Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180208

Bayes Estimations for Parameters of the Inverted Kumarswamy Distribution with Progressive Censoring Scheme

M. Yusuf

Department of Mathematics, Faculty of Science, Helwan University, Ain Helwan, Cairo, Egypt

Received: 27 Aug. 2023, Revised: 17 Dec. 2023, Accepted: 25 Dec. 2023
Published online: 1 Mar. 2024

Abstract: In this paper we develop approximate Bayes estimators of the unknown parameters of the inverted Kumarswamy distribution based on progressive type-II censoring samples. We consider the maximum likelihood and Bayesian estimations of the model with gamma-informative prior distribution for the parameters, as well as the reliability function and reversed hazard rate function. We applied, Lindley's approximation (1980) and Markov Chain Monte Carlo (MCMC) methods. The Bayes estimators have been obtained relative to both symmetric and asymmetric (linex and general entropy) loss functions. Finally, to assess the performance of the proposed estimators, some numerical results with simulation study were reported.

Keywords: Inverted kumaraswamy distribution, progressive censoring, loss functions, Lindley's approximation, MCMC, gamma distribution.
2020 AMS Classification: 62F15, 62N01, 62N02, 62N05.

1 Introduction

During life tests and reliability studies, we consistently observe the prevalence of censoring as a recurring phenomenon. The experimenter might not be able to get total data on disappointment times for every single test unit. For instance, people in a clinical preliminary may pull out from the examination, or the investigation be ended for absence of assets. In a mechanical investigation, units may break unintentionally. Much of the time, be that as it may, the evacuation of units before disappointment is preplanned in request to give investment funds regarding time and cost related with testing. Progressive type-II censoring scheme can be described as follows: Suppose n units are placed on a life test and the experimenter decides before hand the quantity m, the number of failures to be observed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n-R_1-2$ units are randomly removed from the experiment. Finally, at the time of the $m-^{th}$ failure, all the remaining surviving units $R_m = n-m-R_1-\ldots-R_{m-1}$ are removed from the experiment. Therefore, a progressive type-II censoring scheme consists of m, and $R_1,\ldots,R_m$, such that $R_1+\ldots+R_m = n-m$. The m failure times obtained from a progressive Type-II censoring scheme will be denoted by $y_1,\ldots,y_m$. Based on the observed sample $y_1 < y_2 < \ldots < y_m$; from a progressive type-II censoring scheme, $R_1,\ldots,R_m$, the likelihood function can be written as

$$L\left(\underset{\sim}{y}\right) = C\prod_{i=1}^{m} f(y_i)\,[1-F(y_i)]^{R_i}, \tag{1}$$

where;
$C = n(n-R_1-1)\ldots(n-R_1-R_2-\ldots-R_{m-1}-m+1)$.

Al-Fattah et al. [2] obtained the inverted Kumaraswamy distribution which have widely employed in various natural phenomena for instance hydrological data such as daily rainfall and daily stream-flow, atmospheric temperatures and growth models like epidemiology, and in the field of life testing and studies of reliability measures in which the failure time of component is observed to the nearest hours, days or months. Recently many authors have studied the inverted Kumaraswamy distribution; for example, Abu-Moussa

* Corresponding author e-mail: mohammed.yousof@yahoo.com

and El-Din [1] presented on estimation and prediction for the inverted Kumaraswamy distribution based on general progressive censored samples. For more details [See Kumaraswamy [6] , Jones [5], Golizadeh et al. [4], Sindhu et al. [9] and Sharaf EL-Deen et al. [8]]. The reverse hazard functions is represented the instantaneous risk of a state change at time t, given that the event did occur either at time t or a lesser value. In case of any invisible failures present, hazard rate loses its usefulness, since it cannot capture such failures. Thus, there is a need to use another characteristic that known as reversed hazard rate.

2 Maximum Likelihood Estimators (MLEs)

In this section, the MLEs of the unknown parameters, reliability function and reversed hazard rate function based on progressive type-II censored samples are obtained. Assume the failure time distribution to be the inverted kumaraswamy distribution with probability density function (pdf)

$$f(y;\ \eta,\ \kappa) = \eta\ \kappa\ (1+y)^{-(\eta+1)}\left[1-(1+y)^{-\eta}\right]^{\kappa-1}, \quad y>0,\ \eta,\ \kappa>0 \tag{2}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(y;\ \eta,\ \kappa) = \left[1-(1+y)^{-\eta}\right]^{\kappa} \quad ,y>0,\ \eta,\ \kappa>0\ . \tag{3}$$

From (1), (2) and (3), the likelihood function is given by

$$L\left(\underline{y};\ \eta,\ \kappa\right) = C\,\eta^{m}\,\kappa^{m}\prod_{i=1}^{m}(1+y_i)^{-(\eta+1)} \times\prod_{i=1}^{m}\left(1-(1+y_i)^{-\eta}\right)^{\kappa-1}\left[1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}\right]^{R_i}. \tag{4}$$

