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Estimations of stress-strength reliability of Burr XII distribution under accelerated life test model

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Abstract: In this paper, we estimate the parameter of stress-strength reliability ($R = P(X < Y)$) of two independent random variables X and Y , X denote to stress, Y denote to strength and have Burr XII distributions. Based on the assumption of strength and stress variables are subjected to partially step-stress accelerated life test the reliability of a system is discussed. The point estimate of R is computes with maximum likelihood and Bayes estimations. Also, confidence intervals of R are computed with asymptotic distribution, bootstrap technique and Bayesian credible intervals. The reliability of the system under R is computed with respected to numerical example. The results are assessed and compared by constructed Monte Carlo simulation study.

Keywords: Burr XII distribution; Stress-strength reliability; Maximum likelihood estimation; Bootstrap techniques; Bayes estimation.

1 Introduction

In applied science, when the stress is greater than the applied strength the system is failing. The mechanical reliability of a system or units is measuring by estimating R. Practice, if we denoted to the lifetimes of two devices by variables X and Y then the parameter R is used to measure the probability that one fails before the other. For example, in a laboratory test put under failure voltage levels the two types of electrical cable insulation. Increasing voltage stress is applied on the two types electrical cable insulation. Our objective is determining the longer life type of insulation. If, we denoted to the lifetimes of type 1 by X and the lifetimes of type 2 by Y. Hence, the estimate value of $R = P(X > Y)$ take the value greater than 0.5 then Type 1 of insulation is superiority in the form of longevity. The parametric and nonparametric estimate of R discussed early by AL-Hussaine et al. [1]. For extensive review of R see, Nadarajah [2], Mokhlis [3], Kundu and Gupta [4], Kundu and Gupta [5], Krishnamoorthy and Mukherjee [6] and Kundu and Raqab [7]. Recently, this problem discussed by Mahmoud et al. [8], Abd-Elmougod and Abu-Zinadah [9] and Sarhan and Tolba [10].

Under modern technology a highly reliable products are available and the problem of obtaining more information about the lifetime of products is more difficult. Also, a long period of time is used to test units under normal conditions. One way to overcome this problem the authors used censoring schemes. But, the important one in applied science to overcome this problem is accelerated life tests (ALTs). The key word of ALTs was presented by Nelson [11]. Different types of ALTs are available, the first type of ALTs is called constant stress ALTs, in which the stress is kept under constant level of stress through the test. The second type of ALTs is called progressive stress ALTs, in which the stress level is continual increasing the test. The third type of ALTs is called step-stress ALTs. In step-stress ALTs, the stress level is changed through fixed time or number of failure. When the model of ALTs pass through normal and stress conditions then, we mean partially ALTs. In this paper, we adopt partially step-stress ALTs, see Soliman et al. [12] and Al-Essa et al. [13].

Burr XII distribution is one of Burr system which contain twelve distributions with a variety of density shapes, see Burr [14]. Its has applied in reliability studies, quality control, business, chemical engineering, medical study. The random variable X is called Burr XII random variable if and only if has the probability density function

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(PDF) given by

$$f(x) = abx^{b-1}(1+x^b)^{-(a+1)},\ x>0,\ a,b>0, \quad (1)$$

where a and b are shape parameters. Also, the cumulative distribution function (CDF), survival function ($S(.)$) and hazard failure rate function ($H(.)$) given respectively by

$$F(x) = 1-(1+x^b)^{-a}, \quad (2)$$

$$S(t) = (1+t^b)^{-a}, \quad (3)$$

and

$$H(t) = abt^{b-1}(1+t^b)^{-1}. \quad (4)$$

The failure rate function of Burr XII distribution is decreasing function when $b \leq 1$ and unimodal function at $b > 1$. The parameter a don't effect in the shape of failure rate function and the shape parameter b plays an important role for the distribution. Different shapes of PDF and hazard failure rate of Burr XII distribution when b=2.5 and different values of a are represented in Fig 1 and 2. Different authors discussed Burr XII distribution, see Lee et al.[15], Abushal et al. [16] and Ragab et al. [17].

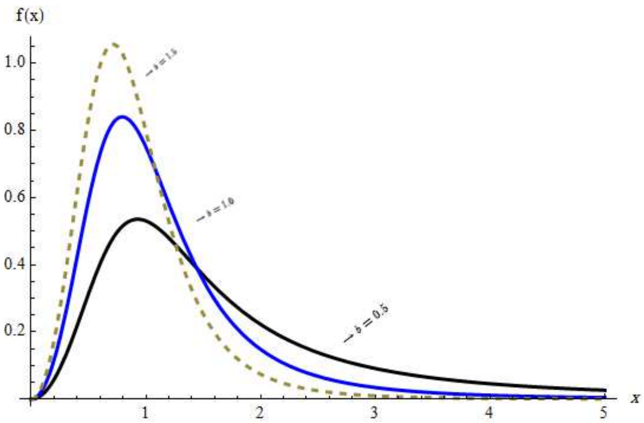

Fig. 1: Different shapes of PDFs for different value of a and $b=3.0$.

We aim to develop estimation problem of system reliability of Bur XII lifetime population. Therefore, we consider independent strength and stress variables X and Y of Burr XII lifetime populations with common one shape parameter and different other shape parameter. The strength and stress data are collected under partially step-stress ALT model. The classical and Bayes methods are used study the estimation of R. The proposed model is

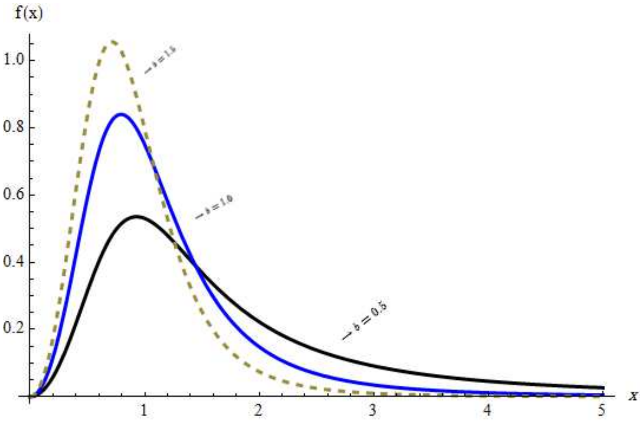

Fig. 2: Different shapes of hazard failure rate functions for different value of a and $b=3.0$.

tested under Monte Carlo simulation study. And, real data is used to determine the reliability of a real populations.

