypothesis is that the four-parameter transformed beta distribution is of no improvement to three parameter Burr distribution for the Danish data. The test statistic is D = 2[3835.119 - 3834.767] = 0.704 and the critical value at a 5% significance level is 3.841. Therefore, the null hypothesis is not rejected as there is not enough evidence to prove that the transformed beta model may be a significant improvement over the Burr model.

In comparison to the class of transformed beta distributions, the class of transformed gamma distribution is of no improvement for the Danish data. There are nine models which outperform the lognormal model, of which four are two-parameter models. Overall, the three-parameter Burr distribution has the best performance for the Danish data.

**Table 5:** Summary of the goodness-of-fit statistics and information criteria for the standard distributions for the Danish fire loss data

| Distribution | p | KS | CvM | AD | NLL | AIC | BIC |
|---|---|---|---|---|---|---|---|
| Burr | 3 | 0.0383 | **0.665** | **3.424** | 3835.12 | **7676.24** | **7693.71** |
| Transformed beta | 4 | 0.0420 | 0.859 | 4.254 | **3834.77** | 7677.53 | 7700.82 |
| Inverse transformed gamma | 3 | **0.0378** | 1.167 | 8.697 | 3931.37 | 7868.75 | 7886.21 |
| Inverse Weibull | 2 | 0.0480 | 2.45 | 17.894 | 3966.83 | 7937.66 | 7949.30 |
| Inverse Burr | 3 | 0.0480 | 2.45 | 17.918 | 3966.88 | 7939.76 | 7957.22 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Inverse paralogistic | 2 | 0.0675 | 3.494 | 32.9 | 4093.32 | 8190.64 | 8202.28 |
| Inverse gamma | 2 | 0.087 | 6.588 | 40.281 | 4097.88 | 8199.76 | 8211.4 |
| Generalized Pareto | 3 | 0.087 | 6.608 | 40.362 | 4098.08 | 8202.16 | 8219.62 |
| Loglogistic | 2 | 0.114 | 5.695 | 52.502 | 4280.59 | 8565.18 | 8576.82 |
| Lognormal | 2 | 0.127 | 14.354 | 85.493 | 4433.89 | 8871.78 | 8883.42 |
| Paralogistic | 2 | 0.165 | 12.080 | 87.576 | 4514.88 | 9033.76 | 9045.41 |
| Inverse Gaussian | 2 | 0.172 | 27.001 | Inf | 4516.31 | 9036.61 | 9048.26 |
| Transformed gamma | 3 | 0.141 | 18.03 | 105.226 | 4579.44 | 9164.88 | 9182.34 |
| Inverse exponential | 1 | 0.208 | 46.212 | 241.47 | 4645.85 | 9293.71 | 9299.53 |
| Inverse Pareto | 2 | 0.208 | 46.199 | 241.418 | 4645.86 | 9295.73 | 9307.37 |
| Pareto | 2 | 0.290 | 39.799 | 223.097 | 5051.91 | 10107.81 | 10119.45 |
| Gamma | 2 | 0.201 | 40.247 | Inf | 5243.03 | 10490.05 | 10501.7 |
| Weibull | 2 | 0.256 | 38.934 | Inf | 5270.47 | 10544.94 | 10556.58 |
| Exponential | 1 | 0.233 | 38.658 | Inf | 5281.29 | 10564.57 | 10570.39 |

Like the taxi claims data parameter estimates in Table 3, the shape parameter $\alpha$ for the Burr distribution in Table 6, is less than 1, which indicates a *very* heavy tailed distribution as discussed previously. Similarly, the shape parameter $\tau$ for the Weibull distribution is also less than 1, which also suggests a heavy tailed distribution.

**Table 6:** Parameter estimates and standard errors in parenthesis for standard distributions for the Danish fire loss data