The logarithm of the likelihood function may then be written as

$$\left.\begin{aligned}\log(L) = \ell = \log[C] + m\log[\eta] + m\log[\kappa] \\ -(\eta+1)\sum_{i=1}^{m}(\log[1+y_i]) \\ +(\kappa-1)\sum_{i=1}^{m}\left(\log\left[1-(1+y_i)^{-\eta}\right]\right) \\ +\sum_{i=1}^{m}\left(R_i\log\left[1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}\right]\right).\end{aligned}\right\} \tag{5}$$

Calculating the first partial derivatives of (5) with respect to η, κ and equating to zero, we obtain the likelihood equation

$$\left.\begin{aligned}\frac{m}{\eta} - \sum_{i=1}^{m}\log[1+y_i] + (-1+\kappa)\sum_{i=1}^{m}\frac{\log[1+y_i](1+y_i)^{-\eta}}{1-(1+y_i)^{-\eta}} \\ -\sum_{i=1}^{m}\frac{\kappa\log[1+y_i]R_i(1+y_i)^{-\eta}\left(1-(1+y_i)^{-\eta}\right)^{-1+\kappa}}{1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}} = 0, \\ \frac{m}{\kappa} + \sum_{i=1}^{m}\log\left[1-(1+y_i)^{-\eta}\right] \\ -\sum_{i=1}^{m}\frac{\log\left[1-(1+y_i)^{-\eta}\right]R_i\left(1-(1+y_i)^{-\eta}\right)^{\kappa}}{1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}} = 0.\end{aligned}\right\} \tag{6}$$

The solution of the non-linear equation (6) is $\hat{\eta}$, $\hat{\kappa}$. The MLEs of the reliability function and reversed hazard rate function are given as

$$\hat{R}(t) = 1-\left[1-(1+t)^{-\hat{\eta}}\right]^{\hat{\kappa}},\ t>0. \tag{7}$$

$$\hat{H}(t) = \hat{\eta}\ \hat{\kappa}(1+t)^{-(\hat{\eta}+1)}\left[1-(1+t)^{-\hat{\eta}}\right]^{-1},\ t>0. \tag{8}$$

3 Bayesian estimation

In this section, the Bayesian estimators of the unknown parameters η, κ of the inverted Kumaraswamy distribution is obtained. Also the reliability function and reversed hazard rate function, based on progressive type-II censoring samples, under symmetric (squared error) and asymmetric (linex and general entropy) loss functions. Lindley's approximation and Markov Chain Monte Carlo (MCMC) methods are obtained.

Assuming that η and κ are independent random variables with gamma informative prior distribution respectively defined by

$$\pi_1(\eta;\ \zeta_1,\ \nu_1) = \frac{e^{-\zeta_1\eta}\zeta_1^{\nu_1}}{\Gamma(\nu_1)}\ \eta^{\nu_1-1},\quad \eta>0,\ (\zeta_1,\ \nu_1>0). \tag{9}$$

$$\pi_2(\kappa;\ \zeta_2,\ \nu_2) = \frac{e^{-\zeta_2\kappa}\zeta_2^{\nu_2}}{\Gamma(\nu_2)}\ \kappa^{\nu_2-1},\quad \kappa>0,\ (\zeta_2,\ \nu_2>0). \tag{10}$$

Then the joint prior distribution for η and κ is defined by

$$\pi(\eta,\ \kappa) = \frac{e^{-\zeta_1\eta-\zeta_2\kappa}\zeta_1^{\nu_1}\zeta_2^{\nu_2}}{\Gamma(\nu_1)\Gamma(\nu_2)}\ \eta^{\nu_1-1}\ \kappa^{\nu_2-1}, \quad \eta>0,\ \kappa>0,\ (\zeta_1,\ \nu_1,\ \zeta_2,\ \nu_2>0). \tag{11}$$

By using equations (4) and (11) we get the posterior distribution of η and κ as follows

$$\pi\left(\eta,\ \kappa|\underline{y}\right)=\frac{\mathrm{e}^{-\zeta_1\eta-\zeta_2\kappa}\eta^{\nu_1+m-1}\kappa^{\nu_2+m-1}\prod_{i=1}^{m}(1+y_i)^{-(\eta+1)}\prod_{i=1}^{m}\left(1-(1+y_i)^{-\eta}\right)^{\kappa-1}\left[1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}\right]^{R_i}}{\int_0^\infty\int_0^\infty\left(\mathrm{e}^{-\zeta_1\eta-\zeta_2\kappa}\eta^{\nu_1+m-1}\kappa^{\nu_2+m-1}\prod_{i=1}^{m}(1+y_i)^{-(\eta+1)}\prod_{i=1}^{m}\left(1-(1+y_i)^{-\eta}\right)^{\kappa-1}\left[1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}\right]^{R_i}\right)d\eta d\kappa}. \tag{12}$$

Integration in equation (12) cannot be obtained in a closed form, so we solve it numerically. In the following subsections we derive Bayesian estimators for the parameters $\eta,\ \kappa,$ the reliability function and the reversed hazard rate function under different loss functions.

