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# Estimation and Prediction for a Fréchet Distribution under Type-II Censoring

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**Abstract:** In this article, we investigate the problem of parameter estimation for the Fréchet distribution and the prediction of new order statistics within the context of Type-II censoring. The Fréchet distribution, widely employed for modeling extreme values in various areas, presents unique complexities when observations are censored. Our main focus is on devising estimators for the scale and shape parameters of the Fréchet distribution by adapting the percentile method for censored samples. Regarding prediction, we generate formulas for predicting future order statistics based on the censored sample and outline the associated calculation procedures. To evaluate the performance of our proposed estimators and predictors in finite samples, we conduct a deep simulation study. Through the application of the proposed methods to a real-world extreme value dataset, we illustrate their practical effectiveness. The results confirm the reliability and accuracy of the developed techniques for parameter estimation and prediction of the Fréchet distribution under Type-II censoring.

**Keywords:** Fréchet distribution, Type-II censoring, Percentile estimators, Maximum likelihood predictor, Conditional median predictor.

## 1 Introduction

The Fréchet distribution, also known as the inverse Weibull distribution, holds significant importance in statistical analysis, finding widespread applications across different fields for modeling extreme events and analyzing datasets involving outliers. It serves as a valuable tool in reliability theory, where it is used as a lifetime distribution. This distribution proves particularly useful in modelling diverse failure characteristics, including infant mortality and wear-out periods (see [1]). Moreover, it finds extensive usage in other fields such as hydrology, finance, environmental sciences, and engineering. In addition to these disciplines, the Fréchet distribution also plays a role in social sciences and various other fields, aiding in the assessment of risks associated with rare and extreme events (see [2]).

Estimation of the parameters of the Fréchet distribution is of particular importance in the effective use of this distribution. Traditionally, parameter estimation has relied on commonly used methods such as the maximum likelihood estimator (MLE) and the method of moments. However, recent developments have introduced novel approaches that provide computationally simpler estimation methods while maintaining efficiency. New methods for estimating the parameters of the Fréchet distribution provide good accuracy, reduced computational aspect, and better handling of small sample sizes. These features contribute to a better understanding of the data containing extreme events.

Several methods have been used in the literature involving estimation parameters of Fréchet distribution. Abbas and Yincai [3] considered the estimation of the scale parameter for Fréchet distribution with a known shape parameter using the MLE, Probability weighted moment method, and Bayes estimation. Sultan et al. [4] discussed the estimation problem of the Fréchet distribution with Bayesian and maximum likelihood approaches under progressive Type-II censoring Ramos et al. [5] presented different estimation methods of the parameters of Frechet distribution including the moment method, L-moment, ordinary and weighted least squares, maximum product of spacing, and Bayesian estimation.

It is known that classical estimation methods such as MLE and moment methods are effective in many cases, but there are limitations to these methods. These challenges can include difficulties in finding local

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maximum or infinite probability functions, as well as cases where the support of a random variable depends on the parameters or when the moments do not exist or exist for specific values of the parameters or are complex functions of the parameters.

In contrast, the percentile method has been proposed as an alternative parameter estimation technique that provides certain advantages over the conventional methods. One of the main advantages is the ease of obtaining estimates, as the method involves finding the quantile function without integration or differentiation. The percentile method can be implemented through two different methods.

The initial approach, which draws parallels to the traditional moment's method, relies on percentiles. A second method involves parameter estimation by minimizing the squared Euclidean distance between sample percentiles and population percentile values. For more detailed information on these approaches, please refer to [6] and [7]. Notably, research indicates that the percentile estimation technique is as efficient as, or even superior to, maximum likelihood and least squares methods, as demonstrated in studies such as [8] and [9]. Turning to the prediction aspect, it is important to note that there is a scarcity of research on the prediction of future observations for the Fréchet distribution under censoring. However, for predicting future order statistics based on other distributions and models, we recommend exploring the work of Raqab and Nagaraja [10], Barakat et al. [11], Abou Ghaida and Baklizi [12], Amleh and Raqab [13] , [14] and [15] and Amleh [16].

In this paper, we investigate the Fréchet distribution within the context of Type-II censoring data, where the experiment halts upon reaching predetermined number of events. Our paper has two primary objectives. First, we introduce various adapted estimators utilizing the percentile estimation method for the Fréchet distribution's parameter. Second, we address the problem of predicting future order statistics using the Fréchet model. Furthermore, we conduct a comparative analysis of these proposed methods through Monte Carlo simulations and by applying them to a real-world dataset.

