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Numerical analysis of approximation error for a nonlinear parabolic problem with terms concentrating in an oscillatory neighborhood of the boundary

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Abstract: Numerically, we verify the theoretical results about the behavior of the solutions of a nonlinear parabolic problem with homogeneous Neumann boundary conditions, when a nonlinear reaction term is concentrated in a neighborhood of the boundary of a domain in $\mathbb{R}^2$, using the finite element method. We assume that this neighborhood shrinks to the boundary as a parameter ε goes to zero. Also, we suppose that the "inner boundary" of this neighborhood presents an oscillatory behavior. We evaluate the error made when the numerical solution of a parabolic problem with nonlinear Neumann boundary conditions is approximated by the family of numerical solutions of the concentrated problem, as ε goes to zero. Numerical results associated with the dynamics of these concentrated problems will be presented as a great novelty.

Keywords: Nonlinear parabolic problem; Concentrated reaction; Oscillatory behavior; Numerical analysis; Error.

1 Introduction

Dynamic systems in infinite dimensional spaces are mathematical models for a large number of problems in applied areas such as physics, biology, chemistry, economics and engineering, among many others. There are several works to study the asymptotic behavior of dynamic systems generated by parabolic partial differential equations in spaces of infinite dimension, see for example [1, 2, 3, 4, 5, 6, 7].

On the other hand, the technique of terms concentrating on the boundary of a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is more recent. The first paper [8] analyzed the limit of the solutions of an elliptic problem when some reaction and potential terms are concentrated in a neighborhood of the boundary $\partial\Omega$ of class $\mathscr{C}^2$ and this neighborhood shrinks to $\partial\Omega$ as a parameter ε goes to zero. Later, the asymptotic behavior of a parabolic problem of the same type was analyzed in [9, 10], where the upper semicontinuity of attractors was proved. In [11] some results of [8] were adapted to a nonlinear elliptic problem posed on an open square Ω in $\mathbb{R}^2$ with terms concentrating in a neighborhood of the boundary and this neighborhood presents a highly oscillatory behavior. Moreover, in [12] was studied a nonlinear parabolic problem still considering concentrated terms and oscillatory behavior, this work will be described in more detail below. We can also cite other more recent works such as [13, 14], where the continuity of the pullback attractors of non-autonomous damped wave equations with terms concentrating on the boundary was studied.

In this work, we numerically verify the theoretical results about the behavior of the solutions of a nonlinear parabolic problem with terms concentrating in an oscillatory neighborhood of the boundary of a domain in $\mathbb{R}^2$, using the finite element method. Note that the numerical behavior was not approached in the works cited above and that is indeed our main contribution in this paper.

To describe the results obtained in [12], let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a $\mathscr{C}^2$ boundary $\partial\Omega$ and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, $T > 0$, be a positive smooth function such that $y \mapsto g(x,y)$ has period $l(x)$ in y for each x, with period $l(x)$ uniformly bounded in $[0,T]$. Given $\varepsilon > 0$ and

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$\nu > 0$, let $g_\varepsilon(\cdot)$ be a positive bounded function which may oscillate as the parameter ε goes to zero. This function has the form

$$g_\varepsilon(s) = g\left(s, \frac{s}{\varepsilon^\nu}\right), \quad s \in [0,T].$$

Let $x, y \in \mathscr{C}^2([0,T])$ such that the curve $\psi(s) = (x(s), y(s))$, $s \in [0,T]$, is a parametrization of class $\mathscr{C}^2$ for the boundary $\partial\Omega$ with $\|\psi'(s)\|_{\mathbb{R}^2} = 1$, for all $s \in [0,T]$. We define the strip with width ε and base $\partial\Omega$ as

$$\omega_\varepsilon = \left\{\xi \in \mathbb{R}^2 : \xi = \psi(s) - t\mathbf{N}(\psi(s)), \quad s \in [0,T] \quad \text{and} \right.$$
$$\left. t \in [0, \varepsilon g_\varepsilon(s))\right\},$$

for sufficiently small ε, say $0 < \varepsilon \leq \varepsilon_0$, where $\mathbf{N}(\psi(s))$ is the unit outward normal vector to $\partial\Omega$. For small ε, ω_ε is a neighborhood of the boundary $\partial\Omega$ on the $\overline{\Omega}$, which shrinks to $\partial\Omega$ as the ε parameter goes to zero. Note that the "inner boundary" of ω_ε,

$$\left\{\xi \in \mathbb{R}^2 : \xi = \psi(s) - \varepsilon g_\varepsilon(s)\mathbf{N}(\psi(s)), \quad s \in [0,T]\right\},$$

presents a highly oscillatory behavior established by function g_ε. Moreover, the height of ω_ε and the amplitude of the oscillations have the same order ε, while the period of the oscillations presents order ε^ν for any $\nu > 0$.