3.1 Bayesian Estimators Under Square Error Loss Function

3.1.1 Bayesian estimator of the shape parameter η

$$\hat{\eta}_{sq}=\mathrm{E}(\eta)=\int_0^\infty\int_0^\infty\eta\left(\begin{array}{c}\left(\begin{array}{l}\mathrm{e}^{-\zeta_1\eta-\zeta_2\kappa}\eta^{\nu_1+m-1}\kappa^{\nu_2+m-1}\prod_{i=1}^{m}(1+y_i)^{-(\eta+1)}\\ \times\prod_{i=1}^{m}\left(1-(1+y_i)^{-\eta}\right)^{\kappa-1}\left[1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}\right]^{R_i}\end{array}\right)\\ \times\left(\int_0^\infty\int_0^\infty\left(\begin{array}{l}\mathrm{e}^{-\zeta_1\eta-\zeta_2\kappa}\eta^{\nu_1+m-1}\kappa^{\nu_2+m-1}\prod_{i=1}^{m}(1+y_i)^{-(\eta+1)}\\ \times\prod_{i=1}^{m}\left(1-(1+y_i)^{-\eta}\right)^{\kappa-1}\left[1-\left(1-(1+y_i)^{-\eta}\right)^{\kappa}\right]^{R_i}\end{array}\right)d\eta d\kappa\right)^{-1}\end{array}\right)d\eta d\kappa. \tag{13}$$

Provided that $E(\eta)$ exists and is finite. This integration cannot be solved analytically, so we use Lindley's Bayes approximation for any function ψ of parameter $\omega,\ \ \omega=(\eta,\ \kappa)$; is defined by

$$E(\psi(\omega)|\underline{y})\approx\psi(\eta,\kappa)+\frac{1}{2}\sum_r\sum_t(\psi_{rt}+2\psi_rQ_t)\sigma_{rt}+\frac{1}{2}\sum_r\sum_t\sum_k\sum_w L_{rtk}\psi_w\sigma_{rt}\sigma_{kw}\qquad\forall\ r,\ t,\ k,\ w=1,2. \tag{14}$$

Where; $Q(\eta,\ \kappa)=\log(\pi(\eta,\ \kappa))$, $Q_1=\frac{\partial Q(\eta,\ \kappa)}{\partial\eta}$, $Q_2=\frac{\partial Q(\eta,\ \kappa)}{\partial\kappa}$, $\psi_1=\frac{\partial\psi(\eta,\ \kappa)}{\partial\eta}$, $\psi_2=\frac{\partial\psi(\eta,\ \kappa)}{\partial\kappa}$, $\psi_{11}=\frac{\partial^2\psi(\eta,\ \kappa)}{\partial\eta^2}$, $\psi_{22}=\frac{\partial^2\psi(\eta,\ \kappa)}{\partial\kappa^2}$, $\psi_{12}=\frac{\partial^2\psi(\eta,\ \kappa)}{\partial\eta\partial\kappa}$, $L_{11}=\frac{\partial^2\ell}{\partial\eta^2}$, $L_{12}=\frac{\partial^2\ell}{\partial\eta\partial\kappa}$, $L_{22}=\frac{\partial^2\ell}{\partial\kappa^2}$, $L_{111}=\frac{\partial^3\ell}{\partial\eta^3}$, $L_{222}=\frac{\partial^3\ell}{\partial\kappa^3}$, $L_{112}=\frac{\partial^3\ell}{\partial\eta^2\partial\kappa}$, $L_{122}=\frac{\partial^3\ell}{\partial\eta\partial\kappa^2}$ and $\sigma_{rt}=(r,t)^{th}$ element ; $r,\ t=1,2$ in the matrix $\begin{pmatrix}-L_{11}&-L_{12}\\-L_{21}&-L_{22}\end{pmatrix}^{-1}$.

Substitution in equation (14), $\psi=\eta$; the Bayesian estimator of the shape parameter η is given as

$$\hat{\eta}_{sq}\approx\eta-Q_1\sigma_{11}-Q_2\sigma_{12}+\frac{1}{2}\left(L_{111}{\sigma_{11}}^2+3L_{112}\sigma_{11}\sigma_{12}+L_{122}\left(\sigma_{22}\sigma_{11}+2{\sigma_{12}}^2\right)+L_{222}\sigma_{12}\sigma_{22}\right).$$

3.1.2 Bayesian estimator of the shape parameter κ

Substitution in equation (14), $\psi=\kappa$; the Bayesian estimator of the parameter κ is given as

$$\hat{\kappa}_{sq}\approx\left(\kappa-Q_1\sigma_{12}-{\sigma_{22}}^2+\frac{1}{2}\left(L_{111}\sigma_{11}\sigma_{12}+3L_{122}\sigma_{12}\sigma_{22}+L_{112}\left(\sigma_{22}\sigma_{11}+2{\sigma_{12}}^2\right)\right)\right).$$