## 2 The Fréchet Distribution and Maximum Likelihood Estimation

The probability density function (pdf) of Fréchet distribution is given by:

$$f(x;\lambda;\beta)=\frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{-\beta-1}e^{-(\frac{x}{\lambda})^{-\beta}},x>0,\lambda,\beta>0, \tag{1}$$

where $\lambda$ is the scale parameter and $\beta$ is the shape parameter. So, if X follows a Fréchet distribution, it will be denoted by $X \backsim Fr(\lambda,\beta)$. The cumulative distribution function (CDF) of a Fréchet distribution is given by:

$$F(x;\lambda;\beta)=e^{-(\frac{x}{\lambda})^{-\beta}},x>0,\lambda,\beta>0. \tag{2}$$

Type-II censoring occurs when a fixed number of items from a sample are observed and the remaining items, which have not yet occurred, are censored. Therefore, if $x=(x_{1:n},x_{2:n},...,x_{r:n})$ is a Type-II censored random sample taken from a Fréchet distribution, then the likelihood function based on this data is given by:

$$L(\lambda,\beta)=\frac{n!}{r!}\prod_{i=1}^{r}f(x_{i:n})[1-F(x_{r:n})]^{n-r},1\leqslant r\leqslant n,$$

Consequently, based on this setup, the likelihood function of the Frechet distribution is obtained as:

$$L(\lambda,\beta)=\frac{\beta^r}{\lambda^r}\prod_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^{-\beta-1}exp\left\{-\prod_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^{-\beta}\right\}\left[1-exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}\right]^{n-r} \tag{3}$$

The associated log-likelihood function is given by:

$$L*(\lambda,\beta)=r(log\beta-log\lambda)-(\beta+1)\times\Sigma_{i=1}^{r}log\left(\frac{x_i}{\lambda}\right)-\Sigma_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^{-\beta}+(n-r)log\left[1-exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}\right] \tag{4}$$

The MLEs of $\lambda$ and $\beta$ can be obtained by maximizing the log-likelihood function in Eq. (4). Thus, the likelihood equations are obtained as:

$$\frac{\partial L*}{\partial\lambda}=\frac{r}{\lambda}+\lambda^{\beta-1}\Sigma_{i=1}^{r}(x_i)^{-\beta}$$

$$\frac{(n-r)\lambda^{\beta-1}exp\{-(\frac{x_{r:n}}{\lambda})^{-\beta}\}}{(t_{r:n})^{\beta}[1-exp\{-(\frac{x_{r:n}}{\lambda})^{-\beta}\}]}=0. \tag{5}$$

$$\frac{\partial L*}{\partial\beta}=\frac{r}{\beta}-\Sigma_{i=1}^{r}log(\frac{x_i}{\lambda})+\Sigma_{i=1}^{r}(\frac{x_i}{\lambda})^{-\beta}log(\frac{x_i}{\lambda})$$

$$-\frac{(n-r)\lambda^{\beta}log(\frac{x_{r:n}}{\lambda})exp\{-(\frac{x_{r:n}}{\lambda})^{-\beta}\}}{(t_{r:n})^{\beta}[1-exp\{-(\frac{t_{r:n}}{\lambda})^{-\beta}\}]}=0. \tag{6}$$

The likelihood equations cannot be solved in closed form. Accordingly, Eq.s (5) and (6) may be handled simultaneously using a numerical technique such as the Newton-Raphson method. In fact, Ramos et al. [5] proved that the MLEs of the Fréchet parameters for complete data exist uniquely. However, it can be shown that the MLEs of the parameters $\beta$ and $\lambda$ under Typ-II censoring also exist uniquely.

## 3 Percentile Methods

Percentiles hold a crucial position in the realm of statistical inference, particularly in contemporary parameter estimation techniques. This methodology mirrors the traditional moment method by equating population percentiles with sample percentiles and subsequently solving these equations concurrently.

### 3.1 Classical Percentile Estimator(PE-I)

In this part of the study, we introduce new percentile estimators depending on the usual quartiles. To proceed, we give the formula of the quantile function for the Fréchet distribution. It is given by:

$$Q(t) = \frac{\lambda}{[ln(\frac{1}{t})]^{\frac{1}{\beta}}}, 0 < t < 1 \tag{7}$$