| **Distribution** | **Parameters** |
|---|---|
| Burr | $\alpha = 0.0878\ (0.0082),\ \gamma = 14.9238\ (1.23225),\ \theta = 0.9209\ (0.0093)$ |
| Transformed beta | $\alpha = 0.07162\ (0.01857), \gamma = 18.0897\ (4.4326),$<br>$\tau = 0.78587\ (0.2205), \theta = 0.93192\ (0.01456)$ |
| Inverse transformed gamma | $\alpha = 0.5550\ (0.0332), \tau = 2.922\ (0.1156),\ \theta = 1.0887\ (0.0255)$ |
| Inverse Weibull | $\tau = 2.0106\ (0.0324), \theta = 1.4395\ (0.015)$ |
| Inverse Burr | $\tau = 1677.347,\ \gamma = 2.011483\ , \theta = 0.03590726$ |
| Inverse paralogistic | $\tau = 2.413\ (0.0352), \theta = 1.1009(0.01294)$ |
| Inverse gamma | $\alpha = 2.7534\ (0.0738), \theta = 4.446\ (0.1307)$ |
| Generalized Pareto | $\alpha = 2.7537,\ \tau = 5029.269, \theta = 0.0009$ |
| Loglogistic | $\gamma = 2.6527\ (0.04503), \theta = 1.77035\ (0.02297)$ |
| Lognormal | $\mu = 0.67185\ (0.01467), \sigma = 0.7323\ (0.0104)$ |
| Paralogistic | $\alpha = 1.8454\ (0.0223), \theta = 2.8063\ (0.0455)$ |
| Inverse Gaussian | $\mu = 3.0630\ (0.0581), \theta = 3.416\ (0.0967)$ |
| Transformed gamma | $\alpha = 25.8583, \tau = 0.255, \theta = 0.000006$ |
| Inverse exponential | $\theta = 1.615\ (0.0324)$ |
| Inverse Pareto | $\tau = 100120.7, \theta = 0.000016$ |
| Pareto | $\alpha = 5.1648\ (0.4223), \theta = 11.8874\ (1.1247)$ |
| Gamma | $\alpha =\ 1.2585\ (0.0320),\theta = 2.4338\ (0.07567)$ |
| Weibull | $\tau = 0.9474\ (0.0113), \theta = 2.9515\ (0.0664)$ |
| Exponential | $\frac{1}{\theta} = 0.3265\ (0.0065)$ |

Of the 19 distributions considered in Table 7 with respect to the Danish data, it is only the inverse transformed gamma distribution which provides theoretical risk estimates close to the empirical risk estimates. The Burr distribution provides theoretical estimates which are larger in comparison to the empirical estimates – which is not ideal as it will lead to over-reserving of funds if used for reserving and consequently less profit. On the other hand, the loglogistic distribution provides devastatingly low risk estimates when compared to the empirical risk estimates. In general, most of these distributions do not capture the tail or the area under the tail adequately. In fact, most of the distributions tend to underestimate the risk metrics, which if used for reserving, may lead to insolvency due to less funds reserved for immediate payment of claims.

**Table 7:** Summary of the empirical risk estimates, estimated risk measures for the standard distributions for the Danish loss data and the percentage deviance with respect to the empirical risk estimates in parenthesis

| | $\mathbf{VaR_{0.95}}$ | $\mathbf{VaR_{0.99}}$ | $\mathbf{TVaR_{0.95}}$ | $\mathbf{TVaR_{0.99}}$ |
|---|---|---|---|---|
| **Empirical estimates** | 8.406298 | 24.61378 | 22.15509 | 54.60396 |
| **Parametric** | | | | |
| Burr | 9.06 (7.8%) | 30.99 (25.9%) | 38.34 (73.1%) | 130.99 (139.9%) |

