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Generalization of the Mehler's formula

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Abstract: We demonstrate that a generalization of the Mehler's formula can be achieved by employing techniques commonly utilized in quantum optics. The methodology involves deriving the Mehler's formula through the solution of a Schrödinger-type equation. The selection of the initial conditions, determines the type of Mehler's formula obtained; for example, the usual Mehler's formula is obtained when the initial condition is the product of two harmonic oscillator base functions. In this article, we focus on investigating specific initial conditions that hold significance within the quantum optics community.

Keywords: Mehler formula, generalized Mehler formula, special functions, Hermite polynomials, generating functions

1 Introduction

The Mehler's formula has been a fundamental tool in the fields of mathematics [1–6], theoretical physics [7], and more specifically, in quantum mechanics [8–11]. However, in recent years, there has been a growing interest for application in various scientific contexts: neural networks, materials science, and chemical birth-death process [12–14], among others. Given that the product of Hermite polynomials leads to the Mehler's formula, Mehler-type formulas have been proposed by generalizing the Hermite polynomials [15–17]. Furthermore, additional extensions have been considered [18, 19].

In the context of quantum mechanics, the Mehler's formula has been used to comprehend the concept of the fractional-order Fourier transform [20], and has been applied in the study of quantum correlations [21, 22]. It has also been employed in solving time-dependent problems [23–26], optical fiber [27], and in the treatment of two-mode squeezed states [28, 29].

The article is organized as follow: In Section 2, we begin obtaining the generating function for Hermite polynomials defining an alternative form of ladder operators. During Section 3, the Mehler's formula is derived using the formal solution of a proposed Schrödinger-type equation. In Section 4, we use the results obtained in the previous section to generalize the Mehler's formula for any initial conditions, e.g., ground states, coherent states, and two arbitrary harmonic oscillator wave functions. Our conclusions are contained in the Section 5.

2 Hermite polynomials

To illustrate the techniques that we will use later to generalize the Mehler's formula, in this section, using those techniques, we will find the well-known generating function of the Hermite polynomials.
Hermite polynomials can be defined in several ways; in this article it is convenient to use the Rodrigues' formula [30]; that is, we will use the following definition

$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n = 0, 1, 2, \ldots. \tag{1}$$

The generating function for the Hermite polynomials is

$$e^{-\alpha^2+2\alpha x} = \sum_{n=0}^{\infty} H_n(x) \frac{\alpha^n}{n!}. \tag{2}$$

The Hermite polynomials satisfy the recurrence relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{3}$$

and

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x). \tag{4}$$

All Hermite polynomials can be generated with the Rodrigues' formula, or with the recurrence relations.

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From the above recurrence relations, we can prove that if we define the normalized simple harmonic oscillator wave functions [30]

$$\varphi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} \mathrm{e}^{-x^2/2} H_n(x), \qquad n = 0,1,2,\dots, \tag{5}$$

the following relations are satisfied

$$\hat{A}^\dagger \varphi_n(x) \equiv \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right) \varphi_n(x) = \sqrt{n+1}\varphi_{n+1}(x), \tag{6}$$

and

$$\hat{A} \varphi_n(x) \equiv \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right) \varphi_n(x) = \sqrt{n}\varphi_{n-1}(x), \tag{7}$$

where $\hat{A}$ and $\hat{A}^\dagger$ are the so-called ladder operators. Introducing the differential operator $\hat{D}$, defined as

$$\hat{D} \equiv \frac{d}{dx}, \tag{8}$$

we can rewrite the Rodrigues' formula, (1), in the form [31]

$$H_n(x) = (-1)^n e^{x^2} \hat{D}^n e^{-x^2} = (-1)^n \left(e^{x^2} \hat{D} e^{-x^2} \right)^n. \tag{9}$$

The operator inside the parenthesis in (9) has the form $e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}$, and we can use the Hadamard's lemma [32–34] which establish that

$$e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} = \hat{B} + \xi \left[\hat{A},\hat{B}\right] + \frac{\xi^2}{2!} \left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \dots, \tag{10}$$

to show that

$$e^{x^2} \hat{D} e^{-x^2} = \hat{D} - 2x, \tag{11}$$

and then, we can write

$$H_n(x) = (-1)^n (D-2x)^n 1. \tag{12}$$

Using this last expression, we can write the right hand side of the generating function (2) as

$$\sum_{n=0}^{\infty} H_n(x) \frac{\alpha^n}{n!} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \left(\hat{D} - 2x\right)^n 1 = e^{-\alpha\left(\hat{D}-2x\right)} 1. \tag{13}$$