The paper is organized with respected to the following sections. In Section 2, The model of stress-strength reliability formulated under partially step-stress ALTs. In Section , The maximum likelihood method to estimate of R with corresponding asymptotic confidence interval. The parametric bootsrap-p and boostrap-t confidence interval of R in Section 4. Bayesian approach to estimate R in Section 5. The proposed model is analyzed under numerical example in Section 5. Monte Carlo simulation studying in Section 6. Finally, Some concluding remarks reported in Section 7.

2 R under Partially Step-stress ALTs Model

Let, a samples of sizes m and n selected randomly from the Burr XII distributions with PDFs $f_x(.)$ and $f_y(.)$,given respectively by

$$f_x(x) = a_1bx^{b-1}(1+x^b)^{-(a_1+1)},\ x>0,\ a_1,\ b>0, \quad (5)$$

and

$$f_y(y) = a_2bx^{b-1}(1+x^b)^{-(a_2+1)},\ x>0,\ a_1,\ b>0, \quad (6)$$

The two samples are tested normal conditions until reaching the time τ then, tested under stress conditions. The collected data from the two sample are given by $\mathbf{X} = (X_1, X_1, ..., X_{k_1}, X_{k_1+1}, ..., X_m)$, $X_{k_1} < \tau < X_{k_1+1}$ and $0 \leq k_1 \leq m$ as well as $\mathbf{Y} = (Y_1, Y_1, ..., Y_{k_2}, Y_{k_2+1}, ..., Y_n)$, $X_{k_2} < \tau < X_{k_2+1}$and $0 \leq k_2 \leq n$. As given by DeGroot and Goel [26]. Under consideration the partially step-stress ALTs the total test time W is given by

$$W = \begin{cases} T, \text{ at } T \leq \tau \\ \tau + \frac{(T-\tau)}{\beta},\ T > \tau \end{cases}, \quad (7)$$

where τ is the stress change time and β is accelerated factor.

2.1 Model assumption

Let, the random variable X (stress variable) has Burr XII with parameters a_1 and b and random variable Y (strength variable) has Burr XII with parameters a_2 and b. The density functions under partially step-stress ALT model are given by

$$f_x(x) = \begin{cases} f_{1x}(x) = a_1 b x^{b-1}(1+x^b)^{-(a_1+1)},\ x \le \tau \\ f_{2x}(x) = a_1 b\beta\,(\tau+\beta(x-\tau))^{b-1} \\ \times(1+(\tau+\beta(x-\tau))^b)^{-(a_1+1)},\ x > \tau \end{cases}, \quad (8)$$

and

$$f_y(y) = \begin{cases} f_{1y}(y) = a_2 b y^{b-1}(1+y^b)^{-(a_2+1)},\ y \le \tau \\ f_{2y}(y) = a_2 b\beta\,(\tau+\theta(y-\tau))^{b-1} \\ \times(1+(\tau+\beta(y-\tau))^b)^{-(a_2+1)},\ y > \tau \end{cases}. \quad (9)$$

The corresponding CDFs are given by

$$F_x(x) = \begin{cases} F_{1x}(x) = 1-(1+x^b)^{-a_1},\ x \le \tau \\ F_{2x}(x) = 1-(1+(\tau+\beta(x-\tau))^b)^{-a_1},\ x > \tau \end{cases}, \quad (10)$$

and

$$F_y(y) = \begin{cases} F_{1y}(y) = 1-(1+y^b)^{-a_2},\ y \le \tau \\ F_{2y}(y) = 1-(1+(\tau+\beta(y-\tau))^b)^{-a_2},\ y > \tau \end{cases}. \quad (11)$$

2.2 Stress-strength model

From Eq.s (8) to (11) the stress-strength reliability model of Burr XII distributions is given by

$$R = P(X<Y) = \int_0^\tau f_{1y}(y)F_{1x}(y)dy + \int_\tau^\infty f_{2y}(y)F_{2x}(y)dy. \quad (12)$$

From (8) to (11) the relation (12) is reduced to

$$R = \int_0^\tau a_2 b y^{b-1}(1+y^b)^{-(a_2+1)}\left[1-(1+y^b)^{-a_1}\right]dy$$
$$+\int_\tau^\infty a_2 b\beta\,(\tau+\beta(y-\tau))^{b-1}(1+(\tau+\theta(y-\tau))^b)^{-(a_2+1)}$$
$$\times\left[1-(1+(\tau+\beta(y-\tau))^b)^{-a_1}\right]dy.$$
$$=\frac{a_1}{a_1+a_2}. \quad (13)$$

3 MLE of R

The non-normalized likelihood function of two samples $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$L(\mathbf{x},\mathbf{y}|\Theta) \propto \prod_{i=1}^{k_1} f_{1x}(x_i) \prod_{i=k_1+1}^{m} f_{2x}(x_i) \prod_{i=1}^{k_2} f_{1y}(y_i) \prod_{i=k_2+1}^{n} f_{2y}(y_i), \quad (14)$$

where $\Theta = \{a_1,\ a_2,\ b,\ \beta\}$ is the parameter vector. (14) is reduced to

$$L(\Theta|\mathbf{x},\mathbf{y}) =$$
$$\prod_{i=1}^{k_1} a_1 b x_i^{b-1}(1+x_i^b)^{-(a_1+1)} \prod_{i=k_1+1}^{m} a_1 b\beta\,(\tau+\beta(x_i-\tau))^{b-1}$$
$$\times(1+(\tau+\beta(x_i-\tau))^b)^{-(a_1+1)} \prod_{i=1}^{k_2} a_2 b y_i^{b-1}(1+y_i^b)^{-(a_2+1)}$$
$$\prod_{i=k_2+1}^{n} a_2 b\beta\,(\tau+\beta(y_i-\tau))^{b-1}(1+(\tau+\beta(y_i-\tau))^b)^{-(a_2+1)},$$

$$L(\Theta|\mathbf{x},\mathbf{y}) = a_1^m a_2^n b^{(m+n)} \beta^{m+n-(k_1+k_2)} \exp\left[(b-1)\left\{\sum_{i=1}^{k_1}\log[x_i]\right.\right.$$
$$\left.+\sum_{i=k_1+1}^{m}\log[\tau+\beta(x_i-\tau)]\right\}-(a_1+1)\left\{\sum_{i=1}^{k_1}\log\left[1+x_i^b\right]\right.$$
$$\left.+\sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right\}+(b-1)$$
$$\times\left\{\sum_{i=1}^{k_1}\log[y_i]+\sum_{i=k_1+1}^{n}\log[\tau\ +\beta(y_i-\tau)]\right\}-(a_2+1)$$
$$\times\left\{\sum_{i=1}^{k_1}\log\left[1+y_i^b\right]+\sum_{i=k_1+1}^{n}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right\}. \quad (15)$$