Now, assume that $x = (x_{1:n}, x_{2:n}, ..., x_{r:n}$ is a Typ-II censored random sample taken from a Fréchet distribution. Following the same approach of Zaka and Akhtar [17], Sampath and Anjana [18], and Bahatti et al. [19]; we choose the lower quartile $Q_1$ and the upper quartile $Q_3$ to be used in obtaining the percentile estimators. Thus, if X is distributed by a Fréchet distribution, based on Eq. [7], the lower and the upper quartiles of X are given as:

$$Q_1 = \frac{\lambda}{[ln(4)]^{\frac{1}{\beta}}}, \tag{8}$$

and

$$Q_3 = \frac{\lambda}{[ln(\frac{4}{3})]^{\frac{1}{\beta}}}, \tag{9}$$

respectively. The technique is mainly depending on computing the population quartiles stated in Eqs. (8) to (9) and compared them to the sample quartiles. Thus, we obtain the following equations

$$\lambda = x_{0.25}[ln(4)]^{\frac{1}{\beta}}, \tag{10}$$

and

$$\lambda = x_{0.75}\left[ln\left(\frac{4}{3}\right)\right]^{\frac{1}{\beta}}, \tag{11}$$

where $x_{0.25}$ is the sample lower quartile and $x_{0.75}$ represents the sample upper quartile. By solving Eq.s (10) and (11), we obtain the percentiles estimators (PE-I) needed to estimate the parameters $\beta$ and $\lambda$ as follows:

$$\widehat{\beta} = \frac{\frac{ln(\frac{4}{3})}{ln(4)}}{ln(\frac{X_{0.75}}{X_{0.25}})}, \tag{12}$$

and,

$$\widehat{\lambda} = X_{0.25}(ln4)^{\frac{1}{\beta}}, \tag{13}$$

respectively.

### 3.2 First Modified percentile estimator (PE-II)

The First modification on the percentile estimator is based on using the geometric mean of the Fréchet distribution and equating it to the sample geometric mean. Now, we formally define a geometric mean of a random variable.

**Definition 1.** *If Y is a non-negative random variable, then its geometric mean (GM) is defined as follows:*

$$GM(Y) = exp\{E[logY]\}.$$

*It is known that the expected value of a Fréchet distribution is defined only if the shape parameter $\beta > 1$, which is not the case for the GM, the following theorem presents a precise expression for the GM of the Fréchet distribution.*

**Theorem 1.***The GM of Fréchet distribution is given by:*

$$GM_F = \lambda e^{\frac{\gamma}{\beta}}, \tag{14}$$

*where $\gamma$ is known as the Euler-Mascheroni Constant, more details on this number can be found in Havil [20].*

*Proof.*

$$E(logX) = \int_0^\infty lnx\frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{-\beta-1} exp\left(-\left(\frac{x}{\lambda}\right)^{-\beta}\right) dx.$$

Letting $t = (\frac{x}{\lambda})^{-\beta}$, the integration is reduced to:

$$E(logX) = \int_0^\infty \ln(\lambda t^{-\frac{1}{\beta}})e^{-t}dt$$
$$= \int_0^\infty ln(\lambda e^{-t})dt - \frac{1}{\beta}\int_0^\infty lnte^{-t}dt = ln\lambda + \frac{\gamma}{\beta}.$$

Here, the second integration

$$\int_0^\infty \ln te^{-t}dt = -\gamma \cong -0.577215,$$

see [20], [21] and [22]. Thus, the $GM_F$ is obtained as

$$GM_F = e^{ln\lambda + \frac{\gamma}{\beta}}$$

The above result ends the proof. Now, using Eq. (14), we get:

$$\lambda e^{\frac{\gamma}{\beta}} = gm_s, \tag{15}$$

where $gm_s$ is the sample GM. Therefore, solving Eq.s (10) and (15) simultaneously, we get:

$$\widehat{\beta} = \frac{\lambda + ln(ln4)}{ln(\frac{gm_s}{x_{0.25}})}. \tag{16}$$

$$\widehat{\lambda} = X_{0.25}(ln4)^{\frac{1}{\beta}}. \tag{17}$$

Eqs. (16) and (17) are the first modified percentile estimators (PE-II) required to estimate the parameters $\beta$ and $\lambda$, respectively.