| | | | | |
|---|---|---|---|---|
| Transformed beta | - | - | - | - |
| Inverse transformed gamma | 7.41 (-11.9%) | 20.01 (-18.7%) | 19.341 (-12.7%) | 52.17 (-4.5%) |
| Inverse Weibull | 6.31 (-24.9%) | 14.19 (-42.3%) | 12.655 (-42.9%) | 28.27 (-48.2%) |
| Inverse Burr | 6.3 (-25.1%) | 14.17 (-42.4%) | 12.642 (-42.9%) | 28.23 (-48.3%) |
| Inverse paralogistic | 5.41 (-35.6%) | 10.66 (-56.7%) | 9.3321 (-57.9%) | 18.24 (-66.6%) |
| Inverse gamma | 6.41 (-23.7%) | 12.55 (-49.0%) | 10.82 (-51.2%) | 20.42 (-62.6%) |
| Generalized Pareto | 6.41 (-23.7%) | 12.55 (-49.0%) | 10.82 (-51.2%) | 20.43 (-62.6%) |
| Loglogistic | 1.71 (-79.7%) | 3.19 (-87.0%) | 2.78 (-87.5%) | 5.14 (-90.6%) |
| Lognormal | 6.53 (-22.3%) | 10.76 (-56.3%) | 9.25 (-58.2%) | 14.2 (-74.0%) |
| Paralogistic | 6.004 (-28.6%) | 10.36 (-57.9%) | 8.98 (-59.5%) | 14.97 (-72.6%) |
| Inverse Gaussian | 8.664 (3.1%) | 14.47 (-41.2%) | 12.31 (-44.4%) | 18.50 (-66.1%) |
| Transformed gamma | 6.735 (-19.9%) | 10.73 (-56.4%) | 9.27 (-58.2%) | 13.71 (-74.9%) |
| Inverse exponential | 31.49 (274.6%) | 160.70 (552.9%) | - | - |
| Inverse Pareto | 31.82 (278.5%) | 162.38 (559.7%) | - | - |
| Pareto | 9.35 (11.2%) | 17.11 (-30.5%) | 14.44 (-34.8%) | 24.07 (-55.9%) |
| Gamma | 8.47 (0.8%) | 12.59 (-48.8%) | 11.03 (-50.2%) | 15.12 (-72.3%) |
| Weibull | 9.4 (11.8%) | 14.79 (-39.9%) | 12.76 (-42.4%) | 18.22 (-66.6%) |
| Exponential | 9.18 (9.2%) | 14.10 (-42.7%) | 12.24 (-44.8%) | 17.17 (-68.6%) |

Based on the PP plots in Figure 5, it is clear that the lognormal and Pareto distributions do not provide an adequate fit for the data, but the Burr distribution seems to fit the middle of the data adequately. Based on the QQ plots in Figure 5, it can be seen that the underlying distribution for the data is heavier than lognormal and Pareto distributions – the plotted points are above the reference line. It can also be seen that the underlying distribution for the data is lighter than the Burr distribution – the plotted points are below the reference line.