The natural logarithms of (15) is reduced to

$$\ell(\Theta|\mathbf{x},\mathbf{y}) = m\log a_1 + n\log a_2 + (m+n)\log b$$
$$+(m+n-(k_1+k_2))\log\beta + (b-1)$$
$$\times\left\{\sum_{i=1}^{k_1}\log[x_i]+\sum_{i=k_1+1}^{m}\log[\tau+\beta(x_i-\tau)]\right\}-(a_1+1)$$
$$\times\left\{\sum_{i=1}^{k_1}\log\left[1+x_i^b\right]+\sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right\}$$
$$+(b-1)\left\{\sum_{i=1}^{k_2}\log[y_i]+\sum_{i=k_2+1}^{n}\log[\tau+\beta(y_i-\tau)]\right\}-(a_2+1)$$
$$\times\left\{\sum_{i=1}^{k_2}\log\left[1+y_i^b\right]+\sum_{i=k_2+1}^{n}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right\} \quad (16)$$

3.1 Point Estimators

In this section, we adopt the classical ML estimators of R as well as adopt Bayesian approach the formulate the point estimate of R.

3.2 Point ML estimate of R

The zero value of the first partially derive of (16) with respected to $a_1$and a_2 are reduced to

$$\frac{\partial \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial a_1}=\frac{m}{a_1}- \left\{\sum_{i=1}^{k_1}\log\left[1+x_i^b\right]+\sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right\}=0,$$

and

$$\frac{\partial \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial a_2}=\frac{n}{a_2}- \left\{\sum_{i=1}^{k_2}\log\left[1+y_i^b\right]+\sum_{i=k_2+1}^{m}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right\}=0,$$

which are reduced to

$$\hat{a}_1(b,\beta)=\frac{m}{\sum_{i=1}^{k_1}\log\left[1+x_i^b\right]+\sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]}, \quad (17)$$

and

$$\hat{a}_2(b,\beta)=\frac{n}{\sum_{i=1}^{k_2}\log\left[1+y_i^b\right]+\sum_{i=k_2+1}^{m}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]}. \quad (18)$$

Also, the zero value of the first partially derive of (16) with respected to b and β are reduced to

$$\begin{aligned}\frac{\partial \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial b}&=\frac{m+n}{b}+\left\{\sum_{i=1}^{k_1}\log[x_i]\right.\\&\left.+\sum_{i=k_1+1}^{m}\log[\tau+\beta(x_i-\tau)]\right\}-(a_1+1)\\&\times\left\{\sum_{i=1}^{k_1}\frac{x_i^b\log x_i}{1+x_i^b}+\sum_{i=k_1+1}^{m}\frac{(\tau+\beta(x_i-\tau))^b\log[\tau+\beta(x_i-\tau)]}{1+(\tau+\beta(x_i-\tau))^b}\right\}\\&+\left\{\sum_{i=1}^{k_2}\log[y_i]+\sum_{i=k_2+1}^{n}\log[\tau+\beta(y_i-\tau)]\right\}-(a_2+1)\\&\times\left\{\sum_{i=1}^{k_2}\frac{x_i^b\log y_i}{1+y_i^b}+\sum_{i=k_2+1}^{n}\frac{(\tau+\beta(y_i-\tau))^b\log[\tau+\beta(y_i-\tau)]}{1+(\tau+\beta(y_i-\tau))^b}\right\}\\&=0, \quad (19)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial \beta}&=\frac{m+n-(k_1+k_2)}{\beta}+(b-1)\\&\times\sum_{i=k_1+1}^{m}\frac{(x_i-\tau)}{\tau+\beta(x_i-\tau)}\\&-(a_1+1)\sum_{i=k_1+1}^{m}\frac{b(x_i-\tau)(\tau+\beta(x_i-\tau))^{b-1}}{1+(\tau+\beta(x_i-\tau))^b}\\&+(b-1)\sum_{i=k_2+1}^{n}\frac{(y_i-\tau)}{\tau+\beta(y_i-\tau)}\\&-(a_2+1)\sum_{i=k_2+1}^{n}\frac{b(y_i-\tau)(\tau+\beta(y_i-\tau))^b}{1+(\tau+\beta(y_i-\tau))^b}=0 \quad (20)\end{aligned}$$

The likelihood equation are reduced to two non-linear equations (19) and (20) solve by using Newton–Raphson iteration to obtain the estimate values $\hat{b}$ and $\hat{\beta}$. Therefore, the ML estimate of the parameters a_1 and a_2 are obtained from (17) and (18). The ML estimate of reliability R is given by

$$\hat{R}=\frac{\hat{a}_1}{\hat{a}_1+\hat{a}_2} \quad (21)$$

3.3 Bayesian estimator of R

In this section, we adopt independent gamma distributions as the prior distributions for the parameters a_1, a_2 and β. Also, the non-informative prior information for the accelerated factor β as follows

$$\begin{aligned}&a_1\propto a_1^{c_1-1}\exp\{-d_1a_1\},\ a_2\propto a_2^{c_2-1}\exp\{-d_2a_2\}\\&b\propto b^{c_3-1}\exp\{-d_3b\}, \quad (22)\end{aligned}$$

and

$$\beta\propto\frac{1}{\beta}. \quad (23)$$

The joint prior distribution is defined by

$$\begin{aligned}\Pi^*(a_1,a_2,b,\beta)&\propto a_1^{c_1-1}a_2^{c_2-1}b^{c_3-1}\beta^{-1}\\&\times\exp\{-d_1a_1-d_2a_2-d_3b\}. \quad (24)\end{aligned}$$

The joint posterior distribution is obtained from (15) and (24) is formulated as

$$\Pi(a_1,a_2,b,\beta|\mathbf{x},\mathbf{y}) \propto a_1^{m+c_1-1} a_2^{nc_2-1} b^{(m+n)+c_3-1} \times \beta^{m+n-(k_1+k_2)-1} \exp\left\{(b-1)\left(\sum_{i=1}^{k_1}\log[x_i] + \sum_{i=k_1+1}^{m}\log[\tau+\beta(x_i-\tau)]\right) - (a_1+1)\left(\sum_{i=1}^{k_1}\log\left[1+x_i^b\right] + \sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right) + (b-1) \times \left(\sum_{i=1}^{k_2}\log[y_i] + \sum_{i=k_2+1}^{n}\log[\tau+\beta(y_i-\tau)]\right) - (a_2+1) \times \left(\sum_{i=1}^{k_2}\log\left[1+y_i^b\right] + \sum_{i=k_2+1}^{n}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right\} - d_1a_1 - d_2a_2 - d_3b\right\}. \quad (25)$$

The joint posterior distribution given by (25) has shown that, the closed for of posterior distribution and the corresponding posterior estimator under any loss function cannot be obtained in the closed form. Therefore, the approximate methods are employed in this cases such as numerical integration, lindely method and Markov chen Monte Carlo method (MCMC). In this section, we employed the MCMC method to obtain the approximate empirical distribution and the corresponding parameters estimate.