### 3.3 Second Modified percentile estimator

The second modification on the percentile estimator is obtained by equating an estimation of the empirical CDF of the first order statistic "$X_{(1)}$" of Fréchet distribution and the value of $F(x_{(1)})$, where $x_{(1)}$ is the minimum of the observed sample $x_{(1)}, x_{(2)}, ..., x_{(n)}$, i.e.

$$\widehat{F}(X_{(1)}) = F(x_{(1)})$$

Following the same technique in [17], [22], and [24] for estimating $F(X_{(1)})$, we obtain

$$\frac{1}{n+1} = e^{-(\frac{x_{(1)}}{\lambda})^{-\beta}}, \tag{18}$$

or equivalently,

$$\lambda = x_{(1)}[ln(n+1)]^{\frac{1}{\beta}}. \tag{19}$$

Therefore, using Eq. (19) along with Eq.(10), the percentile estimators of the parameters $\beta$ and $\lambda$ based on this modification (PE-III) are obtained as

$$\widehat{\beta} = \frac{ln(\frac{ln(n+1)}{ln2})}{ln(\frac{M}{X_{(1)}})} \tag{20}$$

and

$$\widehat{\lambda} = M(ln2)^{\frac{ln(\frac{M}{X_{(1)}})}{ln(\frac{ln(n+1)}{ln2})}}, \tag{21}$$

respectively.

## 4 Prediction of New Order Statistics

This section discusses the prediction of new order statistics of censored units from Fréchet distribution. Suppose that out of $n$ units in the test, $r$ items are observed, say $X = X_{1:n}, X_{2:n}, ..., X_{r:n}$, known as informative sample. Let $X_{s:n}, r+1 \leq s \leq n$ be the unobserved order statistics taken from the same sample. In this setup, we aim to predict the future order statistics $X_{s:n}$ given the observed data $X_{i:n}, 1 \leq i \leq r$.

It is known that the conditional density of $Y = X_{s:n}$ given $X = x = (x_{1:n}, ..., x_{r:n})$ is equivalent to the density of the $(s-r)^{th}$ order statistics out of $(n-r)$ items taken from the distribution with pdf $\varphi(y) = \frac{f(y)}{1-F(x_{r:n})}$ truncated at $x_{r:n}$ . The above result is formulated according to the well-known Markovian property of censored order statistics. Here, $f$ and $F$ represent the pdf and the cdf of the Fréchet distribution, respectively. Thus, $\varphi(y)$ is given by

$$\varphi(y) = \frac{\frac{\beta}{\lambda}(\frac{y}{\lambda})^{-\beta-1}e^{-(\frac{y}{\lambda})^{-\beta}}}{1-e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}}, y > x_{r:n}. \tag{22}$$

### 4.1 Maximum likelihood prediction

Kaminsky and Rhodin [25] used the maximum likelihood principle for the prediction of future random variables as well as estimating the model parameters, the resulting point predictor is known as the maximum-likelihood predictor (MLP). The predictive likelihood function (PLF) of $Y = X_{s:n}$ based on Type-II censored samples may be expressed as:

$$\begin{aligned} L(y,\lambda,\beta|x)\alpha \left(\frac{\beta}{\lambda}\right)\left(\frac{y}{\lambda}\right)^{-\beta-1}\prod_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^{-\beta-1} \\ exp\left(-\Sigma_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^{-\beta}-\left(\frac{y}{\lambda}\right)^{-\beta}\right) \\ \times\left[exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\}-exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}\right]^{s-r-1} \\ \left[1-exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\}\right]^{n-s}. \end{aligned} \tag{23}$$

Hence, the log PLF can be obtained as

$$\begin{aligned} L*\alpha(r+1)(log\beta-log\lambda)-\Sigma_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^{-\beta}-\left(\frac{y}{\lambda}\right)^{-\beta} \\ -(\beta+1)\left[\Sigma_{i=1}^{r}log\left(\frac{x_i}{\lambda}\right)+log\left(\frac{y}{\lambda}\right)\right] \\ +(s-r-1)log\left[exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\}\right. \\ \left.-exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}\right] \\ +(n-s)log\left[1-exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\}\right] \end{aligned} \tag{24}$$

The predictive maximum likelihood estimators (PMLEs) of $\lambda$ and $\beta$ and the MLP of $Y$ are obtained using the predictive likelihood equations (PLEs), as follows