3.1.3 Bayesian estimator of the reliability function $R(t)$

Substitution in equation (14), $\psi = R(t)$; the Bayesian estimator of the reliability function $R(t)$ is given as

$$\hat{R}_{sq} \approx \left(\begin{array}{l} R(t) + Q_1(\psi_1\sigma_{11} + \psi_2\sigma_{12}) + Q_2(\psi_1\sigma_{12} + \psi_2\sigma_{22}) + \frac{1}{2}(\psi_{11}\sigma_{11} + 2\psi_{12}\sigma_{12} + \psi_{22}\sigma_{22}) \\ + \frac{1}{2}\left(\begin{array}{l} L_{111}({\sigma_{11}}^2\psi_1 + \psi_2\sigma_{11}\sigma_{12}) + L_{112}(3\sigma_{11}\psi_1\sigma_{12} + \psi_2(\sigma_{11}\sigma_{22} + 2{\sigma_{12}}^2)) \\ + L_{122}(\psi_1(2{\sigma_{12}}^2 + \sigma_{11}\sigma_{12}) + 3\psi_2\sigma_{22}\sigma_{12}) + L_{222}(\sigma_{22}\psi_1\sigma_{12} + \psi_2{\sigma_{22}}^2) \end{array} \right) \end{array} \right).$$

3.1.4 Bayesian estimator of the reversed hazard rate function $H(t)$

Substitution in equation (14), $\psi = H(t)$; the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{sq} \approx \left(\begin{array}{l} H(t) + Q_1(\psi_1\sigma_{11} + \psi_2\sigma_{12}) + Q_2(\psi_1\sigma_{12} + \psi_2\sigma_{22}) + \frac{1}{2}(\psi_{11}\sigma_{11} + 2\psi_{12}\sigma_{12}) \\ + \frac{1}{2}\left(\begin{array}{l} L_{111}({\sigma_{11}}^2\psi_1 + \psi_2\sigma_{11}\sigma_{12}) + L_{112}(3\sigma_{11}\psi_1\sigma_{12} + \psi_2(\sigma_{11}\sigma_{22} + 2{\sigma_{12}}^2)) \\ + L_{122}(\psi_1(2{\sigma_{12}}^2 + \sigma_{11}\sigma_{12}) + 3\psi_2\sigma_{22}\sigma_{12}) + L_{222}(\sigma_{22}\psi_1\sigma_{12} + \psi_2{\sigma_{22}}^2) \end{array} \right) \end{array} \right).$$

3.2 Bayesian Estimators Under Linear-Exponential Loss Function (LINEX)

3.2.1 Bayesian estimator of the shape parameter η

$$\hat{\eta}_{LINEX} = -\frac{1}{\rho}\log\left[\mathrm{E}\left(e^{-\rho\eta}\right)\right]$$

Provided that $\mathrm{E}(e^{-\rho\eta})$ exists and is finite. Substitution in equation (14), $\psi = e^{-\rho\eta}$; the Bayesian estimator of the parameter η is given by

$$\hat{\eta}_{LINEX} \approx -\frac{1}{\rho}\log\left[\begin{array}{l} e^{-\rho\eta} - \rho e^{-\rho\eta}Q_1\sigma_{11} - \rho e^{-\rho\eta}Q_2\sigma_{12} + \frac{\rho^2 e^{-\rho\eta}\sigma_{11}}{2} \\ \\ -\frac{1}{2}\rho e^{-\rho\eta}\left(L_{111}{\sigma_{11}}^2 + 3L_{112}\sigma_{11}\sigma_{12} + L_{122}(\sigma_{11}\sigma_{22} + 2{\sigma_{12}}^2) + L_{222}\sigma_{12}\sigma_{22}\right) \end{array} \right].$$

3.2.2 Bayesian estimator of the shape parameter κ

Substitution in equation (14), $\psi = e^{-\rho\kappa}$; the Bayesian estimator of the shape parameter κ is given as

$$\hat{\kappa}_{LINEX} \approx -\frac{1}{\rho}\log\left[\begin{array}{l} e^{-\rho\kappa} - \rho e^{-\rho\kappa}Q_1\sigma_{12} - \rho e^{-\rho\kappa}Q_2\sigma_{22} + \frac{\rho^2 e^{-\rho\kappa}\sigma_{22}}{2} \\ \\ -\frac{1}{2}\rho e^{-\rho\kappa}\left(L_{111}\sigma_{11}\sigma_{12} + 3L_{122}\sigma_{22}\sigma_{12} + L_{112}(\sigma_{11}\sigma_{22} + 2{\sigma_{12}}^2) + L_{222}{\sigma_{22}}^2\right) \end{array} \right].$$

3.2.3 Bayesian estimator of the reliability function $R(t)$

Substitution in equation (14), $\psi = e^{-\rho R(t)}$; the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{LINEX} \approx -\frac{1}{\rho}\log\left[\begin{array}{l} e^{-\rho R(t)} + Q_1(\psi_1\sigma_{11} + \psi_2\sigma_{12}) + Q_2(\psi_1\sigma_{12} + \psi_2\sigma_{22}) + \frac{1}{2}(\psi_{11}\sigma_{11} + 2\psi_{12}\sigma_{12} + \psi_{22}\sigma_{22}) \\ + \frac{1}{2}\left(\begin{array}{l} L_{111}({\sigma_{11}}^2\psi_1 + \psi_2\sigma_{11}\sigma_{12}) + L_{112}(3\sigma_{11}\psi_1\sigma_{12} + \psi_2(\sigma_{11}\sigma_{22} + 2{\sigma_{12}}^2)) \\ + L_{122}(\psi_1(2{\sigma_{12}}^2 + \sigma_{11}\sigma_{12}) + 3\psi_2\sigma_{22}\sigma_{12}) + L_{222}(\sigma_{22}\psi_1\sigma_{12} + \psi_2{\sigma_{22}}^2) \end{array} \right) \end{array} \right].$$