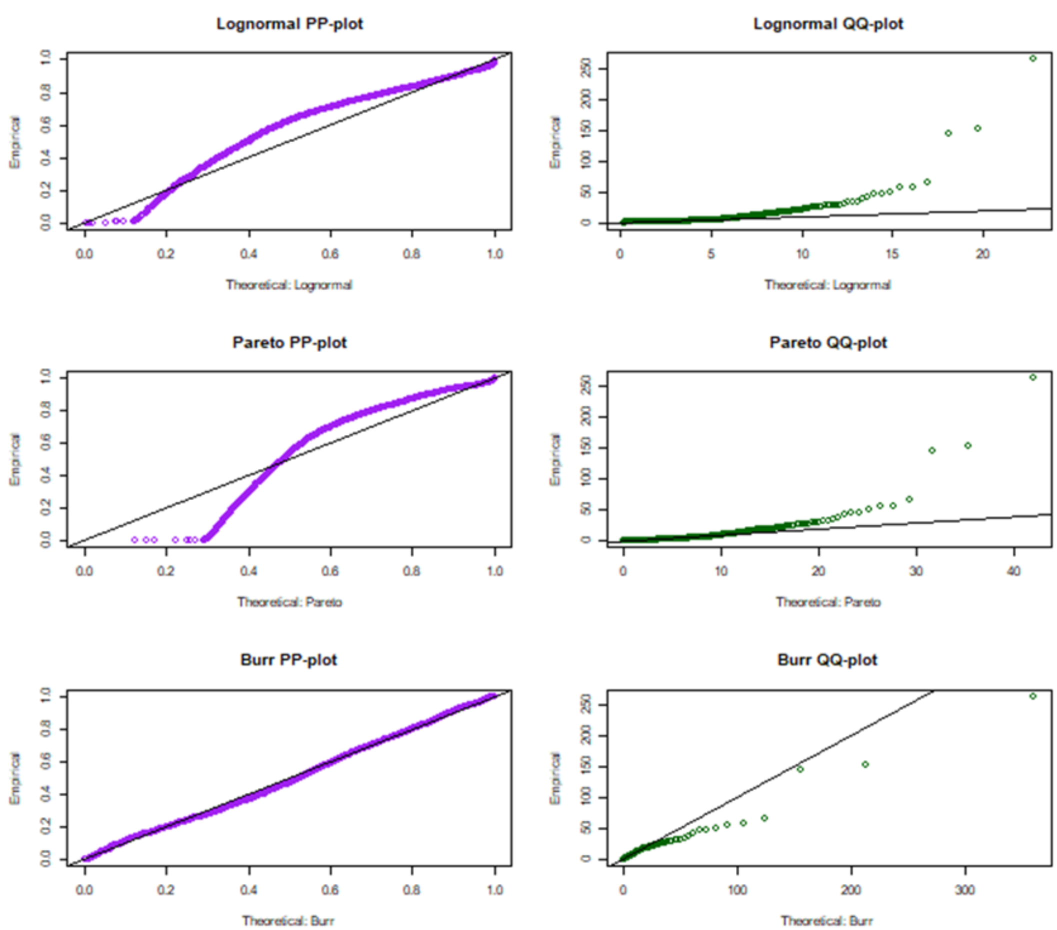


**Fig. 5:** PP plots and QQ plots for the lognormal, Pareto and Burr distributions for the Danish fire loss data

# 4 Conclusion

There have been other studies done to investigate riskiness of different types of data, for instance, [13] conducted a similar analysis using South African Financial Index (J580) using only four distributions (exponential, Weibull, gamma and Burr). Other similar studies in the literature that are consistent with the research conducted here are [14], [15], [16], [17]. Therefore, in an effort to further contribute to the latter publications, this paper considers a larger set of loss distributions, i.e. nineteen in total, using two separate datasets. In general, when fitting the standard distributions to both datasets, the transformed beta family of distributions have the best fit, whereas the transformed gamma family of distributions provide the worst fit. Another observation is that the more parameters the distribution has, the more flexible the distribution is, and the better the fit to the data when compared to the other distributions in the family. In this study, the lognormal distribution and the inverse Gaussian distributions do not have parametric families that they can be related to so we will compare them against each other. For both datasets, the lognormal distribution performs better than the inverse Gaussian or the Pareto distribution. The inverse paralogistic distribution is the only two-parameter distribution that performs better than the lognormal distribution for both datasets. For both datasets, the inverse transformed gamma family is better than the transformed gamma family. When it comes to the risk measures, most of the standard distributions do not adequately estimate the risk measures when compared to the empirical risk estimates.

The best two-parameter distribution for taxi claims is the inverse paralogistic distribution. Overall, based on the BIC, for the taxi claims data it is the four-parameter transformed beta distributions which outperforms the other standard distributions. For the taxi claims, it is the inverse Gaussian distribution which provide risk estimates that are fairly close to the empirical risk estimates. Most of the three-parameter distributions (except the transformed gamma distribution) overestimate the TVaR, suggesting that the area under the tail is not adequately captured by these distributions. It may happen that the third parameter results in a tail that is much heavier than the underlying distribution.

For the Danish data, it is the three-parameter distribution Burr which outperforms the other standard distributions and the best two-parameter distribution for the Danish loss data is the inverse Weibull distribution. For the Danish data, the inverse transformed gamma distribution is the only one that provides very close estimates to the empirical risk estimates, although they are slightly underestimated. The three-parameter Burr distribution significantly overestimates the TVaR, even though this model had the lowest goodness of fit statistics (AD and CvM) and information criteria (AIC and BIC). The risk estimates for the four-parameter transformed beta could not be evaluated and all other distributions models tend to underestimate the risk measures.

For future research ideas related to this research work on loss distributions, we urge the reader to consult [18]'s concluding remarks section for detailed suggestions.

***Conflicts of Interest Statement***

*The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.*

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**Appendix**

The incomplete gamma function is given by

$$\Gamma(\alpha;x)=\frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt\,,\alpha>0,x>0 \tag{A1}$$

where $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx,\alpha>0$. The incomplete beta function is given by

$$\beta(a,b;x)=\beta(a,b)\int_0^x t^{a-1}(1-t)^{b-1}\,dt,a>0,b>0,0<x<1 \tag{A2}$$

where $\beta(a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$ is the beta function with $a>0$ and $b>0$. The moment generating function (mgf) of $X$, denoted by $M_X(t)$ is given by

$$M_X(t)=\mathbb{E}[e^{tX}]=\int_{-\infty}^{\infty} e^{tx}f(x)\,dx. \tag{A3}$$

Suppose that $X$ has a moment generating function $M_X(t)$, then the $k$th raw moment of $X$, $\mathbb{E}[X^k]$ exists and is finite for $k \in \mathbb{N}$ and it is

$$\mathbb{E}[X^k] = \int_{-\infty}^{\infty} x^k f(x)\, dx. \tag{A4}$$

Thus, for the rest of the distributions, see the summary given in Table A1.