3.3.1 Posterior full conditional distributions

The full conditional distributions from the joint posterior distribution (25) formulated by

$$\Pi_1(a_1|b,\beta,\mathbf{x},\mathbf{y}) \propto a_1^{m+c_1-1} \exp\left\{-a_1\left(d_1 + \sum_{i=1}^{k_1}\log\left[1+x_i^b\right] + \sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right)\right\}, \quad (26)$$

$$\Pi_1(a_2|b,\beta,\mathbf{x},\mathbf{y}) \propto a_2^{n+c_2-1} \exp\left\{-a_2\left(d_2 + \sum_{i=1}^{k_2}\log\left[1+y_i^b\right] + \sum_{i=k_2+1}^{n}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right)\right\}. \quad (27)$$

$$\Pi_3(b|a_1,a_2,\beta,\mathbf{x},\mathbf{y}) \propto b^{(m+n)+c_3-1} \exp\left\{-d_3b + b\left(\sum_{i=1}^{k_1}\log[x_i] + \sum_{i=k_1+1}^{m}\log[\tau+\beta(x_i-\tau)]\right) - (a_1+1)\left(\sum_{i=1}^{k_1}\log\left[1+x_i^b\right] + \sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right) + b\left(\sum_{i=1}^{k_1}\log[y_i] + \sum_{i=k_1+1}^{n}\log[\tau+\beta(y_i-\tau)]\right) - (a_2+1)\left(\sum_{i=1}^{k_1}\log\left[1+y_i^b\right] + \sum_{i=k_1+1}^{n}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right)\right\}, \quad (28)$$

and

$$\Pi_4(\beta|a_1,a_2,b,\mathbf{x},\mathbf{y}) \propto \beta^{m+n-(k_1+k_2)-1} \times \exp\left\{(b-1)\left(\sum_{i=k_1+1}^{m}\log[\tau+\beta(x_i-\tau)]\right) - (a_1+1)\left(\sum_{i=k_1+1}^{m}\log\left[1+(\tau+\beta(x_i-\tau))^b\right]\right) + (b-1)\left(\sum_{i=k_1+1}^{n}\log[\tau+\beta(y_i-\tau)]\right) - (a_2+1)\left(\sum_{i=k_1+1}^{n}\log\left[1+(\tau+\beta(y_i-\tau))^b\right]\right)\right\}. \quad (29)$$

The full conditional distributions have shown that, two gamma distribution are obtained as (26) and (27). Also, two more general function its plots similar normal distribution. Therefore the empirical distribution is obtained by adopted the MCMC method (MH under Gibbs algorithms) as the following algorithms.

3.3.2 MH under Gibbs algortms

In the problem at hand MH algorithms under Gibbs sample is more suitable to generate empirical posterior distribution and the corresponding estimate, see Metropolis et al. [30] as the following algorithms.

Algorithm 1 (Empirical posterior distribution and Bayes estimate of R)

1.Put s=1 and begin with the initial value $a_{1=}^{(0)}\hat{a}_1$, $a_{2=}^{(0)}\hat{a}_2$, $b_{=}^{(0)}\hat{b}$ and $\beta_{=}^{(0)}\hat{\beta}$.
2.By using gamma distribution (26) and (27) generate the iterate values $a_1^{(s)}$, $a_2^{(s)}$.
3.by using MH algorithms generate the values $b^{(s)}$, $\beta^{(s)}$ from (28) and (29) under normal proposal distribution.
1.Compute $R^{(s)} = \frac{a_1^{(s)}}{a_1^{(s)}+a_2^{(s)}}$.
5.put $s = s+1$.
6.Steps from 2 to 5 are repeat **MB** times.
2.Report $R^{(1)}, R^{(2)}, ..., R^{(\mathbf{MB})}$.
3.Compute The Bayes estimate of R by

$$\hat{R}_B = \frac{1}{\mathbf{MB}-\mathbf{MB}^*}\sum_{i=\mathbf{MB}^*+1}^{\mathbf{MB}} R^{(i)}, \quad (30)$$

where $\mathbf{MB}^*$ is the first iteration needed to reach the stationary distribution.

4.Compute The Bayes variance of R by

$$\hat{R}_B = \frac{1}{\mathbf{MB}-\mathbf{MB}^*} \sum_{i=\mathbf{MB}^*+1}^{\mathbf{MB}} (R^{(i)} - \hat{R}_B)^2. \tag{31}$$

3.4 interval Estimate of R

In this section, we discuss interval estimation of the model parameters by, approximate ML confidence interval, bootstrap confidence interval and credible intervals of R.

3.5 Approximate confidence interval of R

From the second derivative with respected to a_1 and a_2 of the log-likelihood function (16), we obtain

$$I_{a_1a_1} = \frac{\partial^2 \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial a_1^2} = \frac{-m}{a_1^2}, \tag{32}$$

$$I_{a_2a_2} = \frac{\partial^2 \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial a_2^2} = \frac{-n}{\alpha_2^2}, \tag{33}$$

$$I_{a_1a_2} = I_{a_2a_1} = \frac{\partial^2 \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial a_1 \partial a_2} = \frac{\partial^2 \ell(\Theta|\mathbf{x},\mathbf{y})}{\partial a_2 \partial a_2} = 0. \tag{34}$$

The Fisher information matrix (FIM) of the vector $\Psi = (a_1, a_2)$ is defined by

$$\text{FIM}(a_1,a_2) = \begin{pmatrix} \frac{-m}{a_1^2} & 0 \\ 0 & \frac{-n}{\alpha_2^2} \end{pmatrix}, \tag{33}$$

and the inverse one of FIM is given by

$$\text{FIM}^{-1}(a_1,a_2) = \begin{pmatrix} \frac{a_1^2}{m} & 0 \\ 0 & \frac{a_2^2}{n} \end{pmatrix}. \tag{35}$$

Also, delta method is used to obtain the variance of R by

$$V_R = \left(\frac{\partial R}{a_1}\right)^2 \text{FIM}_{11}^{-1} + \left(\frac{\partial R}{a_2}\right)^2 \text{FIM}_{22}^{-1}. \tag{36}$$

Hence,

$$V_R = \left(\frac{m+n}{mn}\right) \left(\frac{a_1 a_2}{(a_1+a_2)^2}\right)^2. \tag{37}$$

Hence, (1-2α)100% confidence interval of R is given by

$$(\hat{R} - Z_\alpha \sqrt{V_R},\ \hat{R} + Z_\alpha \sqrt{V_R}), \tag{38}$$

where Z_α is percentile normal variate $N(0,\ 1)$ and confidence level given by γ.