3.2.4 Bayesian estimator of the reversed hazard rate function $H(t)$

Substitution in equation (14), $\psi = e^{-\rho H(t)}$; the Bayesian estimator of the hazard rate function $H(t)$ is given as

$$\hat{H}_{LINEX} \approx -\frac{1}{\rho}\log\left[\begin{array}{l} e^{-\rho H(t)} + Q_1\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}\right) + Q_2\left(\psi_1\sigma_{12}+\psi_2\sigma_{22}\right) + \frac{1}{2}\left(\psi_{11}\sigma_{11}+2\psi_{12}\sigma_{12}+\psi_{22}\sigma_{22}\right) \\ +\frac{1}{2}\left(\begin{array}{l} L_{111}\left({\sigma_{11}}^2\psi_1+\psi_2\sigma_{11}\sigma_{12}\right)+L_{112}\left(3\sigma_{11}\psi_1\sigma_{12}+\psi_2\left(\sigma_{11}\sigma_{22}+2{\sigma_{12}}^2\right)\right) \\ +L_{122}\left(\psi_1\left(2{\sigma_{12}}^2+\sigma_{11}\sigma_{12}\right)+3\psi_2\sigma_{22}\sigma_{12}\right)+L_{222}\left(\sigma_{22}\psi_1\sigma_{12}+\psi_2{\sigma_{22}}^2\right)\end{array}\right)\end{array}\right].$$

3.3 Bayesian Estimators Under General Entropy Loss Function

3.3.1 Bayesian estimator of the shape parameter η

$$\hat{\eta}_{Gentropy} = \left[\mathrm{E}\left(\eta^{-q}\right)\right]^{\frac{-1}{q}}.$$

Provided that $\mathrm{E}\left(\eta^{-q}\right)$ exists and is finite. Substitution in equation (14), $\psi = \eta^{-q}$; the Bayesian estimator of the parameter η is given by

$$\hat{\eta}_{Gentropy} \approx \left[\begin{array}{l} \eta^{-q}+q\eta^{-q-1}Q_1\sigma_{11}+q\eta^{-q-1}Q_2\sigma_{12}+\frac{q(q+1)\eta^{-q-2}\sigma_{11}}{2} \\ -\frac{1}{2}q\eta^{-q-1}\left(L_{111}{\sigma_{11}}^2+3L_{112}\sigma_{11}\sigma_{12}+L_{122}\left(\sigma_{22}\sigma_{11}+2{\sigma_{12}}^2\right)+L_{222}\sigma_{12}\sigma_{22}\right)\end{array}\right]^{\frac{-1}{q}}.$$

3.3.2 Bayesian estimator of the shape parameter κ

Substitution in equation (14), $\psi = \kappa^{-q}$; the Bayesian estimator of the shape parameter κ is given by

$$\hat{\kappa}_{Gentropy} \approx \left[\begin{array}{l} \kappa^{-q}+q\kappa^{-q-1}Q_1\sigma_{12}+q\kappa^{-q-1}Q_2\sigma_{22}+\frac{q(q+1)\kappa^{-q-2}\sigma_{22}}{2} \\ +\frac{1}{2}q\kappa^{-q-1}\left(L_{111}\sigma_{11}\sigma_{12}+3L_{112}\left(\sigma_{11}\sigma_{22}+2{\sigma_{12}}^2\right)+3L_{122}\sigma_{22}\sigma_{12}+L_{222}{\sigma_{22}}^2\right)\end{array}\right]^{\frac{-1}{q}}.$$

3.3.3 Bayesian estimator of the reliability function $R(t)$

Substitution in equation (14), $\psi = R(t)^{-q}$; the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{Gentropy} \approx \left[\begin{array}{l} (R(t))^{-q} + Q_1\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}\right) + Q_2\left(\psi_1\sigma_{12}+\psi_2\sigma_{22}\right) + \frac{1}{2}\left(\psi_{11}\sigma_{11}+2\psi_{12}\sigma_{12}+\psi_{22}\sigma_{22}\right) \\ +\frac{1}{2}\left(\begin{array}{l} L_{111}\left({\sigma_{11}}^2\psi_1+\psi_2\sigma_{11}\sigma_{12}\right)+L_{112}\left(3\sigma_{11}\psi_1\sigma_{12}+\psi_2\left(\sigma_{11}\sigma_{22}+2{\sigma_{12}}^2\right)\right) \\ +L_{122}\left(\psi_1\left(2{\sigma_{12}}^2+\sigma_{11}\sigma_{12}\right)+3\psi_2\sigma_{22}\sigma_{12}\right)+L_{222}\left(\sigma_{22}\psi_1\sigma_{12}+\psi_2{\sigma_{22}}^2\right)\end{array}\right)\end{array}\right]^{\frac{-1}{q}}.$$