We have obtained the exponential of the sum of two operators that do not commute, $\hat{D}$ and x. Using the Baker-Hausdorff formula [35], the above exponential can be factorized in the product of two exponentials; indeed, if in that formula we make the identifications

$$A \to 2\alpha x, \qquad B \to -\alpha\hat{D}, \tag{14}$$

we have that

$$[A,B] = -2\alpha^2 \left[x,\hat{D}\right] = 2\alpha^2, \tag{15}$$

and we get

$$\sum_{n=0}^{\infty} H_n(x) \frac{\alpha^n}{n!} = e^{-\alpha^2} e^{2\alpha x} e^{\alpha\hat{D}} 1. \tag{16}$$

Using now the obvious fact that $e^{\alpha\hat{D}}1 = 1$, we finally obtain the desired result

$$\sum_{n=0}^{\infty} H_n(x) \frac{\alpha^n}{n!} = e^{-\alpha^2 + 2\alpha x}, \tag{17}$$

which is the generating function for Hermite polynomials, as we wanted to show.

3 Mehler's formula

The Mehler's formula [36–39] establishes that the function

$$S(x,y;\rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{\rho^2\left(x^2+y^2\right) - 2\rho xy}{1-\rho^2} \right], \tag{18}$$

can be expanded in terms of Hermite polynomials, as

$$S(x,y;\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{2^n n!} H_n(x) H_n(y). \tag{19}$$

In this section, we will show how the Mehler's formula can be obtained using the methods of quantum optics, outlined in the previous section. The idea of proving this well-known and perfectly well-proven formula in this way, is to illustrate the method that we will use in its generalization.

First, we multiply (19) by a product of Gaussian functions in x and y, and use the eigenfunctions of the harmonic oscillator (5), to write

$$e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} S(x,y;\rho) = \sqrt{\pi} \sum_{n=0}^{\infty} \rho^n \varphi_n(x) \varphi_n(y). \tag{20}$$

As a key point in this work, we introduce the Schrödinger-type equation

$$i\frac{\partial \Phi(x,y,t)}{\partial t} = \left(y\hat{p}_x + x\hat{p}_y\right) \Phi(x,y,t), \tag{21}$$

where, as usual, $\hat{p}_\xi = -i\partial/\partial\xi$, $(\xi = x,y)$, and t is a real parameter. The formal solution of this Schrodinger-like equation is

$$\Phi(x,y,t) = e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)} \Phi_0(x,y), \tag{22}$$

where $\Phi_0(x,y) = \Phi(x,y,t=0)$ is the initial condition.

Next, we will show that with the appropriate initial condition, the solution of this equation is essentially the function $S(x,y,t)$ on the left side of the Mehler's formula, Eq. (18). For that, we write the formal solution (22) as

$$\Phi(x,y,t) = e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)} \Phi_0(x,y) e^{it\left(y\hat{p}_x + x\hat{p}_y\right)} e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)} 1, \tag{23}$$

where we have introduced the unit operator $\hat{I}$, written as $\hat{I} = e^{it\left(y\hat{p}_x + x\hat{p}_y\right)} e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)}$ at the end.
Using again the Hadamard's lemma [32,33], Eq. (10), we can demonstrate that

$$e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)} x e^{it\left(y\hat{p}_x + x\hat{p}_y\right)} = x\cosh t - y\sinh t, \tag{24}$$

and that

$$e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)} y e^{it\left(y\hat{p}_x + x\hat{p}_y\right)} = y\cosh t - x\sinh t. \tag{25}$$

From the definition of the operators $\hat{p}_x$ and $\hat{p}_y$, which are derivatives, it is obvious that

$$e^{-it\left(y\hat{p}_x + x\hat{p}_y\right)} 1 = 1. \tag{26}$$

Thus, we can cast (22) as

$$\Phi(x,y,t) = \Phi_0(x\cosh t - y\sinh t, y\cosh t - x\sinh t). \tag{27}$$

We now use the special initial condition mentioned above, which is

$$\Phi_0(x,y) = \varphi_0(x)\varphi_0(y), \tag{28}$$

where $\varphi_0(\xi)$ is the harmonic oscillator eigenfunction (5) with $n = 0$, that is just a Gaussian function. Hence,

$$\Phi(x,y) = \frac{1}{\sqrt{\pi}} e^{-\frac{(x\cosh t - y\sinh t)^2 + (y\cosh t - x\sinh t)^2}{2}}. \tag{29}$$

Making the identification $\tanh(t) = \rho$, and after some simple algebra, we arrive to

$$\Phi(x,y) = \frac{1}{\sqrt{\pi}} \exp\left[-\frac{1}{2}\frac{(y-\rho x)^2 + (x-\rho y)^2}{1-\rho^2}\right]. \tag{30}$$