In particular, in [12] was studied the behavior, for small ε, of the solutions of the nonlinear parabolic problem with homogeneous Neumann boundary conditions given by

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + u_\varepsilon = \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} f(u_\varepsilon), & \text{in } (0,\infty) \times \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{N}} = 0, & \text{on } (0,\infty) \times \partial\Omega \\ u_\varepsilon(0) = \varphi_0 \in H^1(\Omega), & \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^2$ function, $\chi_{\omega_\varepsilon}$ is the characteristic function of the set ω_ε and we refer to the term $\dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} f(u_\varepsilon)$ as the nonlinear reaction concentrating on the region ω_ε.

The authors showed that the limit problem of the concentrated problem (1) is given by the following parabolic problem with nonlinear Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - \Delta u_0 + u_0 = 0, & \text{in } (0,\infty) \times \Omega \\ \dfrac{\partial u_0}{\partial \mathbf{N}} = \mu f(u_0), & \text{on } (0,\infty) \times \partial\Omega \\ u_0(0) = \varphi_0 \in H^1(\Omega), & \end{cases} \tag{2}$$

where the boundary coefficient $\mu \in L^\infty(\partial\Omega)$ is related to the periodic function g as follows

$$\mu(s) = \mu(\psi(s)) = \frac{1}{l(s)} \int_0^{l(s)} g(s,\tau)\, d\tau, \quad \forall s \in (0,T). \tag{3}$$

Under the usual assumptions of regularity and dissipativeness, the authors proved the existence and upper semicontinuity of the family of attractors in $H^1(\Omega)$ with respect to ε. And, assuming hiperbolicity of the equilibria of the limit problem (2), they also proved the lower semicontinuity of the attractors.

In this paper, some numerical simulations will be presented to verify some theoretical results about the behavior of the solutions of (1) and (2) as ε goes to zero, in a particular case. For this, we consider $\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$, $f(u) = u(1-u)$ for $u \in \mathbb{R}$, $\varphi_0(x,y) = \sin(x^2+y^2)$ for $(x,y) \in \Omega$, and $g_\varepsilon(s) = 2 + \cos\left(\dfrac{s}{\varepsilon}\right)$ for $s \in [0,2\pi]$ and $0 < \varepsilon \leq \varepsilon_0$, that is, the oscillatory function g_ε presents a purely periodic behavior with $1 \leq g_\varepsilon(s) \leq 3$, for all $s \in [0,2\pi]$. In this case, by (3) we get the constant boundary coefficient $\mu = 2$.

We take the following parametrization

$$\omega_\varepsilon = \left\{\xi \in \mathbb{R}^2 : \xi = ((1-t)\cos(s), (1-t)\sin(s)),\right.$$
$$\left. s \in [0,2\pi] \quad \text{and} \quad t \in [0, \varepsilon g_\varepsilon(s))\right\},$$

with $0 < \varepsilon \leq \varepsilon_0$, for sufficiently small ε_0.

We are interested in analyzing, as ε goes to zero, how the family of numerical solutions of the following concentrated nonlinear parabolic problem with homogeneous Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + u_\varepsilon = \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} u_\varepsilon(1-u_\varepsilon), & \text{in } (0,\infty) \times \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{N}} = 0, & \text{on } (0,\infty) \times \partial\Omega \\ u_\varepsilon(0) = \sin(x^2+y^2), & \end{cases} \tag{4}$$

converges to the solution of the following parabolic problem with nonlinear Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - \Delta u_0 + u_0 = 0, & \text{in } (0,\infty) \times \Omega \\ \dfrac{\partial u_0}{\partial \mathbf{N}} = 2u_0(1-u_0), & \text{on } (0,\infty) \times \partial\Omega \\ u_0(0) = \sin(x^2+y^2). & \end{cases} \tag{5}$$

We will refer to $\varepsilon = 0$ for the limit problem (5). We will build differents "inner boundaries" on ω_ε. The Figure1 illustrates an example of this construction. Then, for each of these "inner boundaries", we will use a finite element scheme to discretize Ω together with ω_ε. So, for sufficiently small ε, we will evaluate the error made when we approximate the unique solution of (5) by solutions of the problem (4).

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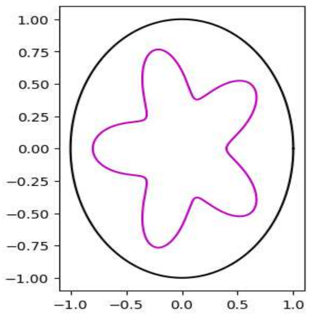

Fig. 1: The domain $\overline{\Omega}$, in black, and "inner boundary" of ω_ε, in magenta, for $\varepsilon = 0.2$.