3.3.4 Bayesian estimator of the reversed hazard rate function $H(t)$

Substitution in equation (14), $\psi = H(t)^{-q}$; the Bayesian estimator of the hazard rate function $H(t)$ is given as

$$\hat{H}_{Gentropy} \approx \left[\begin{array}{l} (H(t))^{-q} + Q_1\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}\right) + Q_2\left(\psi_1\sigma_{12}+\psi_2\sigma_{22}\right) + \frac{1}{2}\left(\psi_{11}\sigma_{11}+2\psi_{12}\sigma_{12}+\psi_{22}\sigma_{22}\right) \\ +\frac{1}{2}\left(\begin{array}{l} L_{111}\left({\sigma_{11}}^2\psi_1+\psi_2\sigma_{11}\sigma_{12}\right)+L_{112}\left(3\sigma_{11}\psi_1\sigma_{12}+\psi_2\left(\sigma_{11}\sigma_{22}+2{\sigma_{12}}^2\right)\right) \\ +L_{122}\left(\psi_1\left(2{\sigma_{12}}^2+\sigma_{11}\sigma_{12}\right)+3\psi_2\sigma_{22}\sigma_{12}\right)+L_{222}\left(\sigma_{22}\psi_1\sigma_{12}+\psi_2{\sigma_{22}}^2\right)\end{array}\right)\end{array}\right]^{\frac{-1}{q}}.$$

4 Simulation studies

To demonstrate the importance of the results obtained in the preceding sections, simulation studies are conducted. For this purpose, by using Monte Carlo method, with fixed sample size n (the total items put in a life test), with constant censoring scheme, where $R_1 = R_2 = R_3 = ... = R_m$, where m is the sample size of progressively censored from the sample of size n.
The following algorithm is used to generate sample based on progressive type-II censoring scheme, based on any continuous df F, see Balakrishnan and Aggarwala [3].

1. Generate m independent Uniform (0,1) observations $W_1, ..., W_m$.
2. Set $V_i = W_i^{1/\gamma_i}$, $\gamma_i = \left(i + \sum_{j=m-i+1}^{m} R_j \right)$ for $i = 1, 2, ..., m$.
3. $U_i = 1 - V_m V_{m-1} ... V_{m-i+1}$, $i = 1, 2, ..., m$.
4. Set $X_i = F^{-1}(U_i)$, then X_i, for $i = 1, 2, ..., m$, is the progressive type-II censoring scheme based on the df F.
5. We repeated steps 1,2,3 and 4 (10000) times, for different values of n and m.

Estimation average $= \frac{\sum_{i=1}^{10000} \widehat{\theta}_i}{10000}$, mean square error $= \frac{\sum_{i=1}^{10000} \left(\hat{\theta}_i - \theta\right)^2}{10000}$, where, θ is the parameter and $\widehat{\theta}$ is the estimator.
All the computations are prepared by Mathematica 11.
Since the non-linear equations (6) are not solvable analytically, numerical methods can be used, as Newton Raphson method with initial values closed to real values of the parameters.

Throughout this section we will use the following abbreviations:

1. ML : means that the estimate by using the (MLE),
2. B_{Sq} : means that the estimate under squared error loss function,
3. $B_{Lx,\rho=2}$: means that the estimate under linex loss function at $\rho = 2$,
4. $B_{Lx,\rho=5}$: means that the estimate under linex loss function at $\rho = 5$,
5. $B_{Ge,q=3}$: means that the estimate under general entropy loss function at $q = 3$,
6. $B_{Ge,q=7}$: means that the estimate under general entropy loss function at $q = 7$.

From tables 1, 2, 3 and 4, we observe that the MLE and Bayes estimates of the parameters η, κ, the reliability and the reversed hazard rate functions are very good in terms of $MSEs$. As the number of items n and effective sample size m increase, $MSEs$ of all estimates decrease as expected and increase the accuracy of estimators. In general, the Bayesian estimators have $MSEs$ less than that of the MlE.

5 Concluding remarks

In this paper, assuming a good lifetime model, we consider the problem of estimating the unknown parameters η, κ as well as the reliability and reversed hazard rate functions, using progressive type-II censored samples. This censoring plan has points of interest as far as decreasing test time. We derived MLE and Bayes estimators of the parameter η, κ the reliability and the reversed hazard rate functions using gamma informative prior, under both symmetric (squared error) and asymmetric (general entropy and linex) loss functions. These estimates cannot be obtained in closed form, but can be computed numerically. It is clear that the proposed Bayes estimators perform very well for different n and m. As expected, the Bayes estimators based on informative prior perform are much better than the estimators based on MLE in terms of $MSEs$. The simulation also stresses the importance of linex and general entropy loss functions as asymmetric loss functions, in the case studied.

Table 1. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameters η, κ, the reliability function and the reversed hazard rate function when $\eta = 0.7,\ \kappa = 1.2,\ \zeta_1 = 1,\ \nu_1 = 2,\ \zeta_2 = 2,\ \nu_2 = 4$.