Now, we go back to the formal solution (22), and we are going to express it in a way that is essentially the right hand side of the Mehler's formula, (18). For that, we introduce the common ladder operators for x and y [40,41],

$$\hat{a} = \frac{x + i\hat{p}_x}{\sqrt{2}}, \qquad \hat{a}^\dagger = \frac{x - i\hat{p}_x}{\sqrt{2}}, \tag{31}$$

and

$$\hat{b} = \frac{y + i\hat{p}_y}{\sqrt{2}}, \qquad \hat{b}^\dagger = \frac{y - i\hat{p}_y}{\sqrt{2}}, \tag{32}$$

and we cast the formal solution, Eq. (22), as

$$\Phi(x,y,t) = \exp\left[t\left(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\right)\right]\Phi_0(x,y)\,. \tag{33}$$

In Appendix A, we prove that the evolution operator $\hat{U}(t) = \exp\left[t\left(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\right)\right]$ can be factorized as

$$\hat{U}(t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger} e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}\right)} e^{-\tanh(t)\hat{a}\hat{b}}. \tag{34}$$

We apply this evolution operator to the special initial condition $\Phi_0(x,y) = \varphi_0(x)\varphi_0(y)$. As $\hat{a}\varphi_0(x) = 0$ and $\hat{b}\varphi_0(y) = 0$, it is easy to convince yourself that

$$\Phi(x,y,t) = \frac{1}{\cosh(t)} e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\varphi_0(x)\varphi_0(y); \tag{35}$$

expanding in Taylor series the exponential, and using the fact that $\hat{a}^\dagger$ and $\hat{b}^\dagger$ commute, we obtain

$$\Phi(x,y,t) = \frac{1}{\cosh t}\sum_{n=0}^{\infty}\frac{\tanh^n(t)}{n!}\hat{a}^{\dagger n}\hat{b}^{\dagger n}\varphi_0(x)\varphi_0(y). \tag{36}$$

We know that $\hat{a}^{\dagger n}\varphi_0(x) = \sqrt{n!}\varphi_n(x)$ and that $\hat{b}^{\dagger n}\varphi_0(y) = \sqrt{n!}\varphi_n(y)$, so

$$\Phi(x,y,t) = \frac{1}{\cosh(t)}\sum_{n=0}^{\infty}\tanh^n(t)\varphi_n(x)\varphi_n(y). \tag{37}$$

Substituting back the harmonic oscillator eigenfunctions, (5), we arrive to

$$\Phi(x,y,t) = \frac{e^{-\frac{x^2+y^2}{2}}}{\sqrt{\pi}\cosh t}\sum_{n=0}^{\infty}\frac{\tanh^n t}{2^n n!}H_n(x)H_n(y). \tag{38}$$

Remembering the identification $\rho = \tanh(t)$, we get

$$\Phi(x,y,t) = \frac{1}{\sqrt{\pi}}\sqrt{1-\rho^2}\,e^{-\frac{x^2+y^2}{2}}\sum_{n=0}^{\infty}\frac{\rho^n}{2^n n!}H_n(x)H_n(y). \tag{39}$$

Finally, we match Eqs. (30) and (39) to reproduce the Mehler's formula.

4 Generalization of the Mehler's formula

The generalizations of the Mehler's formula that we present below are obtained by changing the initial condition of the Schrödinger type equation, Eq. (21), and employing the same tactic as for the usual Mehler's formula; i.e., using in one side the solution (27), and in the other side the solution obtained with the evolution operator (34). First, we will use a ground state times a coherent state, then the product of two coherent states, and finally, the product of two arbitrary harmonic oscillator eigenfunctions.

4.1 Ground and coherent states

We start with the case where the initial condition is the product of a ground state and a coherent state, such that initial condition is

$$\Phi_0(x,y) = \varphi_0(x)\psi_\alpha(y), \tag{40}$$

where $\psi_\alpha(y)$ is the wave function of the coherent state. Employing Eq. (27), we obtain

$$\Phi(x,y,t) = \varphi_0(x\cosh t - y\sinh t)\psi_\alpha(y\cosh t - x\sinh t). \tag{41}$$

Besides, from Eq. (34),

$$\Phi(x,y,t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger} e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}\right)} e^{-\tanh(t)\hat{a}\hat{b}}\varphi_0(x)\psi_\alpha(y). \tag{42}$$

In the equation above, we proceed to apply each of the operators from right to left. As, $\hat{a}\varphi_n(x) = 0$, we have $e^{-\tanh(t)\hat{a}\hat{b}}\varphi_0(x)\psi_\alpha(y) = \varphi_0(x)\psi_\alpha(y)$, and

$$\Phi(x,y,t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger} e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger+\hat{b}^\dagger\hat{b}\right)}\varphi_0(x)\psi_\alpha(y). \tag{43}$$