This paper is organized as follows: in Section 2, we will obtain some technical results to ensure that the problems (4) and (5) have a unique global solution. In Section 3, we will show the numerical scheme to numerical verification of some results obtained in Section 2 and [12].

2 Existence and uniqueness of solutions

Before we analyze the numerical behavior, we will need to show that the problems (4) and (5) have a unique global solution on suitable spaces. For this, we will prove some technical results.

Initially, we denote by $H^{\alpha,p}(\Omega)$ the Bessel Potentials spaces, where $\alpha \leq 1$ and $1 < p < 1$. We note that $H^{0,p}(\Omega) = L^p(\Omega)$ and $H^{\alpha,2}(\Omega) = H^\alpha(\Omega)$. Note that the regularity of Ω and standard trace theory imply that for any function $v \in H^{\alpha,p}(\Omega)$, with $\alpha > \frac{1}{p}$, the trace of v, denoted by $\gamma(v)$, is well defined and it lies in $L^q(\partial\Omega)$, provided $\alpha - \frac{n}{p} \geq -\frac{(n-1)}{q}$, for $\Omega \subset \mathbb{R}^n$; in our case $n = 2$. Moreover, the trace operator $\gamma : H^{\alpha,p}(\Omega) \to L^q(\partial\Omega)$ is continuous linear. For more details on the standard trace theory in regular domains, see [15, 16].

We will get a result about concentrated integrals, which is an adaptation of [12, Lemma 2.1].

Lemma 1. *Let $v \in H^\alpha(\Omega)$ with $\frac{1}{2} < \alpha \leq 1$ e $\alpha - 1 \geq -\frac{1}{q}$. Then, for sufficiently small ε_0, there exists a constant $C > 0$ independent of ε and v such that for any $0 < \varepsilon \leq \varepsilon_0$, we have*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q d\xi \leq C\|v\|^q_{H^\alpha(\Omega)}.$$

Proof. We notice that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q d\xi \leq \frac{1}{\varepsilon}\int_{r_\varepsilon} |v|^q d\xi,$$

where r_ε is given by

$$r_\varepsilon = \{\xi \in \mathbb{R}^2 : \ \xi = ((1-3\varepsilon)\cos(s), (1-3\varepsilon)\sin(s)), \ s \in [0, 2\pi]\},$$

with $0 < \varepsilon \leq \varepsilon_0$, as illustrated in Figure 2.

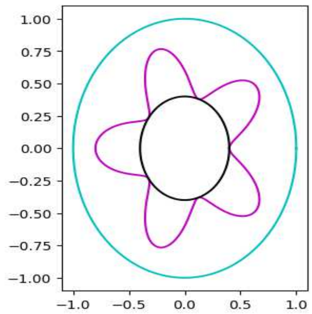

Fig. 2: The domain $\overline{\Omega}$, in cyan, the "inner boundaries" of ω_ε and r_ε, in magenta and black, respectively, for $\varepsilon = 0.2$.

Thus, the result follows from [8, Lemma 2.1]. □

The problems (4) and (5) can be written in abstract forms. To do this, we introduce the following continuous bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$a(u,v) = \int_\Omega (\nabla u \nabla v + uv)\, dxdy, \quad \forall u, v \in H^1(\Omega).$$

Moreover, the bilinear form a is uniformly coercive in $H^1(\Omega)$.

Thus, we can define the linear operator $A : H^1(\Omega) \subset H^{-1}(\Omega) \to H^1(\Omega)$ by

$$\langle Au, v\rangle_{-1,1} = a(u,v), \quad \forall v \in H^1(\Omega).$$

Remark. This operator A can also be considered as going from $H^{2-\alpha}(\Omega) \subset H^{-\alpha}(\Omega)$ into $H^{-\alpha}(\Omega)$, with $\frac{1}{2} < \alpha \leq 1$. With some abuse of notation we will identify all different realizations of this operator and we will write all of them as A. So, the operator $A : H^{2-\alpha}(\Omega) \subset H^{-\alpha}(\Omega) \to H^{-\alpha}(\Omega)$, $\frac{1}{2} < \alpha \leq 1$, is invertible, selfadjoint and positive, thus a sectorial operator with spectrum contained in the subset $(0, \infty) \subset \mathbb{R}$.

Now, we define the abstract maps associated to the nonlinearities in the problems (4) and (5). Initially, for each $0 < \varepsilon \leq \varepsilon_0$, we define $F_\varepsilon : H^1(\Omega) \to H^{-\alpha}(\Omega)$, with $\frac{1}{2} < \alpha \leq 1$, by

$$\langle F_\varepsilon(u), \phi\rangle = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} u(1-u)\phi\, d\xi, \tag{6}$$

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for all $u \in H^1(\Omega)$ and for all $\phi \in H^\alpha(\Omega)$.