**Table A1:** Different properties of the 19 standard loss distributions

| **Distribution** | **Parameters** | **PDF** | **CDF** | $\mathbb{E}[X^k]$ |
|---|---|---|---|---|
| Burr | $\alpha>0, \gamma>0, \theta>0$ | $\frac{\alpha\gamma\left(\frac{x}{\theta}\right)^{\gamma}}{x[1+\left(\frac{x}{\theta}\right)^{\gamma}]^{\alpha+1}}$ | $1-u^{\alpha}, \quad u=\frac{1}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$ | $\frac{\theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\alpha)}, -\gamma<k<\alpha\gamma$ |
| Exponential | $\theta>0$ | $\frac{e^{-\frac{x}{\theta}}}{\theta}$ | $1-e^{-\frac{x}{\theta}}$ | $\begin{cases}\theta^k\Gamma(k+1), & k>-1\\ \theta^k k!, & \text{if } k \text{ is a positive integer}\end{cases}$ |
| Gamma | $\alpha>0, \theta>0$ | $\frac{\left(\frac{x}{\theta}\right)^{\alpha}e^{-\frac{x}{\theta}}}{x\Gamma(\alpha)}$ | $\Gamma\left(\alpha;\frac{x}{\theta}\right)$ | $\begin{cases}\frac{\theta^k\Gamma(\alpha+k)}{\Gamma(\alpha)}, & k>-\alpha\\ \theta^k(\alpha+k-1)\cdots\alpha, & \text{if } k \text{ is a positive integer}\end{cases}$ |
| Weibull | $\tau>0, \theta>0$ | $\frac{\tau\left(\frac{x}{\theta}\right)^{\tau}e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x}$ | $1-e^{-\left(\frac{x}{\theta}\right)^{\tau}}$ | $\theta^k\Gamma\left(1+\frac{k}{\tau}\right), k>-\tau$ |
| Generalized Pareto | $\alpha>0, \tau>0, \theta>0$ | $\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\frac{\theta^{\alpha}x^{\tau-1}}{(x+\theta)^{\alpha+\tau}}$ | $\beta(\tau,\alpha;u), \quad u=\frac{x}{x+\theta}$ | $\begin{cases}\frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, & -\tau<k<\alpha\\ \frac{\theta^k\tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)}, & \text{if } k \text{ is a positive integer}\end{cases}$ |
| Inverse Burr | $\tau>0, \gamma>0, \theta>0$ | $\frac{\tau\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x[1+\left(\frac{x}{\theta}\right)^{\gamma}]^{\tau+1}}$ | $u^{\tau}, \quad u=\frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$ | $\frac{\theta^k\Gamma(\tau+\frac{k}{\gamma})\Gamma(1-\frac{k}{\gamma})}{\Gamma(\tau)}, -\tau\gamma<k<\gamma$ |
| Inverse Exponential | $\theta>0$ | $\frac{\theta e^{-\frac{\theta}{x}}}{x^2}$ | $e^{-\frac{\theta}{x}}$ | $\theta^k\Gamma(1-k), k<1$ |
| Inverse Gamma | $\alpha>0, \theta>0$ | $\frac{\left(\frac{\theta}{x}\right)^{\alpha}e^{-\frac{\theta}{x}}}{x\Gamma(\alpha)}$ | $1-\Gamma\left(\alpha;\frac{\theta}{x}\right)$ | $\begin{cases}\frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}, & k<\alpha\\ \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)}, & \text{if } k \text{ is a positive integer}\end{cases}$ |
| Inverse Gaussian | $\mu>0, \theta>0$ | $\left(\frac{\theta}{2\pi x^3}\right)^{\frac{1}{2}}e^{-\frac{\theta z^2}{2x}}, \quad z=\frac{x-\mu}{\mu}$ | $\Phi\left[z\left(\frac{\theta}{x}\right)^{\frac{1}{2}}\right] + e^{\frac{2\theta}{\mu}}\Phi\left[-y\left(\frac{\theta}{x}\right)^{\frac{1}{2}}\right], \quad z=\frac{x-\mu}{\mu}$ | $\sum_{n=0}^{k-1}\frac{(k+n-1)!}{(k-n-1)!\,n!}\frac{\mu^{n+k}}{(2\theta)^n}, k=1,2,\ldots,$ |
| Inverse Paralogistic | $\tau>0, \theta>0$ | $\frac{\tau^2\left(\frac{x}{\theta}\right)^{\tau^2}}{x\left[1+\left(\frac{x}{\theta}\right)^{\tau}\right]^{\tau+1}}$ | $u^{\tau}, \quad u=\frac{\left(\frac{x}{\theta}\right)^{\tau}}{1+\left(\frac{x}{\theta}\right)^{\tau}}$ | $\frac{\theta^k\Gamma\left(\tau+\frac{k}{\tau}\right)\Gamma\left(1-\frac{k}{\tau}\right)}{\Gamma(\tau)}, -\tau^2<k<\tau$ |
| Inverse Pareto | $\tau>0, \theta>0$ | $\frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}$ | $\left(\frac{x}{x+\theta}\right)^{\tau}$ | $\begin{cases}\frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, & -\tau<k<1\\ \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, & \text{if } k \text{ is a negative integer}\end{cases}$ |
| Inverse Transformed Gamma | $\alpha>0, \theta>0, \tau>0$ | $\frac{\tau u^{\alpha}e^{-u}}{x\Gamma(\alpha)}, u=\left(\frac{\theta}{x}\right)^{\tau}$ | $1-\Gamma(\alpha;u)$ | $\frac{\theta^k\Gamma(\alpha-\frac{k}{\tau})}{\Gamma(\alpha)}, k<\alpha\tau$ |
| Inverse Weibull | $\tau>0, \theta>0$ | $\frac{\tau\left(\frac{\theta}{x}\right)^{\tau}e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x}$ | $e^{-\left(\frac{\theta}{x}\right)^{\tau}}$ | $\theta^k\Gamma\left(1-\frac{k}{\tau}\right), k<\tau$ |
| Loglogistic | $\gamma>0, \theta>0$ | $\frac{\gamma\left(\frac{x}{\theta}\right)^{\gamma}}{x[1+\left(\frac{x}{\theta}\right)^{\gamma}]^2}$ | $\frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$ | $\theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right), -\gamma<k<\gamma$ |
| Lognormal | $\mu>0, \sigma>0$ | $\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{z^2}{2}}=\frac{\phi(z)}{\sigma x}, \quad z=\frac{lnx-\mu}{\sigma}$ | $\Phi(z)$ | $e^{k\mu+\frac{1}{2}k^2\sigma^2}$ |