Applying the following operator in a right-to-left sequence (see Appendix B), we arrive at

$$\Phi(x,y,t) = f(\alpha,t)e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\varphi_0(x)\psi_\beta(y), \tag{44}$$

where

$$f(\alpha,t) = \frac{1}{\cosh t}\exp\left(-\frac{|\alpha|^2}{2}+\frac{|\beta|^2}{2}\right), \tag{45}$$

with $\beta = \alpha/\cosh t$.
In Appendix C, we show that the action of the remainder operator leads us to

$$\Phi(x,y,t) = \frac{e^{\frac{-|\alpha|^2}{2}}}{\cosh(t)}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\tanh^j(t)}{\sqrt{j!}}\frac{\beta^m}{m!}\sqrt{(j+m)!}\varphi_j(x)\varphi_{j+m}(y). \tag{46}$$

Returning to the Hermite polynomials using (5),

$$\Phi(x,y,t) = \frac{e^{-\frac{x^2+y^2}{2}}e^{\frac{-|\alpha|^2}{2}}}{\sqrt{\pi}\cosh(t)}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\tanh^j(t)}{j!m!}\frac{\beta^m}{2^{j+\frac{m}{2}}}H_j(x)H_{j+m}(y). \tag{47}$$

To obtain the final generalized form of the Mehler's formula, we need to equate this last expression with the equation obtained substituting the explicit wave functions for $\varphi_0(x)$ and for $\psi_\alpha(y)$ in Eq.(41), and make the identification $\tanh(t) = \rho$. We write the explicit result only in the case in which α is real,

$$\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{\rho^2\left(x^2+y^2\right)+2\rho xy}{1-\rho^2}\right] \times\exp\left[-\frac{\alpha^2}{2}+\frac{\sqrt{2}\alpha\left(y+\rho x\right)}{\sqrt{1-\rho^2}}\right] = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\rho^j\alpha^m\left(1-\rho^2\right)^{m/2}}{j!m!2^{j+m/2}}H_j(x)H_{j+m}(y). \tag{48}$$

Note that if $\alpha = 0$ in the expression above, we recover the formula (18) as it should be, because in that case the Mehler's generalized formula must be reduced to the traditional one.

4.2 Coherent states

We consider now the case when the initial condition is the product of two coherent states; i.e.,

$$\Phi_0(x,y) = \psi_{\alpha_x}(x)\psi_{\alpha_y}(y). \tag{49}$$

According to Eq. (27), we get

$$\Phi(x,y,t) = \psi_{\alpha_x}(x\cosh t - y\sinh t)\psi_{\alpha_y}(y\cosh t - x\sinh t). \tag{50}$$

Furthermore, from (34) we obtain

$$\Phi(x,y,t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger+\hat{b}^\dagger\hat{b}\right)}e^{-\tanh(t)\hat{a}\hat{b}}\psi_{\alpha_x}(x)\psi_{\alpha_y}(y). \tag{51}$$

Similarly to the previous case, we need to apply each of the operators in the above expression. Thus, after applying the first one, we arrive to

$$\Phi(x,y,t) = e^{-\tanh(t)\alpha_x\alpha_y}e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger+\hat{b}^\dagger\hat{b}\right)}\psi_{\alpha_x}(x)\psi_{\alpha_y}(y). \tag{52}$$

Upon applying the second operator, we can write

$$\Phi(x,y,t) = f(\alpha_x,\alpha_y,t)e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\psi_{\beta_x}(x)\psi_{\beta_y}(y), \tag{53}$$

with

$$f(\alpha_x,\alpha_y,t) = \frac{e^{-\tanh(t)\alpha_x\alpha_y}}{\cosh(t)}e^{-\frac{|\alpha_x|^2}{2}-\frac{|\alpha_y|^2}{2}+\frac{|\beta_x|^2}{2}+\frac{|\beta_y|^2}{2}}, \tag{54}$$

$\beta_x = \frac{\alpha_x}{\cosh(t)}$, and $\beta_y = \frac{\alpha_y}{\cosh(t)}$.
Expanding the last operator in its Taylor series,

$$\Phi(x,y,t) = f(\alpha_x,\alpha_y,t)\sum_{j=0}^{\infty}\frac{\tanh^j(t)}{j!}(\hat{a}^\dagger\hat{b}^\dagger)^j\psi_{\beta_x}(x)\psi_{\beta_y}(y). \tag{55}$$

Then,it is easy to see that

$$\Phi(x,y,t) = \frac{e^{-\tanh(t)\alpha_x\alpha_y}}{\cosh(t)}e^{-\frac{|\alpha_x|^2}{2}-\frac{|\alpha_y|^2}{2}}\sum_{j=0}^{\infty}\sum_{n,m=0}^{\infty}\frac{\tanh^j(t)}{j!}\frac{\beta_x^n\beta_y^m}{n!m!}\sqrt{(j+n)!(j+m)!}\varphi_{j+n}(x)\varphi_{j+m}(y). \tag{56}$$