We define $F_0 : H^1(\Omega) \to H^{-\alpha}(\Omega)$, with $\frac{1}{2} < \alpha \leq 1$, by

$$\langle F_0(u), \phi \rangle = \int_{\partial\Omega} 2\gamma(u)(1-\gamma(u))\gamma(\phi)\, dS, \qquad (7)$$

for all $u \in H^1(\Omega)$ and for all $\phi \in H^\alpha(\Omega)$, where dS is the surface measure associated to $\partial\Omega$ and γ is the trace operator.

Therefore, the problems (4) and (5) can be written in the following abstract form

$$\begin{cases} \dfrac{du_\varepsilon(t)}{dt} + Au_\varepsilon(t) = F_\varepsilon(u_\varepsilon(t)), \quad t>0 \quad \text{and} \quad 0 \leq \varepsilon \leq \varepsilon_0 \\ u_\varepsilon(0) = \varphi_0 \in H^1(\Omega), \end{cases} \qquad (8)$$

where $\varphi_0(x,y) = \sin(x^2+y^2)$, for $(x,y) \in \Omega$.

Next, we study the behavior of the maps F_ε defined in (6) and (7). In particular, we conclude that the abstract parabolic problem (8) is well posed in $H^1(\Omega)$.

Proposition 1. *Suppose $\frac{1}{2} < \alpha \leq 1$, then:*

(a) There exists $K>0$ independent of ε such that

$$\|F_\varepsilon(u)\|_{H^{-\alpha}(\Omega)} \leq K\|u\|_{H^1(\Omega)}\left(1+\|u\|_{H^1(\Omega)}\right),$$

for all $u \in H^1(\Omega)$ and $0 \leq \varepsilon \leq \varepsilon_0$.

In particular, the map $F_\varepsilon : H^1(\Omega) \to H^{-\alpha}(\Omega)$ is bounded, uniformly in ε, in bounded sets of $H^1(\Omega)$.

(b) There exists $L>0$ independent of ε such that

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{H^{-\alpha}(\Omega)} \leq L\|u-v\|_{H^1(\Omega)}\left(1+\|u\|_{H^1(\Omega)}+\|v\|_{H^1(\Omega)}\right),$$

for all $u, v \in H^1(\Omega)$.

In particular, for each $0 \leq \varepsilon \leq \varepsilon_0$, the map $F_\varepsilon : H^1(\Omega) \to H^{-\alpha}(\Omega)$ is locally Lipschitz, uniformly in ε.

(c) For each $u \in H^1(\Omega)$, we have

$$\|F_\varepsilon(u) - F_0(u)\|_{H^{-\alpha}(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Furthermore, this limit is uniform for u in bounded sets of $H^1(\Omega)$.

Proof. (a) Given $u \in H^1(\Omega)$ and $0 \leq \varepsilon \leq \varepsilon_0$, we have

$$\|F_\varepsilon(u)\|_{H^{-\alpha}(\Omega)} = \sup_{\substack{\phi \in H^\alpha(\Omega) \\ \|\phi\|_{H^\alpha(\Omega)}=1}} |\langle F_\varepsilon(u), \phi\rangle|.$$

For each $0 < \varepsilon \leq \varepsilon_0$, $\phi \in H^\alpha(\Omega)$ and $u \in H^1(\Omega)$, using (6) we have

$$\begin{aligned} |\langle F_\varepsilon(u), \phi\rangle| &\leq \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u||1-u||\phi|\, d\xi \\ &\leq \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|\,|\phi|\, d\xi + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|^2\,|\phi|\, d\xi = I_1 + I_2. \end{aligned}$$

In the case of I_1, using the Cauchy-Schwartz inequality and Lemma 1, we obtain

$$\begin{aligned} I_1 &\leq \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|^2\, d\xi\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^2\, d\xi\right)^{\frac{1}{2}} \\ &\leq C_1\|u\|_{H^1(\Omega)}\|\phi\|_{H^\alpha(\Omega)}, \end{aligned}$$

where $C_1 > 0$ is independent of ε. Similarly for I_2, we obtain

$$\begin{aligned} I_2 &\leq \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u|^4\, d\xi\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^2\, d\xi\right)^{\frac{1}{2}} \\ &\leq C_2\|u\|^2_{H^1(\Omega)}\|\phi\|_{H^\alpha(\Omega)}, \end{aligned}$$

where $C_2 > 0$ is independent of ε.

Consequently, there exists $K_1 > 0$ independent of ε such that

$$|\langle F_\varepsilon(u), \phi\rangle| \leq K_1\|u\|_{H^1(\Omega)}\left(1+\|u\|_{H^1(\Omega)}\right)\|\phi\|_{H^\alpha(\Omega)}, \qquad (9)$$

for $0 < \varepsilon \leq \varepsilon_0$.