3.6 Bootstrap confidence interval of R

Bootstrap technique is commonly method not only in parameters estimation but it used to predict of the variance and bias of an estimators as well as in testing hypothesis. More information about parametric and nonparametric bootstrap techniques presented by Davison and Hinkley [27] and Efron [28]. In this section, we adopt parametric percentile technique (bootstrap-p), see DiCiccio and Efron [29] as follows.

3.7 Algorithm 2: (Approximate bootstrap-p confidence interval)

1For given the stress change time τ and strength stress samples $\mathbf{X} = (X_1, X_1, ..., X_{k_1}, X_{k_1+1}, ..., X_m)$, and $\mathbf{Y} = (Y_1, Y_1, ..., Y_{k_2}, Y_{k_2+1}, ..., Y_n)$ determine the integer m, n, K_1 and k_2.

1.Compute the ML estimate of the model parameters value $\hat{a}$, $\hat{a}_2$, $\hat{b}$ and $\hat{\beta}$.

2Under the same value of m, n and τ generate bootstrap sample $\mathbf{X}^*= (X_1^*, X_2^*, ..., X_{k_1}^*, X_{k_1+1}^*, ..., X_m^*)$ and $\mathbf{Y}^*=(Y_1^*, Y_2^*, ..., Y_{k_2}^*, Y_{k_2+1}^*, ..., Y_n^*)$.

3Determine the integer value k_1 and k_2.

4Compute bootstrap sample estimate $\hat{a}^*$, $\hat{a}_2^*$, $\hat{b}^*$ and $\hat{\beta}^*$.

5Compute the bootstrap estimate value of R, $R^* = \frac{\hat{a}_1^*}{\hat{a}_1^*+\hat{a}_2^*}$.

6Repeat steps (2-5) $\mathbf{S}$ times, we obtain the sample estimates of R, say $R^{*(1)}$, $R^{*(2)}$, ..., $R^{*(\mathbf{S})}$.

7The sample estimates of R, put in ascending order $R^*_{(1)}$, $R^*_{(2)}$, ..., $R^*_{(\mathbf{S})}$.

8The empirical CDF of ordered values $R^*_{(1)}$, $R^*_{(2)}$, ..., $R^*_{(\mathbf{S})}$ is defined by ,

$$\Gamma(w) = P(\hat{R}^* < w) \text{ and } \hat{R}^*_{\text{boot-p}} = \Gamma^{-1}(w). \tag{39}$$

9The bootstrap-p confidence intervals of with 2α confidence level is given by

$$\left(\hat{R}^*_{\text{boot-p}}(\alpha),\ \hat{R}^*_{\text{boot-p}}(1-\alpha)\right). \tag{40}$$

3.8 Bayesian credible interval of R

Algorithm 3 (**Credible interval of** R)

1.Put the sample $R^{(1+\mathbf{MB}^*)}, R^{(2)}, \ldots, R^{(\mathbf{MB})}$ in ascending order $R_{(1)}, R_{(2)}, \ldots, R_{(\mathbf{MB}-\mathbf{MB}^*)}$.

2.The empirical CDF of ordered values $R_{(1)}$, $R_{(2)}, \ldots, R_{(\mathbf{MB}-\mathbf{MB}^*)}$ is defined by ,

$$\Gamma\text{B}(w) = P(\hat{R}_B < w) \text{ and } \hat{R}_{CI} = \Gamma\text{B}^{-1}(w). \tag{41}$$

3.The Bayes $(1-2\alpha)100\%$ credible intervals of R is given by

$$\left(R_{\alpha(\mathbf{MB}-\mathbf{MB}^*)},\ R_{(1-\alpha)(\mathbf{MC}-\mathbf{MC}^*)}\right). \tag{42}$$

4 Numerical Example

To illustrate the developed results in this paper, we consider a numerical samples generated from Burr XII distributions. Therefore, we consider two populations Burr XII with shape parameters $\{a_1,b\}$=$\{0.8,\ 0.5\}$ and $\{a_2,b\}$=$\{1.5,\ 0.5\}$.to discuss the inference procedures. Also, the sample size $m = n$ =30, accelerated factor $\beta = 1.5$ and stress change time $\tau = 0.5$. The strength data generated from Burr XII with $\{a_1,b\}$=$\{0.8,\ 0.5\}$ is given by

{0.00104922, 0.0190217, 0.0235457, 0.0335899, 0.0975236, 0.0975334, 0.124916, 0.14277, 0.471222, 0.923666, 1.12178, 1.15805, 1.2345, 2.68943, 3.23917, 6.78473, 12.8286,

17.3145, 42.4134, 74.8762, 142.408, 143.452, 282.632, 362.374, 368.707, 849.952, 2383.04, 3007.98, 13082., 13587.6}.

Also The stress data generated from Burr XII with $\{a_1, b\}=\{1.5, 0.5\}$ is given by

{**0.00000626096, 0.00218342, 0.00471568, 0.00503699, 0.00554835, 0.00587211, 0.0153165, 0.0241679, 0.0245518, 0.0251434, 0.0724508, 0.136352, 0.14772, 0.178771, 0.271778, 0.299763, 0.541363, 0.668674, 0.734493, 0.743836, 0.754889, 1.52874, 1.71577, 3.41646, 3.47571, 4.74769, 5.23083, 7.45187, 86.7267, 321.488**}.

Under change stress time $\tau = 0.5$ and accelerated factor $\beta = 1.5$ two data transformed to **X** ={ **0.00104922, 0.0190217, 0.0235457, 0.0335899, 0.0975236, 0.0975334, 0.124916, 0.14277, 0.471222, 0.782444, 0.914519, 0.938699, 0.989669, 1.95962, 2.32612, 4.68982, 8.71909, 11.7096, 28.4423, 50.0842, 95.105, 95.8016, 188.588, 241.749, 245.971, 566.801, 1588.86, 2005.49, 8721.51, 9058.58**} and

Y ={**0.00000626096, 0.00218342, 0.00471568, 0.00503699, 0.00554835, 0.00587211, 0.0153165, 0.0241679, 0.0245518, 0.0251434, 0.0724508, 0.136352, 0.14772, 0.178771, 0.271778, 0.299763, 0.527576, 0.612449, 0.656329, 0.662557, 0.669926, 1.18583, 1.31051, 2.4443, 2.48381, 3.33179, 3.65389, 5.13458, 57.9844, 214.492**}.