Finally, using (5), we write the last equation in terms of the Hermite polynomials

$$\Phi(x,y,t) = \frac{e^{-\tanh(t)\alpha_x\alpha_y}e^{-\frac{x^2+y^2}{2}}}{\sqrt{\pi}\cosh(t)}e^{-\frac{|\alpha_x|^2}{2}-\frac{|\alpha_y|^2}{2}}\sum_{j=0}^{\infty}\sum_{n,m=0}^{\infty}\frac{\tanh^j(t)}{j!n!m!}\frac{\beta_x^n\beta_y^m}{2^{j+\frac{n+m}{2}}}H_{j+n}(x)H_{j+m}(y). \tag{57}$$

Like in the previous case, to generalize the Mehler's formula, it is necessary to equate the above expression with Eq. (50), when we substitute the explicit wave functions for $\psi_{\alpha_x}(x)$ and $\psi_{\alpha_y}(y)$, and set $\tanh(t) = \rho$. We will provide the explicit result only in the case where α_x and α_y are real, which is

$$\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{\rho^2\left(x^2+y^2\right)+2\rho xy}{1-\rho^2}\right]$$
$$\exp\left[-\frac{\alpha_x^2+\alpha_y^2}{2}+\frac{\sqrt{2}(1+\rho)(x\alpha_x+y\alpha_y)}{\sqrt{1-\rho^2}}+\rho\alpha_x\alpha_y\right]$$
$$=\sum_{j=0}^{\infty}\sum_{n,m=0}^{\infty}\frac{\rho^j\alpha_x^n\alpha_x^m\left(1-\rho^2\right)^{\frac{n+m}{2}}}{j!n!m!2^{j+\frac{n+m}{2}}}H_{j+n}(x)H_{j+m}(y). \quad (58)$$

In Fig. 1, we plot the square of the absolute value of the field in Eq. (50) for different values of α_x and α_y; we display four different cases in which we can readily observe displacement, rotation and slight compression in each of the states, based on the values of α_x and α_y; these changes occur due to the value assigned to t. Furthermore, for longer times, this compression becomes increasingly conspicuous, primarily because the term $\exp\left[\tanh(t)\hat{a}^\dagger\hat{b}^\dagger\right]$ tends to diverge exponentially.

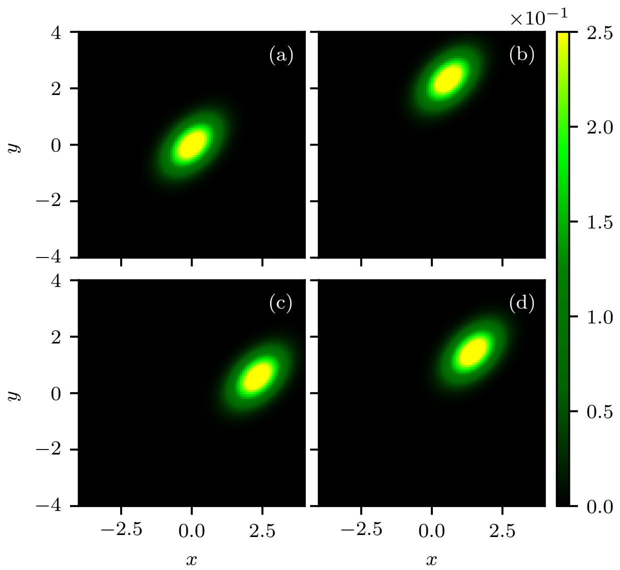

Fig. 1: The square of the absolute value of the field in Eq. (50) for different values of α_x and α_y at $t = 0.25$; (a) $\alpha_x = \alpha_y = 0$, (b) $\alpha_x = 0$ and $\alpha_y = 2$, (c) $\alpha_x = 2$ and $\alpha_y = 0$, (d) $\alpha_x = 1$ and $\alpha_y = 1$.

4.3 Two harmonic oscillator wave functions

Finally, we delve into the case of two harmonic oscillator wave functions; for this, we take our initial condition as

$$\Phi_0(x,y) = \varphi_n(x)\varphi_m(y), \quad (59)$$

where the functions $\varphi_j(x)$ are given by (5).
From Eqs. (27) and (34), we have

$$\Phi(x,y,t) = \varphi_n(x\cosh t - y\sinh t)\varphi_m(y\cosh t - x\sinh t), \quad (60)$$

and

$$\Phi(x,y,t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger+\hat{b}^\dagger\hat{b}\right)}e^{-\tanh(t)\hat{a}\hat{b}}\varphi_n(x)\varphi_m(y), \quad (61)$$

respectively. In the same manner as in the previous cases, we have to apply each of the operators in the last expression; after applying the first one, we get

$$\Phi(x,y,t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger+\hat{b}^\dagger\hat{b}\right)}\sum_{j=0}^{\min(n,m)}\frac{[-\tanh(t)]^j}{j!}\sqrt{\frac{n!m!}{(n-j)!(m-j)!}}\varphi_{n-j}(x)\varphi_{m-j}(y); \quad (62)$$

as we have $(n-j)!$ and $(m-j)!$ in the denominator, the sum over j can be extended to infinity, giving

$$\Phi(x,y,t) = e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}e^{-\ln(\cosh(t))\left(\hat{a}\hat{a}^\dagger+\hat{b}^\dagger\hat{b}\right)}\sum_{j=0}^{\infty}\frac{(-1)^j\tanh^j(t)}{j!}\sqrt{\frac{n!m!}{(n-j)!(m-j)!}}\varphi_{n-j}(x)\varphi_{m-j}(y). \quad (63)$$