$B_{Lx,\rho=2}$	$B_{Lx,\rho=5}$	$B_{Ge,q=3}$	$B_{Ge,q=7}$	B_{Sq}	ML	m	n
The average, MSEs of the estimators of parameter η							
0.74138 (0.00117)	0.73562 (0.00081)	0.72330 (0.00026)	0.70887 (0.00004)	0.73371 (0.00071)	0.71713 (0.00979)	50	100
0.76737 (0.00364)	0.76835 (0.00373)	0.73024 (0.00054)	0.72221 (0.00023)	0.75141 (0.00196)	0.72486 (0.02153)	25	50
The average, MSEs of the estimators of parameter κ							
1.32198 (0.01207)	1.37245 (0.03721)	1.2539 (0.00174)	1.2263 (0.00020)	1.26085 (0.00006)	1.23427 (0.02233)	50	100
1.40031 (0.03641)	1.12779 (0.02140)	1.25570 (0.00192)	1.23664 (0.00060)	1.26891 (0.00005)	1.22818 (0.04018)	25	50
The average, MSEs of the estimators of relaibility function R(t=3)=0.43536							
0.43310 (0.00004)	0.42668 (0.00016)	0.42869 (0.00012)	0.41956 (0.00040)	0.43442 (0.00002)	0.43956 (0.00116)	50	100
0.42108 (0.00034)	0.40866 (0.00095)	0.41294 (0.00071)	0.39708 (0.00180)	0.42366 (0.00025)	0.43421 (0.00280)	25	50
The average, MSEs of the estimators of reversed hazard rate function H(t=3)=0.12813							
0.13206 (0.00006)	0.13129 (0.00003)	0.13068 (0.00005)	0.12873 (0.00007)	0.13215 (0.00007)	0.13008 (0.00008)	50	100
0.13130 (0.00003)	0.13159 (0.00004)	0.12860 (0.00007)	0.12675 (0.00002)	0.13149 (0.00004)	0.12812 (0.00004)	25	50

Table 2. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameters η, κ, the reliability function and the reversed hazard rate function when $\eta = 0.7,\ \kappa = 1.5,\ \zeta_1 = 1,\ \nu_1 = 2,\ \zeta_2 = 2,\ \nu_2 = 4$.

$B_{Lx,\rho=2}$	$B_{Lx,\rho=5}$	$B_{Ge,q=3}$	$B_{Ge,q=7}$	B_{Sq}	ML	m	n
The average, MSEs of the estimators of parameter η							
0.75275 (0.00207)	0.72147 (0.00020)	0.73750 (0.00092)	0.69853 (0.00007)	0.75753 (0.00253)	0.73562 (0.00989)	50	100
0.76091 (0.00291)	0.75498 (0.00228)	0.73010 (0.00053)	0.71703 (0.00011)	0.77172 (0.00419)	0.73353 (0.01630)	25	50
The average, MSEs of the estimators of parameter κ							
1.77866 (0.07016)	1.08642 (0.00214)	1.57415 (0.00346)	1.51457 (0.00005)	1.61664 (0.00005)	1.56558 (0.03409)	50	100
2.3268 (0.71099)	0.96036 (0.09194)	1.5867 (0.00511)	1.54754 (0.00109)	1.67854 (0.00004)	1.58579 (0.07138)	25	50
The average, MSEs of the estimators of relaibility function R(t=3)=0.51055							
0.50397 (0.00013)	0.50938 (0.00003)	0.50054 (0.00022)	0.50394 (0.00013)	0.50519 (0.00011)	0.50908 (0.00128)	50	100
0.50513 (0.00011)	0.49602 (0.00038)	0.49885 (0.00028)	0.48654 (0.00083)	0.50750 (0.00006)	0.51391 (0.00247)	25	50
The average, MSEs of the estimators of reversed hazard rate function H(t=3)=0.16016							
0.16472 (0.00008)	0.16505 (0.00001)	0.16294 (0.00003)	0.16166 (0.00006)	0.16488 (0.00009)	0.16262 (0.00006)	50	100
0.16788 (0.00003)	0.16816 (0.00004)	0.16425 (0.00002)	0.15596 (0.00008)	0.16821 (0.00004)	0.16443 (0.00003)	25	50

Table 3. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameters η, κ, the reliability function and the reversed hazard rate function when $\eta = 0.9$, $\kappa = 1.2$, $\zeta_1 = 1$, $\nu_1 = 2$, $\zeta_2 = 2$, $\nu_2 = 4$.

$B_{Lx,\rho=2}$	$B_{Lx,\rho=5}$	$B_{Ge,q=3}$	$B_{Ge,q=7}$	B_{Sq}	ML	m	n
The average, MSEs of the estimators of parameter η							
0.96374 (0.00297)	0.93341 (0.00059)	0.94441 (0.00124)	0.90890 (0.00001)	0.96633 (0.00325)	0.94035 (0.01567)	50	100
0.99701 (0.00770)	0.92189 (0.00016)	0.95800 (0.00238)	0.88004 (0.00083)	1.00548 (0.00925)	0.95995 (0.03975)	25	50
The average, MSEs of the estimators of parameter κ							
1.29566 (0.00696)	1.37622 (0.02725)	1.2581 (0.00210)	1.22871 (0.00027)	1.27265 (0.00007)	1.24206 (0.02095)	50	100
1.35943 (0.02185)	1.68111 (0.18061)	1.27847 (0.00439)	1.19889 (0.00017)	1.31227 (0.00003)	1.25972 (0.05447)	25	50
The average, MSEs of the estimators of relaibility function R(t=3)=0.33384							
0.32268 (0.00021)	0.32327 (0.00019)	0.31612 (0.00043)	0.31346 (0.00055)	0.32411 (0.00016)	0.33043 (0.00121)	50	100
0.31216 (0.00062)	0.31450 (0.00051)	0.30048 (0.00133)	0.30028 (0.00134)	0.31496 (0.00049)	0.32681 (0.00298)	25	50
The average, MSEs of the estimators of reversed hazard rate function H(t=3)=0.10877							
0.10978 (0.00006)	0.11036 (0.00002)	0.10858 (0.00003)	0.10816 (0.00002)	0.10985 (0.00002)	0.10916 (0.00006)	50	100
0.10927 (0.00003)	0.10969 (0.00003)	0.10692 (0.00002)	0.10567 (0.00001)	0.10941 (0.00002)	0.10822 (0.00006)	25	50