From the last data, we observe $k_1 = 9$ and $k_2 = 16$. The point estimate of R under ML and Bayes method and the corresponding ACI, Boot-p CI, and CI are reported in Table 8.

5 Mote Carlo Studying

In this section, we are presented Monte Carlo simulation study to assessed the quality of developed results in this work of estimation R. Therefore, we assess the point estimate of different methods, ML, bootstrap and Bayes estimation by computing mean estimate (ME) and mean square error (MSE). But, the approximate confidence interval (ACI), two bootstrap confidence intervals (CIs), and Bayes credible intervals (BCI) of R are assessed by computing mean interval length (MIL) and probability coverage (PC). In our studying, we test the results for different parameter values, different acceleration factor and different stress change time τ. Also, we adopt sample sizes described in the tables. The following algorithms describe the outline of this study.

Algorithm 2: (Mote Carlo simulation study)

1:Begin with indicator $s = 1$.
2:For given m,n, τ and parameter vector $\Theta = \{a_1, a_2, b, \beta\}$ generate two samples $\mathbf{X} = (X_1, X_1, ..., X_{k_1}, X_{k_1+1}, ..., X_m)$ and $\mathbf{Y} = (Y_1, Y_1, ..., Y_{k_2}, Y_{k_2+1}, ..., Y_n)$.
3:Form (21) and (30) compute the ML and Bayes point estimates
4:From (38), (40) and (42) compute interval estimates by, approximate ML confidence interval, percentile bootstrap confidence interval and credible interval.
5:Put $s = s+1$.
6:Repeat the Step 2 to 5 1000 times.
7:Compute the the ME and MSE from the relations

$$\mathrm{ME} = \frac{1}{1000}\sum_{i=1i}^{1000} R^{(i)}, \tag{433}$$

and

$$\mathrm{MSE}(R) = \frac{1}{1000}\sum_{i=1i}^{1000}\left(R^{(i)} - \mathrm{ME}\right)^2, \tag{44}$$

8:Compute the value of MIL and PC and all results are reported in tables from 2 to 5.

Numerical discussion: All the results of simulation study have shown that, the model of estimation the reliability of system present a good results. Also, some points are observed from Tables 2 to 5 are given as follows

1.The MSEs and MILs are decreasing when sample sizes are increasing.
2.Bayes point estimate with respected to MSE serve well than ML estimate.
3.Bayesian credible interval with expected to MIL and PC serve well than Approximate confidence interval and percentile bootstrap confidence interval.
4.PCs are closed to proposed one at a large sample size.
5.The proposed inference methods of R present consistent results.

6.The large value of τ serve well than small value of τ.

6 Conclusions

In this paper, we discussed the problem of estimation the parameter of stress strength reliability when two variables strength and stress have Burr XII distribution. This problem is discussed under partially step-stress ALT model. We exposed the estimation problem of R by different method of estimation. We applied ML and Bayes methods to obtain the point estimates of R. Also, interval estimate of R discussed under Approximate ML confidence interval, percentile bootstrap confidence interval as well as Bayesian credible interval interval. The numerical results of numerical example and simulation study have shown that, each methods serve well.

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Table 1: The point and interval estimate of (**R**)

$\widehat{R}_{\text{Exact}}$	$\widehat{R}_{\text{ML}}$	$\widehat{R}_{\text{Bayes}}$	ACI	Boot-p CI	CI
0.3478	0.2701	0.3208	(0.1703, 0.3698)	(0.2752, 0.4215)	(0.2100, 0.3978)

Table 2: ME, MSR, MIL and PC of R =0.25 when $\Theta = \{0.5, 1.5, 1.5, 1.5\}$

τ	(m, n)	Point estimate				Interval estimate					
		ME		MSE		MIL			CP		
		ML	Bayes	ML	Bayes	ACI	Boot-p	CI	ACI	Boot-p	CI
0.5	(15,15)	0.243	0.245	0.0345	0.0287	0.354	0.374	0.347	(0.89)	(0.88)	(0.90)
	(15,25)	0.255	0.253	0.0324	0.0256	0.342	0.361	0.328	(0.90)	(0.91)	(0.92)
	(25,15)	0.254	0.251	0.0321	0.0253	0.337	0.354	0.319	(0.91)	(0.90)	(0.93)
	(25,25)	0.247	0.253	0.0301	0.0228	0.319	0.327	0.302	(0.92)	(0.93)	(0.94)
	(40,40)	0.245	0.249	0.0272	0.0201	0.298	0.311	0.289	(0.93)	(0.92)	(0.96)
	(40,60)	0.248	0.251	0.0261	0.0189	0.291	0.304	0.275	(0.91)	(0.93)	(0.95)
	(60,40)	0.250	0.249	0.0257	0.0183	0.292	0.301	0.269	(0.94)	(0.97)	(0.94)
	(60,60)	0.252	0.253	0.0235	0.0159	0.281	0.292	0.261	(0.93)	(0.93)	(0.95)
1.5	(15,15)	0.245	0.254	0.0337	0.0281	0.347	0.371	0.342	(0.90)	(0.90)	(0.91)
	(15,25)	0.242	0.251	0.0317	0.0251	0.338	0.347	0.325	(0.92)	(0.90)	(0.93)
	(25,15)	0.255	0.252	0.0317	0.0248	0.334	0.351	0.320	(0.933)	(0.92)	(0.92)
	(25,25)	0.248	0.251	0.0295	0.0224	0.313	0.324	0.303	(0.91)	(0.94)	(0.96)
	(40,40)	0.247	0.252	0.0268	0.0197	0.293	0.35	0.284	(0.92)	(0.94)	(0.95)
	(40,60)	0.251	0.247	0.0254	0.0182	0.288	0.301	0.271	(0.94)	(0.92)	(0.94)
	(60,40)	0.253	0.252	0.0251	0.0180	0.288	0.303	0.264	(0.93)	(0.94)	(0.95)
	(60,60)	0.250	0.251	0.0232	0.0154	0.278	0.290	0.254	(0.96)	(0.97)	(0.94)