Applying the second operator, we obtain

$$\Phi(x,y,t) = \frac{1}{\cosh^{n+m+1}(t)}e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\sum_{j=0}^{\infty}\frac{(-1)^j\tanh^j(t)\cosh^{2j}(t)}{j!}\sqrt{\frac{n!m!}{(n-j)!(m-j)!}}\varphi_{n-j}(x)\varphi_{m-j}(y). \quad (64)$$

Applying the third operator, we arrive to

$$\Phi(x,y,t) = \frac{\sqrt{n!m!}}{\cosh^{n+m+1}(t)}\sum_{j,k=0}^{\infty}\frac{(-1)^j\sqrt{(n-j+k)!(m-j+k)!}}{j!k!(n-j)!(m-j)!}\tanh^{j+k}(t)\cosh^{2j}(t)\varphi_{n-j+k}(x)\varphi_{m-j+k}(y). \quad (65)$$

Finally, substituting in (60) and in (65) the explicit expression for the quantum harmonic oscillator eigenfunctions, Eq. (5), making $\tanh(t) = \rho$, and equating those two expressions, we get the following generalization of the Mehler's formula,

$$\frac{1}{(1-\rho^2)^{\frac{n+m+1}{2}}} \exp\left[-\frac{\rho^2\left(x^2+y^2\right)-2\rho xy}{1-\rho^2}\right] H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right) H_m\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) = \sum_{j,k=0}^{\infty} \frac{(-1)^j n!m!}{2^{k-j} j!k!(n-j)!(m-j)!} \frac{\rho^{j+k}}{(1-\rho^2)^{j/2}} H_{n-j+k}(x) H_{m-j+k}(y). \quad (66)$$

If we chose $n = m = 0$ in (66), we return to the standard Mehler's formula Eq. (19), as must be.
In Fig. 2, we depict the square of the absolute value of the field in Eq. (60) for different values of n and m at $t = 0.25$; we observe that the states are compressed according to the value of t that we have chosen. Furthermore, for larger times, this compression increases, as in the previous case. In addition, the values of n and m indicate how many spots will be generated.

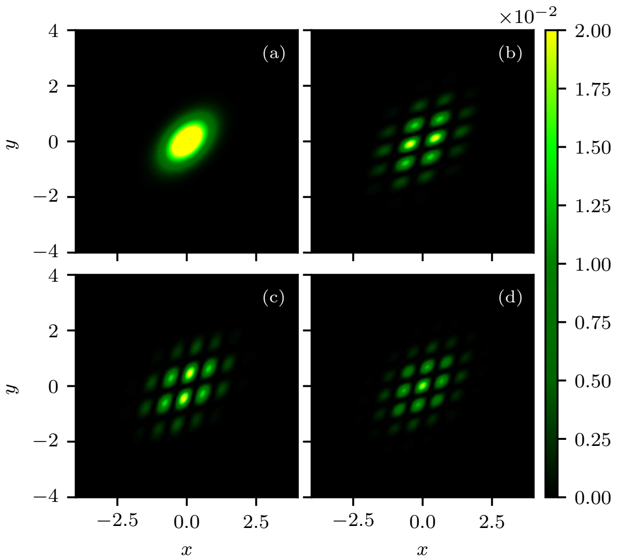

Fig. 2: The square of the absolute value of the field in Eq. (60) for different values of n and m at $t = 0.25$: (a) $n = m = 0$, (b) $n = 5$ and $m = 10$, (c) $n = 10$ and $m = 10$, (d) $n = 10$ and $m = 10$

5 Conclusions

The generalizations of the Mehler's formula broaden its scope and applicability, enabling us to address a variety of initial conditions and more complex quantum phenomena.

In summary, this paper presents three generalizations of the Mehler's formula:
(1) The first one is obtained when a product of the ground state of the harmonic oscillator and a coherent state is considered as initial state. The generalization obtained is presented in expression (48).
(2) The generalization formula (58) is obtained when the initial condition is the product of two coherent states with real amplitudes.
(3) If the initial state is the product of two eigenfunctions of the harmonic oscillator, we acquire the third and final generalization of the Mehler's formula. The new formula is given in (66).
These generalizations are achieved through the formal solution of a Schrödinger-type equation, along with the utilization of well-established tools in quantum optics. Furthermore, we apply it to various initial conditions, thereby demonstrating its utility and practicality.