In the case $\varepsilon = 0$, for each $\phi \in H^\alpha(\Omega)$ and $u \in H^1(\Omega)$, using (7) we have

$$\begin{aligned} |\langle F_0(u), \phi\rangle| &\leq 2\int_{\partial\Omega} |\gamma(u)||1-\gamma(u)||\gamma(\phi)|\, dS \\ &\leq 2(I_3 + I_4), \end{aligned}$$

with

$$I_3 = \int_{\partial\Omega} |\gamma(u)|\,|\gamma(\phi)|\, dS, \quad I_4 = \int_{\partial\Omega} |\gamma(u)|^2\,|\gamma(\phi)|\, dS.$$

In the case of I_3, using the Cauchy-Schwartz inequality and trace theorem, that is, the continuity of the trace operator $\gamma : H^\alpha(\Omega) \to L^2(\partial\Omega)$, with $\frac{1}{2} < \alpha \leq 1$, we have there exists $C_3 > 0$ such that

$$\begin{aligned} I_3 &\leq \left(\int_{\partial\Omega} |\gamma(u)|^2\, dS\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |\gamma(\phi)|^2\, dS\right)^{\frac{1}{2}} \\ &\leq C_3\|u\|_{H^1(\Omega)}\|\phi\|_{H^\alpha(\Omega)}. \end{aligned}$$

While for I_4, using also the continuity of the trace operator $\gamma : H^1(\Omega) \to L^4(\partial\Omega)$, we have there exists $C_4 > 0$ such that

$$\begin{aligned} I_4 &\leq \left(\int_{\partial\Omega} |\gamma(u)|^4\, dS\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |\gamma(\phi)|^2\, dS\right)^{\frac{1}{2}} \\ &\leq C_4\|u\|^2_{H^1(\Omega)}\|\phi\|_{H^\alpha(\Omega)}. \end{aligned}$$

Therefore, there exists $K_2 > 0$ such that

$$|\langle F_0(u), \phi\rangle| \leq K_2\|u\|_{H^1(\Omega)}\left(1+\|u\|_{H^1(\Omega)}\right)\|\phi\|_{H^\alpha(\Omega)}. \qquad (10)$$

Now, taking the supreme, in (9) and (10), over all function $\phi \in H^\alpha(\Omega)$ such that $\|\phi\|_{H^\alpha(\Omega)} = 1$, we obtain there exists $K > 0$ independent of ε such that

$$\|F_\varepsilon(u)\|_{H^{-\alpha}(\Omega)} \leq K\|u\|_{H^1(\Omega)}\left(1+\|u\|_{H^1(\Omega)}\right),$$

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for all $u \in H^1(\Omega)$ and $0 \le \varepsilon \le \varepsilon_0$.

(b) Given $u, v \in H^1(\Omega)$ and $0 \le \varepsilon \le \varepsilon_0$, we have

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{H^{-\alpha}(\Omega)} = \sup_{\substack{\phi \in H^\alpha(\Omega) \\ \|\phi\|_{H^\alpha(\Omega)}=1}} |\langle F_\varepsilon(u) - F_\varepsilon(v), \phi\rangle| .$$

For each $0 < \varepsilon \le \varepsilon_0$, $\phi \in H^\alpha(\Omega)$ and $u, v \in H^1(\Omega)$, using (6) we have

$$\begin{aligned} &|\langle F_\varepsilon(u) - F_\varepsilon(v), \phi\rangle| \\ &\le \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u-v||\phi|\, d\xi + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u^2 - v^2||\phi|\, d\xi = I_5 + I_6. \end{aligned}$$

Using the Cauchy-Schwartz inequality and Lemma 1, there exists $C_1 > 0$ independent of ε such that

$$\begin{aligned} I_5 &\le \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u-v|^2\, d\xi\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^2\, d\xi\right)^{\frac{1}{2}} \\ &\le C_1 \|u-v\|_{H^1(\Omega)} \|\phi\|_{H^\alpha(\Omega)} . \end{aligned}$$

For I_6, using the Cauchy-Schwartz inequality and Lemma 1, there exists $C_2 > 0$ independent of ε such that

$$\begin{aligned} I_6 &\le \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u-v|^2 |u+v|^2\, d\xi\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^2\, d\xi\right)^{\frac{1}{2}} \\ &\le C_2 \|u-v\|_{H^1(\Omega)} \|u+v\|_{H^1(\Omega)} \|\phi\|_{H^\alpha(\Omega)} \\ &\le C_2 \|u-v\|_{H^1(\Omega)} \left(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\right) \|\phi\|_{H^\alpha(\Omega)} . \end{aligned}$$

Therefore, there exists $L_1 > 0$ independent of ε such that

$$|\langle F_\varepsilon(u) - F_\varepsilon(v), \phi\rangle| \le L_1 \|u-v\|_{H^1(\Omega)} \left(1 + \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\right) \|\phi\|_{H^\alpha(\Omega)} , \tag{11}$$

for $0 < \varepsilon \le \varepsilon_0$.