Table 3: ME, MSR, MIL and PC of R =0.456 when $\Theta = \{1.0, 1.2, 2.0, 1.5\}$

	(m, n)	Point estimate				Interval estimate					
		ME		MSE		MIL			CP		
		ML	Bayes	ML	Bayes	ACI	Boot-p	CI	ACI	Boot-p	CI
0.4	(15,15)	0.465	0.446	0.0554	0.0503	0.614	0.635	0.601	(0.90)	(0.90)	(0.91)
	(15,25)	0.461	0.459	0.0542	0.0494	0.597	0.624	0.587	(0.91)	(0.92)	(0.93)
	(25,15)	0.447	0.454	0.0513	0.0452	0.575	0.600	0.542	(0.92)	(0.91)	(0.96)
	(25,25)	0.458	0.455	0.0497	0.0428	0.562	0.589	0.529	(0.96)	(0.93)	(0.94)
	(40,40)	0.455	0.453	0.0481	0.0407	0.549	0.565	0.511	(0.92)	(0.94)	(0.93)
	(40,60)	0.447	0.452	0.0458	0.0382	0.520	0.541	0.491	(0.92)	(0.93)	(0.93)
	(60,40)	0.457	0.449	0.0451	0.0374	0.511	0.529	0.482	(0.93)	(0.97)	(0.94)
	(60,60)	0.452	0.451	0.0432	0.0344	0.482	0.503	0.454	(0.94)	(0.92)	(0.95)
0.8	(15,15)	0.463	0.448	0.0550	0.0501	0.68	0.631	0.597	(0.90)	(0.92)	(0.91)
	(15,25)	0.459	0.447	0.0537	0.0491	0.592	0.619	0.583	(0.92)	(0.91)	(0.94)
	(25,15)	0.448	0.451	0.0508	0.0448	0.571	0.601	0.539	(0.94)	(0.93)	(0.94)
	(25,25)	0.459	0.449	0.0492	0.0423	0.557	0.583	0.524	(0.92)	(0.94)	(0.93)
	(40,40)	0.452	0.457	0.0478	0.0402	0.544	0.561	0.507	(0.94)	(0.93)	(0.96)
	(40,60)	0.449	0.451	0.0456	0.0378	0.514	0.537	0.487	(0.93)	(0.93)	(0.95)
	(60,40)	0.458	0.454	0.0447	0.0371	0.506	0.524	0.479	(0.94)	(0.94)	(0.93)
	(60,60)	0.453	0.454	0.0429	0.0341	0.478	0.501	0.451	(0.92)	(0.97)	(0.94)

References

[1] E.K. AL-Hussaine, M.A. Ali-Mousa, K.S. Sultan, Parametric and non-parameteric estimation of $P[Y < X]$ for finite mixtures of lognormal components, Communications in Statistics-Theory and Methods 26 (1997) 1269-1289.

[2] Nadarajah, Reliability for Laplace distribution, Mathematical Problems in Engineering 2 (2004) 169-183.

[3] N.A. Mokhlis, Reliability of a stress-strength model with Burr type III distributions, Communications in Statistics-Theory and Methods 34 (7) (2005) 1643-1657.

[4] D. Kundu, R.D. Gupta, Estimation of $P(X < Y)$ for generalized exponential distribution, Metrika 61 (2005) 291-308.

[5] D. Kundu, R.D. Gupta, Estimation of $P[Y < X]$ for Weibull distributions, IEEE Transactions on Reliability 55 (2) (2006) 270-280.

[6] K. Krishnamoorthy, S. Mukherjee, H. Guo, Inference on reliability in two-parameter exponential stress-strength

Table 4: ME, MSR, MIL and PC of R =0.692 when $\Theta = \{1.8, 0.8, 2.0, 2.0\}$

	(m, n)	Point estimate				Interval estimate					
		ME		MSE		MIL			CP		
		ML	Bayes	ML	Bayes	ACI	Boot-p	CI	ACI	Boot-p	CI
0.6	(15,15)	0.723	0.718	0.0987	0.0882	0.877	0.899	0.754	(0.90)	(0.89)	(0.90)
	(15,25)	0.715	0.712	0.0971	0.0858	0.860	0.877	0.741	(0.91)	(0.90)	(0.92)
	(25,15)	0.711	0.704	0.0949	0.0836	0.834	0.845	0.722	(0.92)	(0.91)	(0.91)
	(25,25)	0.681	0.698	0.0915	0.0811	0.805	0.817	0.697	(0.93)	(0.92)	(0.94)
	(40,40)	0.687	0.692	0.0901	0.0798	0.792	0.804	0.681	(0.93)	(0.93)	(0.92)
	(40,60)	0.695	0.691	0.0882	0.0769	0.761	0.779	0.648	(0.92)	(0.96)	(0.94)
	(60,40)	0.687	0.692	0.0855	0.0742	0.741	0.762	0.631	(0.93)	(0.94)	(0.96)
	(60,60)	0.693	0.691	0.0831	0.0718	0.718	0.741	0.611	(0.92)	(0.93)	(0.95)
1.2	(15,15)	0.719	0.717	0.0982	0.0875	0.873	0.894	0.751	(0.91)	(0.90)	(0.92)
	(15,25)	0.712	0.709	0.0965	0.0853	0.856	0.872	0.738	(0.92)	(0.92)	(0.93)
	(25,15)	0.708	0.701	0.0946	0.0833	0.831	0.841	0.718	(0.94)	(0.92)	(0.94)
	(25,25)	0.683	0.696	0.0911	0.0804	0.801	0.813	0.692	(0.93)	(0.94)	(0.95)
	(40,40)	0.689	0.693	0.0897	0.0794	0.789	0.801	0.678	(0.92)	(0.94)	(0.93)
	(40,60)	0.693	0.693	0.0877	0.0764	0.757	0.776	0.649	(0.94)	(0.93)	(0.96)
	(60,40)	0.689	0.693	0.0851	0.0738	0.735	0.754	0.632	(0.92)	(0.92)	(0.95)
	(60,60)	0.691	0.689	0.0817	0.0714	0.715	0.742	0.607	(0.96)	(0.94)	(0.93)