A Factorization of $\hat{U}(t)$

In this appendix, we factorize the evolution operator $e^{t(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})}$.
To achieve the factorization, we propose the ansatz

$$\hat{U}(t) = e^{t(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})} = e^{f_1(t)\hat{a}^\dagger\hat{b}^\dagger} e^{f_2(t)(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b})} e^{f_3(t)\hat{a}\hat{b}}, \quad (67)$$

where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are functions of time to be determined.
The derivative of the evolution operator with respect to t gives, on the one hand,

$$\frac{d\hat{U}(t)}{dt} = (\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})\hat{U}(t), \quad (68)$$

and, on the other hand,

$$\begin{aligned}\frac{d\hat{U}(t)}{dt} =& \frac{df_1}{dt}\hat{a}^\dagger\hat{b}^\dagger e^{f_1(t)\hat{a}^\dagger\hat{b}^\dagger} e^{f_2(t)(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b})} e^{f_3(t)\hat{a}\hat{b}} \\ &+ \frac{df_2}{dt} e^{f_1(t)\hat{a}^\dagger\hat{b}^\dagger} (\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}) e^{f_2(t)(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b})} e^{f_3(t)\hat{a}\hat{b}} \\ &+ \frac{df_3}{dt} e^{f_1(t)\hat{a}^\dagger\hat{b}^\dagger} e^{f_2(t)(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b})} \hat{a}\hat{b} e^{f_3(t)\hat{a}\hat{b}}. \end{aligned} \quad (69)$$

By properly inserting the identity operator, written in different ways several times, the expression above can be copied as

$$\begin{aligned}\frac{d\hat{U}(t)}{dt} =& \left[\frac{df_1}{dt}\hat{a}^\dagger\hat{b}^\dagger + \frac{df_2}{dt} e^{f_1(t)\hat{a}^\dagger\hat{b}^\dagger} (\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}) e^{-f_1(t)\hat{a}^\dagger\hat{b}^\dagger} \right. \\ &\left. + \frac{df_3}{dt} e^{f_1(t)\hat{a}^\dagger\hat{b}^\dagger} e^{f_2(t)(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b})} \hat{a}\hat{b} e^{-f_2(t)(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b})} e^{-f_1(t)\hat{a}^\dagger\hat{b}^\dagger}\right] \\ &\hat{U}(t). \end{aligned} \quad (70)$$

Using above the Hadamard lemma, Eq. (10), and equating the result obtained with the derivative in (68), we get the set of coupled differential equations

$$1 = \frac{df_1}{dt} - 2f_1\frac{df_2}{dt} + f_1^2\frac{df_3}{dt}e^{-2f_2}, \tag{71a}$$

$$0 = \frac{df_2}{dt} - f_1\frac{df_3}{dt}e^{-2f_2}, \tag{71b}$$

$$-1 = \frac{df_3}{dt}e^{-2f_2}, \tag{71c}$$

subject to the initial conditions $f_1(0) = f_2(0) = f_3(0) = 0$. It is not difficult to show that the solution of the above system of equations is

$$f_1(t) = \tanh t,\ f_2(t) = -\ln(\cosh t),\ f_3(t) = -\tanh t. \tag{72}$$

B Appendix B

In this appendix, we show that

$$e^{-\ln(\cosh t)\left(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}\right)}\varphi_0(x)\psi_\alpha(y) = f(\alpha,t)\varphi_0(x)\psi_\beta(y), \tag{73}$$

where

$$f(\alpha,t) = \frac{1}{\cosh t}e^{\frac{1}{2}\left(-|\alpha|^2+|\beta|^2\right)}, \tag{74}$$

and $\beta = \alpha/\cosh t$.

As the operators $\hat{a}$ and $\hat{a}^\dagger$ commute with $\hat{b}$ and $\hat{b}^\dagger$, we can write

$$e^{-\ln(\cosh t)\left(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}\right)} = e^{-\ln(\cosh t)\hat{a}\hat{a}^\dagger}e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}; \tag{75}$$

thus,

$$e^{-\ln(\cosh t)\left(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b}\right)}\varphi_0(x)\psi_\alpha(y) = e^{-\ln(\cosh t)\hat{a}\hat{a}^\dagger}\varphi_0(x)e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}\psi_\alpha(y). \tag{76}$$

Since $[\hat{a},\hat{a}^\dagger] = 1$, we can cast the first term in the right hand side of the equation above as

$$e^{-\ln(\cosh t)\hat{a}\hat{a}^\dagger}\varphi_0(x) = e^{-\ln(\cosh t)\left(\hat{a}^\dagger\hat{a}+1\right)}\varphi_0(x), \tag{77}$$

and because $\hat{a}^\dagger\hat{a}\,\varphi_0(x) = 0$, we get

$$e^{-\ln(\cosh t)\hat{a}\hat{a}^\dagger}\varphi_0(x) = e^{-\ln(\cosh t)}\varphi_0(x) = \frac{\varphi_0(x)}{\cosh t}. \tag{78}$$