In the case $\varepsilon = 0$, for each $\phi \in H^\alpha(\Omega)$ and $u, v \in H^1(\Omega)$, using (7) we have

$$\begin{aligned} |\langle F_0(u) - F_0(v), \phi\rangle| &\le 2\int_{\partial\Omega} |\gamma(u) - \gamma(v)||\gamma(\phi)|\, dS \\ &\quad + 2\int_{\partial\Omega} |(\gamma(u))^2 - (\gamma(v))^2||\gamma(\phi)|\, dS \\ &= 2(I_7 + I_8) . \end{aligned}$$

Using the Cauchy-Schwartz inequality, trace theorem and similar arguments to those used to estimate the integrals I_5 and I_6, we can get $C_3, C_4 > 0$ such that

$$I_7 \le C_3 \|u-v\|_{H^1(\Omega)} \|\phi\|_{H^\alpha(\Omega)}$$

and

$$I_8 \le C_4 \|u-v\|_{H^1(\Omega)} \left(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\right) \|\phi\|_{H^\alpha(\Omega)} .$$

Therefore, there exists $L_2 > 0$ such that

$$|\langle F_0(u) - F_0(v), \phi\rangle| \le L_2 \|u-v\|_{H^1(\Omega)} \left(1 + \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\right) \|\phi\|_{H^\alpha(\Omega)} . \tag{12}$$

Thus, taking the supreme, in (11) and (12), over all function $\phi \in H^\alpha(\Omega)$ such that $\|\phi\|_{H^\alpha(\Omega)} = 1$, we obtain there exists $L > 0$ independent of ε such that

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{H^{-\alpha}(\Omega)} \le L\|u-v\|_{H^1(\Omega)} \left(1 + \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\right),$$

for all $u, v \in H^1(\Omega)$ and $0 \le \varepsilon \le \varepsilon_0$.

(c) Initially, we take $\frac{1}{2} < \alpha_0 \le 1$.

Note that, for each $u \in H^1(\Omega)$ and $\phi \in H^{\alpha_0}(\Omega)$, we have

$$|\langle F_\varepsilon(u) - F_0(u), \phi\rangle| = \left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} u(1-u)\phi\, d\xi - \int_{\partial\Omega} 2\gamma(u)(1-\gamma(u))\gamma(\phi)\, dS\right| .$$

From [12, Lemma 2.2], for each $\phi \in H^{\alpha_0}(\Omega)$, we obtain

$$\langle F_\varepsilon(u), \phi\rangle \to \langle F_0(u), \phi\rangle, \quad \text{as } \varepsilon \to 0. \tag{13}$$

Moreover, fixing $u \in H^1(\Omega)$ and using item (a), we have that the set $\{F_\varepsilon(u) \in H^{-\alpha_0}(\Omega) : 0 < \varepsilon \le \varepsilon_0\}$ is equicontinuous. Thus, the limit (13) is uniform for ϕ in compact sets of $H^{\alpha_0}(\Omega)$. Hence, choosing α_0 such that $\frac{1}{2} < \alpha_0 < \alpha \le 1$, we have that the embedding $H^\alpha(\Omega) \hookrightarrow H^{\alpha_0}(\Omega)$ is compact, and then, in particular,

$$\|F_\varepsilon(u) - F_0(u)\|_{H^{-\alpha}(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0. \tag{14}$$

Now, we will show that the limit (14) is uniform for $u \in H^1(\Omega)$, with $\|u\|_{H^1(\Omega)} \le R$ for some $R > 0$. Let $u_n \rightharpoonup u_0$ in $H^1(\Omega)$, as $n \to \infty$. Since $H^1(\Omega) \hookrightarrow H^s(\Omega)$ with compact embedding, for $s < 1$, we have $u_n \to u_0$ in $H^s(\Omega)$, as $n \to \infty$.