Table 4. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameters η, κ, the reliability function and the reversed hazard rate function when $\eta = 0.9$, $\kappa = 1.5$, $\zeta_1 = 1$, $\nu_1 = 2$, $\zeta_2 = 2$, $\nu_2 = 4$.

$B_{Lx,\rho=2}$	$B_{Lx,\rho=5}$	$B_{Ge,q=3}$	$B_{Ge,q=7}$	B_{Sq}	ML	m	n
The average, MSEs of the estimators of parameter η							
0.95893 (0.00247)	0.91475 (0.00003)	0.94267 (0.00112)	0.89346 (0.00024)	0.97462 (0.00427)	0.94319 (0.01734)	50	100
1.00307 (0.00878)	0.92720 (0.00032)	0.97195 (0.00392)	0.89376 (0.00078)	1.04754 (0.01912)	0.98601 (0.03760)	25	50
The average, MSEs of the estimators of parameter κ							
1.67679 (0.02624)	1.35712 (0.00231)	1.56526 (0.00250)	1.50056 (0.00021)	1.61521 (0.00005)	1.56024 (0.03801)	50	100
1.94485 (0.20576)	1.20846 (0.03256)	1.62662 (0.01234)	1.48771 (0.00078)	1.7546 (0.00003)	1.64163 (0.09672)	25	50
The average, MSEs of the estimators of relaibility function R(t=3)=0.39817							
0.38770 (0.00020)	0.39236 (0.00009)	0.38233 (0.00038)	0.38407 (0.00032)	0.38910 (0.00016)	0.39435 (0.00168)	50	100
0.37791 (0.00058)	0.37729 (0.00061)	0.36780 (0.00117)	0.36385 (0.00145)	0.38069 (0.00045)	0.38834 (0.00269)	25	50
The average, MSEs of the estimators of reversed hazard rate function H(t=3)=0.13597							
0.13762 (0.00008)	0.13801 (0.00004)	0.13623 (0.00002)	0.13544 (0.00003)	0.13773 (0.00001)	0.13672 (0.00002)	50	100
0.13923 (0.00003)	0.13694 (0.00001)	0.13647 (0.00002)	0.13224 (0.00002)	0.13944 (0.00004)	0.13778 (0.00003)	25	50

Acknowledgement

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] Abu-Moussa, M.H., El-Din, M.M.M. On Estimation and Prediction for the Inverted Kumaraswamy Distribution Based on General Progressive Censored Samples. Pakistan Journal of Statistics and Operation Research 14, 717-736. (2018).

[2] Al-Fattah, A.M., EL-Helbawy, A., Al-Dayian, G. Inverted Kumaraswamy distribution: Properties and estimation. Pak. J. Statist, 33(1), 37-61. (2017).

[3] Balakrishnan, N. and Aggarwala, R. Progressive Censoring: Theory, Methods and Applications, Birkhauser, Boston. (2000).

[4] Gholizadeh, R., Mastani Shirazi, A., Mosalmanza, S. Classical and Bayesian Estimations on the Kumaraswamy Distribution using Grouped and Ungrouped Data under Difference Loss Functions. Journal of Applied Sciences, 11(12), 2154-2162. (2011).

[5] Jones, M.C. Kumaraswamy distribution: A beta-type distribution with some tractability advantages. Statistical Methodology 6, 7081. (2009).

[6] Kumaraswamy, P. A generalized probability density function for double-bounded random processes. Journal of Hydrology 46, 7988. (1980).

[7] Lindley, D. V. Approximate Bayesian Methods, Trabajos de Estadistica Y de Investigacion Operativa, 31:1, 223-245. (1980).

[8] Sharaf El-Deen, M., Al-Dayian, G., EL-Helbawy, A. Statistical Inference for Kumaraswamy Distribution Based on Generalized Order Statistics with Applications. British Journal of Mathematics & Computer Science, 4, 1710?1743. (2014).

[9] Sindhu, T.N., Feroze, N., Aslam, M.S. Bayesian Analysis of the Kumaraswamy Distribution under Failure Censoring Sampling Scheme. International Journal of Advanced Science and Technolog, 51, 39-58. (2013).

M. Yusuf is Assistant Professor of Mathematical Statistics at Helwan University, Egypt. He is referee of several international journals in the frame of mathematical statistics. His main research interests are: Bayesian Analysis, Order Statistics, Generalized Order Statistics, Statistical Inference, and Simulation.