Table 5: ME, MSR, MIL and PC of R =0.800 when $\Theta = \{2.0, 0.5, 1.5, 2.0\}$

	(m, n)	Point estimate				Interval estimate					
		ME		MSE		MIL			CP		
		ML	Bayes	ML	Bayes	ACI	Boot-p	CI	ACI	Boot-p	CI
0.8	(15,15)	0.828	0.819	0.1425	0.1142	0.899	0.912	0.847	(0.89)	(0.90)	(0.91)
	(15,25)	0.823	0.814	0.1382	0.1013	0.851	0.882	0.801	(0.90)	(0.92)	(0.93)
	(25,15)	0.817	0.812	0.1365	0.0987	0.803	0.818	0.782	(0.92)	(0.91)	(0.94)
	(25,25)	0.812	0.808	0.1247	0.0966	0.795	0.801	0.764	(0.93)	(0.93)	(0.96)
	(40,40)	0.808	0.803	0.1120	0.0918	0.766	0.779	0.732	(0.92)	(0.94)	(0.94)
	(40,60)	0.795	0.802	0.1050	0.0899	0.742	0.731	0.707	(0.93)	(0.92)	(0.95)
	(60,40)	0.798	0.798	0.0999	0.0865	0.717	0.708	0.697	(0.92)	(0.92)	(0.94)
	(60,60)	0.802	0.801	0.0966	0.0824	0.704	0.701	0.678	(0.96)	(0.93)	(0.93)
1.3	(15,15)	0.823	0.814	0.1421	0.1139	0.897	0.908	0.843	(0.90)	(0.92)	(0.92)
	(15,25)	0.818	0.812	0.1379	0.1011	0.848	0.877	0.798	(0.92)	(0.91)	(0.94)
	(25,15)	0.813	0.810	0.1364	0.0981	0.797	0.812	0.777	(0.93)	(0.93)	(0.96)
	(25,25)	0.808	0.805	0.1241	0.0962	0.788	0.795	0.760	(0.92)	(0.94)	(0.95)
	(40,40)	0.807	0.804	0.1117	0.0913	0.759	0.772	0.724	(0.95)	(0.93)	(0.93)
	(40,60)	0.797	0.801	0.1047	0.0893	0.738	0.729	0.704	(0.94)	(0.94)	(0.96)
	(60,40)	0.799	0.781	0.0996	0.0855	0.713	0.701	0.692	(0.95)	(0.96)	(0.94)
	(60,60)	0.801	0.800	0.0964	0.0825	0.697	0.699	0.674	(0.93)	(0.92)	(0.95)

model, Metrika 65 (3) (2007) 261-273.

[7] D. Kundu, M.Z. Raqab, Estimation of $R = P[Y < X]$ for three-parameterWeibull distribution, Statistics and Probability Letters 79 (2009) 1839-1846.

[8] M. A. W. Mahmoud, R. M. EL-Sagheer, and M. M. M. El- Sagheer, Inference on the stress-strength model from Weibull gamma distribution," Applied Mathematics & Information Sciences, vol. 11, no. 3, pp. 737-744, 2017.

[9] G. A. Abd-Elmougod and H. H. Abu-Zinadah, "Estimation of P $[Y < X]$ for two-parameter bathtub shape distribution using records. Journal of Computational and Theoretical Nanoscience, 14, 1127-1135(2017).

[10] A. M. Sarhan and A. H. Tolba. Stress-strength reliability under partially accelerated life testing using Weibull model. Scientific African, 20 (2023).

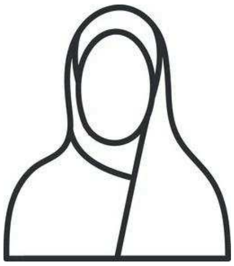
[11] W. Nelson, Accelerated life testing - step-stress models and data analyses," IEEE Transactions on Reliability, vol. R-29, no. 2, pp. 103-108, 1980.

[12] Soliman A. A. , Abd-Elmougod G. A. and Al- Sobhi M. M. (2017). Estimation in step-stress partially accelerated life tests for the Chen distribution using progressive Type-II censoring. Applied Mathematics & Information Sciences, 11, 325-332.

[13] Laila A. Al-Essa, Ahmed A. Soliman , Gamal A. Abd-Elmougod, Huda M. Alshanbari. (2023). Copula approach for dependent competing risks of generalized half-logistic distributions under accelerated censoring data, Symmetry, 15(2), 564.

[14] Burr, I. W. (1942). Cumulative frequency functions. Annals of Mathematical Statistics, 13, 215-232.

[15] Lee, W. C. , Wu, J. W, Hong, C. W. (2009). Assessing the lifetime performance index of products from progressively type II right censored data using Burr XII model. Mathematics and Computers in Simulation, 79 2167-2179.
[16] Tahani. A. Abushal , A. A. Soliman and G. A. Abd-Elmougod (2021). Statistical inferences of Burr XII lifetime models under joint Type-1 competing risks samples. Journal of Mathmetics, 2021, 1-16.
[17] Mahmoud Ragab, Aisha Fayomi, Ali Algarni, G. A. Abd-Elmougod, Neveen Sayed-Ahmed, S. M. Abo-Dahab , and S. Abdel-Khalek (2021). Statistical Analysis of Joint Type-I Generalized Hybrid Censoring Data from Burr XII Lifetime Distributions. Complexity, https://doi.org/10.1155/2021/5543187.
[18] Lomax, K.S., 1954. Business failures: another example of the analysis of failure data. J. Amer. Statist. Assoc. 49 (268), 847–852.
[19] Marshall, A.W., Olkin, I., 2007. Life Distributions. Structure of Nonparametric, Semiparametric, and Parametric Families. Springer, New York, NY.
[20] Chahkandi, M., Ganjali, M., 2009. On some lifetime distributions with decreasing failure rate. Comput. Statist. Data Anal. 53 (12), 4433–4440.
[21] Bryson, M.C., 1974. Heavy-tailed distributions: properties and tests. Technometrics 16 (1), 61–68.
[22] Raqab, M.Z., Asgharzadeh, A., Valiollahi, R., 2010. Prediction for Pareto distribution based on progressively type-II censored samples. Comput. Statist. Data Anal. 54 (7), 1732–1743.
[23] Cramer E. and Schmiedt A.B. (2011). Progressively Type-II censored competing risks data from Lomax distributions, Comput. Statist. Data Anal. 55, 1285–1303.
[24] Tahani. A. Abushal , A. A. Soliman and G. A. Abd-Elmougod (2021). Inference of partially observed causes for failure of Lomax competing risks model under type-II generalized hybrid censoring scheme. Alexandria Engineering Journal, 61, 5427-5439.
[25] Abdulaziz S.Alghamdia, Mubarak H.Elhafiana, Hassan M.Aljohanib and G.A.Abd-Elmougod (2021). Estimations of accelerated Lomax lifetime distribution with a dependent competing risks model under type-I generalized hybrid censoring scheme. Alexandria Engineering Journal. 61, 6489-6499.
[26] M. H. DeGroot and P. K. Goel, "Bayesian estimation and optimal designs in partially accelerated life testing," Naval Research Logistics Quarterly, vol. 26, no. 2, pp. 223–235, 1979.
[27] Davison, A. C.and and Hinkley, D. V. (1997). Bootstrap Methods and their Applications, 2nd, Cambridge University Press, Cambridge United Kingdom.
[28] Efron, B. (1982). The jackknife, the bootstrap and other resampling plans. In: CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Phiadelphia, PA ; 38.
[29] T. J. DiCiccio and B. Efron, "Bootstrap confidence intervals," Statistical Science, vol. 11, no. 3, pp. 189–228, 1996.
[30] Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N., Teller, A.H., Teller, E., (1953). Equations of state calculations by fast computing machines. Journal Chemical Physics 21, 1087–1091.



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