Proceeding similarly to item (b) with $\frac{1}{2} < s < 1$, we get

$$\|F_\varepsilon(u_n) - F_\varepsilon(u_0)\|_{H^{-\alpha}(\Omega)} \to 0, \quad \text{as } n \to \infty.$$

Therefore, for each $0 < \varepsilon \le \varepsilon_0$, $F_\varepsilon : H^1(\Omega) \to H^{-\alpha}(\Omega)$ is continuous in $H^1(\Omega)$ with the weak topology. Hence, F_ε is uniformly continuous in compact sets of $H^1(\Omega)$ with the weak topology. We note that the set $\bar{B}_R(0) = \left\{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \le R\right\}$, with $R > 0$, is compact in $H^1(\Omega)$ with the weak topology. From this and (14), we obtain that the limit (14) is uniform in $\bar{B}_R(0)$. □

*Remark.*The Remark 2 and item (b) of the Proposition 1 imply the local existence and uniqueness of solutions of the problem (8) (or equivalently of (4) and (5)), see [5, Theorem 3.3.3]. Using also item (a) of the Proposition 1, we obtain that the local solution of (8) is globally defined and we have well defined semigroup in $H^1(\Omega)$, see [5, Corolary 3.3.5]. Moreover, item (c) of the Proposition 1 and [12, Proposition 4.1] ensure that the limit problem of the concentrated problem (4) is given by the parabolic

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problem with nonlinear Neumann boundary conditions (5). Finally, as done in [12], we can show the continuity of the family of attractors of (8) with respect to ε, for this, we need to study the behavior of Fréchet derivatives of the abstract nonlinearities F_ε, defined by (6) and (7), and we analyze the linearized problems.

3 Numerical Scheme

In Section 2, we establish an abstract form for the problems (4) and (5) given by (8). Using this same abstract form, we want to establish a Finite Element Scheme to obtain a numerical aproach to the solutions of the problems (4) and (5). The development of such scheme applied to parabolic problems can be influenced by work [17], where different Galerkin methods are applied to a certain parabolic problem with nonlinear Neumann boundary conditions.

Furthermore, more recent works such as [18, 19] analyze the convergence of certain algorithms based on the Finite Difference Method for a certain class of problems, which the Laplacian is perturbed by a value ε. These types of problems, however, are not related to the problems (4) and (5) since the nature of ε in the problems (4) and (5) is very different from that studied in [18, 19].

That said, our work does not try to explore the characteristics of the proposed algorithm but rather to numerically verify some theoretical results obtained in Section 2. In addition, as far as we know, there is no previous bibliography related to numerical behavior of the problems (4) and (5).

First, we need to approximate the time derivative. We consider an equally spaced partition for the time interval $[0,1]$ with $n+1$-elements and denote by t_k a representative element of this partition given by $t_k = t_0 + kh$, with $k \in \{0,\ldots,n\}$, where $t_0 = 0$, $t_n = 1$ and $h = \dfrac{t_n - t_0}{n}$. We denote for u_ε^k, $0 \le \varepsilon \le \varepsilon_0$, the value of the numerical solution of (8) at the time $t = t_k$. Then, using a Backward Difference Form, we can write the equation (8) as

$$\frac{u_\varepsilon^{k+1} - u_\varepsilon^k}{h} + Au_\varepsilon^{k+1} = F_\varepsilon(u_\varepsilon^{k+1}), \quad 0 \le \varepsilon \le \varepsilon_0, \qquad (15)$$

where $u_\varepsilon^0 = u_\varepsilon(0) = \varphi_0$.

Next, we need to discretize the spatial domain $\overline{\Omega}$. To do this, we are going to create a m-elements circular mesh for $\overline{\Omega}$ using the functions provided by the FEniCS Python package. Then, we are going to approximate the test function $v \in H^1(\Omega)$ over this mesh by a function space consisting in Lagrange polynomials of order 2. Such approximation will be denoted by $\widetilde{v}$. For more information about the FEniCS documentation, see [20].

With these steps, we can write the variational discrete form of the problem (15) with $0 < \varepsilon \le \varepsilon_0$, or the Finite Element Method Form for the problem (4), by

$$\frac{u_\varepsilon^{k+1} - u_\varepsilon^k}{h} + a\left(u_\varepsilon^{k+1}, \widetilde{v}\right) = \left\langle F_\varepsilon(u_\varepsilon^{k+1}), \widetilde{v}\right\rangle, \quad 0 < \varepsilon \le \varepsilon_0, \quad (16)$$

and the variational discrete form of the problem (15) with $\varepsilon = 0$, or the Finite Element Method Form for the problem (5), by

$$\frac{u_0^{k+1} - u_0^k}{h} + a\left(u_0^{k+1}, \widetilde{v}\right) = \left\langle F_0(u_0^{k+1}), \widetilde{v}\right\rangle, \quad \varepsilon = 0. \quad (17)$$

Taking $n = 3$, $m = 45$ and $h = \dfrac{1}{3}$. For each point $(t_k, x_j, y_j) \in [0,1] \times \overline{\Omega}$, with $k \in \{0,1,2,3\}$ and $j \in \{0,1,\ldots,44\}$, and $0 \le \varepsilon \le \varepsilon_0$, let us denote by $u_\varepsilon(t_k, x_j, y_j) = u_{j_\varepsilon}^k$ the numerical solution of the problems (16) and (17) at the time $t = t_k$. Denoting by κ the "inner boundary" of ω_ε with a tolerance of 10^{-40}, we use a finite element scheme and proceed to analyze the behavior of the zeros of the equation (16) and how they behave with respect to the zeros of the equation (17), as ε goes to zero. For more information about the discretization of time dependent PDEs, see [21].