On the other hand, to deal with the term $e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}\psi_\alpha(y)$, we write the coherent state $\psi_\alpha(y)$ in terms of the harmonic oscillator eigenfuntions $\varphi_n(y)$ as

$$\psi_\alpha(y) = e^{-\frac{|\alpha|^2}{2}}\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\varphi_j(y), \tag{79}$$

and then

$$\begin{aligned} e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}\psi_\alpha(y) &= e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}e^{-\frac{|\alpha|^2}{2}}\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\varphi_j(y) \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}\varphi_j(y); \end{aligned} \tag{80}$$

we have that $\hat{b}^\dagger\hat{b}\,\varphi_j(y) = j\varphi_j(y)$, thus

$$\begin{aligned} e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}\psi_\alpha(y) &= e^{-\frac{|\alpha|^2}{2}}\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}e^{-\ln(\cosh t)j}\varphi_j(y) \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\frac{1}{\cosh^j t}\varphi_j(y) \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{\alpha}{\cosh t}\right)^j\varphi_j(y) \\ &= e^{-\frac{|\alpha|^2}{2}}e^{\frac{|\frac{\alpha}{\cosh t}|^2}{2}}e^{-\frac{|\frac{\alpha}{\cosh t}|^2}{2}}\sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{\alpha}{\cosh t}\right)^j\varphi_j(y). \end{aligned} \tag{81}$$

Hence,

$$e^{-\ln(\cosh t)\hat{b}^\dagger\hat{b}}\psi_\alpha(y) = e^{\frac{1}{2}\left(-|\alpha|^2+|\beta|^2\right)}\psi_\beta(y). \tag{82}$$

Finally, using (78) and (82), we get (73), as wished.

C Appendix C

In this appendix, we prove that

$$e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\varphi_0(x)\psi_\beta(y) = e^{-\frac{|\beta|^2}{2}}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\sqrt{(j+m)!}}{\sqrt{j!}m!}\tanh^j(t)\beta^m\varphi_j(x)\varphi_{j+m}(y). \tag{83}$$

Using the definition of the exponential operator in terms of its Taylor's series, we have

$$e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\varphi_0(x)\psi_\beta(y) = \left[\sum_{j=0}^{\infty}\frac{\tanh^j(t)}{j!}\left(\hat{a}^\dagger\hat{b}^\dagger\right)^j\right]\varphi_0(x)\psi_\beta(y); \tag{84}$$

since $\hat{a}^\dagger$ and $\hat{b}^\dagger$ commute, we can write

$$e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\varphi_0(x)\psi_\beta(y) = \sum_{j=0}^{\infty}\frac{\tanh^j(t)}{j!}\hat{a}^{\dagger j}\varphi_0(x)\hat{b}^{\dagger j}\psi_\beta(y). \tag{85}$$

We have already used in Appendix B that $\hat{a}^{\dagger j}\varphi_0(x) = \sqrt{j!}\varphi_j(x)$, so

$$e^{\tanh(t)\hat{a}^\dagger\hat{b}^\dagger}\varphi_0(x)\psi_\beta(y) = \sum_{j=0}^{\infty}\frac{\tanh^j(t)}{\sqrt{j!}}\varphi_j(x)\hat{b}^{\dagger j}\psi_\beta(y). \tag{86}$$

We turn now to the term $\hat{b}^{\dagger j}\psi_\beta(y)$. For that, we use the expansion of the coherent wave function $\psi_\beta(y)$ in terms of the eigenfunctions of the harmonic oscillator; i.e.,

$$\psi_\beta(y) = e^{-\frac{|\beta|^2}{2}}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}\varphi_m(y), \tag{87}$$

in such a way that

$$\begin{aligned}\hat{b}^{\dagger j}\psi_\beta(y) =&\hat{b}^{\dagger j}e^{-\frac{|\beta|^2}{2}}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}\varphi_m(y)\\ =&e^{-\frac{|\beta|^2}{2}}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}\hat{b}^{\dagger j}\varphi_m(y);\end{aligned} \tag{88}$$

using the well know fact that

$$\hat{b}^{\dagger j}\varphi_m(y) = \sqrt{\frac{(j+m)!}{m!}}\varphi_{j+m}(y), \tag{89}$$

we get

$$\hat{b}^{\dagger j}\psi_\beta(y) = e^{-\frac{|\beta|^2}{2}}\sum_{m=0}^{\infty}\frac{\beta^m}{m!}\sqrt{(j+m)!}\varphi_{j+m}(y), \tag{90}$$

which substituted in (86) gives us (83), as we were looking.

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