| | | | | |
|---|---|---|---|---|
| Paralogistic | $\alpha > 0, \theta > 0$ | $\frac{\alpha^2 \left(\frac{x}{\theta}\right)^{\alpha}}{x[1+\left(\frac{x}{\theta}\right)^{\alpha}]^{\alpha+1}}$ | $1-u^{\alpha}, \quad u=\frac{1}{1+\left(\frac{x}{\theta}\right)^{\alpha}}$ | $\frac{\theta^k \Gamma(1+\frac{k}{\alpha})\Gamma(\alpha-\frac{k}{\alpha})}{\Gamma(\alpha)}, -\alpha < k < \alpha^2$ |
| Pareto | $\alpha > 0, \theta > 0$ | $\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$ | $1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$ | $\begin{cases} \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, & -1 < k < \alpha \\ \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, & \text{if } k \text{ is a positive integer} \end{cases}$ |
| Transformed Beta | $\alpha > 0, \theta > 0, \gamma > 0,$ $\tau > 0$ | $\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\gamma \left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{\alpha+\tau}}$ | $\beta(\tau,\alpha;u), u=\frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$ | $\frac{\theta^k \Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\alpha)\Gamma(\tau)}, -\tau\gamma < k < \alpha\gamma$ |
| Transformed Gamma | $\alpha > 0, \tau > 0, \theta > 0$ | $\frac{\tau u^{\alpha} e^{-u}}{x\Gamma(\alpha)}, \quad u=\left(\frac{x}{\theta}\right)^{\tau}$ | $\Gamma(\alpha;u)$ | $\frac{\theta^k \Gamma(\alpha+\frac{k}{\tau})}{\Gamma(\alpha)}, k > -\alpha\tau$ |