In the case of the equation (17), we execute a Newton solution scheme for a diffusion equation and nonhomogeneous Newmann type boundary conditions, with $u_{j0}^0 = 0.3$ as seed value and absolute tolerance of 10^{-1}, thus we obtain the following heat map given in Figure 3.

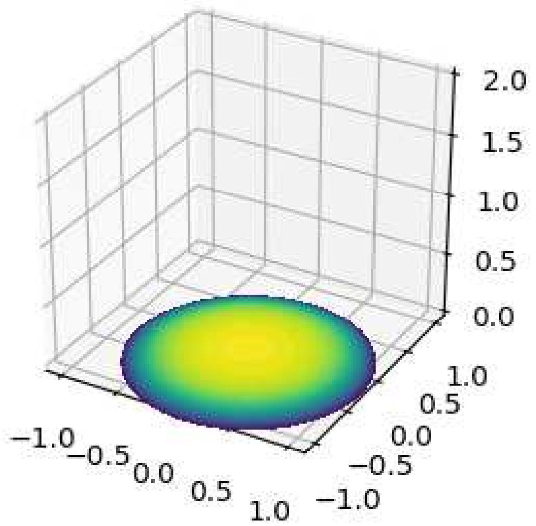

Fig. 3: Heat map for the limit equation (17).

Choosing, arbitrarily, the following values for $\varepsilon \in \{0.1, 0.01, 0.001, 0.0001\} \subset (0, \varepsilon_0]$, we execute a Newton solution scheme for the equation (16), with $u_{j0}^0 = 0.3$ and an aboslute tolerance of 10^{-1}, thus we obtain the following results given in Table 1.

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Table 1: Evolution of the error for the equation (16), as ε goes to zero.

ε	$\left\|u^k_{j\varepsilon} - u^k_{j0}\right\|_{H^1(\Omega)}$
0.1	1.50
0.01	1.74
0.001	0.37
0.0001	0.37

As we can see in Table 1, when ε is sufficiently small, the error made in the approximation remains constant around 0.37. Taking a tolerance of 10^{-1} for the error, our stopping criterion is satisfied. Thus, we can observe that, as ε goes to zero, the solutions of the problem (16) converge in $H^1(\Omega)$ to the solution of the problem (17). We can see this graphically by considering the differents heat maps generated for the problem (16), as ε goes to zero. For example, the Figure 4 illustrates the heat maps for ε equals to 0.1, 0.01, 0.001 and 0.0001, respectively.

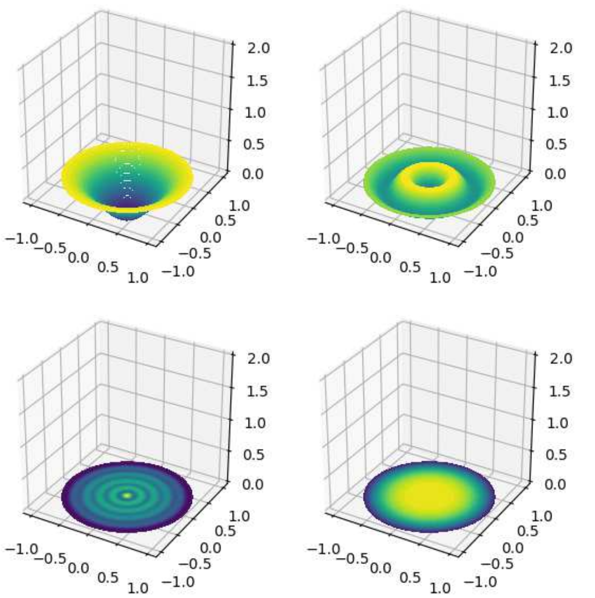

Fig. 4: Evolution of the heat map for the equation (16), as ε goes to zero.

Note that the use of the numerical scheme (15) constitutes a numerical verification of some results obtained in Section 2 and work [12], thanks to the work [8], where the concentration technique was developed. In addition, the numerical behavior of the solutions of nonlinear parabolic problems with terms concentrating on the boundary already opens a brand new way of analyzing and verifying the dynamics of such problems since, up to now, no work had been done, to the best of our knowledge, using this approach.

With this in mind and noting that the problem (4) represents a problem based on a Logistic Model, this approach opens the possibility of studying population dynamics and how the problem (5) is based on the Helmoltz equation, then also this approach, together with the technique of concentrated integrals, would allow the use of these results to study various electromagnetic phenomena modeled through such an equation.

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