$$\begin{aligned} \frac{\partial L*}{\partial\beta} = \frac{r+1}{\beta}-\Sigma_{i=1}^{r}log(\frac{x_i}{\lambda})-log(\frac{y}{\lambda})+\Sigma_{i=1}^{r}(\frac{x_i}{\lambda})^{-\beta} \\ \times log(\frac{x_i}{\lambda})+(\frac{x_i}{\lambda})^{-\beta}log(\frac{x_i}{\lambda})+(s-r-1) \\ \times\left(\frac{(\frac{y}{\lambda})^{-\beta}exp\{-(\frac{\gamma}{\lambda})^{-\beta}\}log(\frac{\gamma}{\lambda})-(\frac{x_{r:n}}{\lambda})^{-\beta}}{exp\{-(\frac{y}{\lambda})^{-\beta}\}-exp\{-(\frac{x_{r:n}}{\lambda})^{-\beta}\}}\right. \\ \left.-\frac{exp\{(\frac{x_{r:n}}{\lambda})^{-\beta}\}log(\frac{x_{r:n}}{\lambda})}{exp\{-(\frac{y}{\lambda})^{-\beta}\}-exp\{-(\frac{x_{r:n}}{\lambda})^{-\beta}\}}\right) \\ +(n-s)\frac{(\frac{y}{\lambda})^{-\beta}exp\{-(\frac{y}{\lambda})^{-\beta}\}log(\frac{y}{\lambda})}{1-exp\{-(\frac{y}{\lambda})^{-\beta}\}} = 0; \end{aligned} \tag{25}$$

$$\frac{\partial L*}{\partial \lambda} = -\frac{\beta(r+1)}{\lambda} - \beta \Sigma_{i=1}^{r} (\frac{x_{i:n}}{\lambda})^{-\beta-1} (\frac{x_{i:n}}{\lambda^2}) - \beta(\frac{y}{\lambda})^{-\beta-1}(\frac{y}{\lambda^2}) + (s-r-1) \times \left( \frac{\beta\left(\frac{y}{\lambda}\right)^{-\beta-1} exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\}}{exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\} - exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}} - \frac{\beta\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta-1} exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}}{exp\left\{-\left(\frac{y}{\lambda}\right)^{-\beta}\right\} - exp\left\{-\left(\frac{x_{r:n}}{\lambda}\right)^{-\beta}\right\}} \right) + (n-s)\frac{\beta(\frac{y}{\lambda})^{-\beta-1} exp\{-(\frac{y}{\lambda})^{-\beta}\}}{1-exp\{-(\frac{y}{\lambda})^{-\beta}\}} = 0; \quad (26)$$

$$\frac{\partial L*}{\partial y} = -\frac{\beta+1}{y} + \frac{\beta}{\lambda}(\frac{y}{\lambda})^{-\beta-1} + (s-r-1) \times \frac{\frac{\beta}{\lambda}(\frac{y}{\lambda})^{-\beta-1} exp\{-(\frac{y}{\lambda})^{-\beta}\}}{exp\{-(\frac{y}{\lambda})^{-\beta}\} - exp\{-(\frac{x_{r:n}}{\lambda}^{-\beta}\}} + (n-s)\frac{\frac{\beta}{\lambda}(\frac{\beta}{\lambda})^{-\beta-1} exp\{-(\frac{y}{\lambda})^{-\beta}\}}{1-exp\{-(\frac{y}{\lambda})^{-\beta}\}} = 0. \quad (27)$$

As equations (25) – (27) cannot be solved explicitly, we can use some numerical methods such as Newton –Raphson method to solve them simultaneously, resulting in finding the MLP for $Y$ and the PMLEs of $\lambda$ and $\beta$. The so obtained MLP of Y is going to be denoted by $\widehat{Y}_{MLP}$.

### 4.2 Conditional Median Prediction

Raqab and Nagaraja [10] proposed a new point predictor known as the conditional median predictor (CMP). The formulation of the CMP of $Y, \widehat{Y}_{CMP}$, is based on obtaining the median of the conditional distribution of $Y$ given $X = x, i.e.$

$$P(Y \leq \widehat{Y}_{CMP}|X = x) = P_0(Y \leq \widehat{Y}_{CMP} \geq X = x)$$

Now, according to the conditional distribution of Y given X=x, we may obtain

$$P(Y \leq \widehat{Y}|X = x) = P_0\left(\frac{e^{-(\frac{Y}{\lambda})^{-\beta}} - e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}}{1-e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}} \geq \frac{e^{-(\frac{\widehat{Y}}{\lambda})^{-\beta}} - e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}}{1-e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}} \mid X = x\right).$$

It can be shown that, given X=x, the distribution of the random variable

$$Q = \frac{e^{-(\frac{Y}{\lambda})^{-\beta}} - e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}}{1-e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}},$$

is a beta distribution with parameters $s-r$ and $n-s+1$, denoted by Beta $(s-r, n-s+1)$. Thus, supposing that $B$ follows Beta $(s-r, n-s+1)$ , and $M_B$ represents the median of $B$, the $CMP$ of $Y$ can be obtained as

$$\widehat{Y}_{CMP} = \lambda\left\{-log\left[M_B + (1-M_B)e^{-(\frac{x_{r:n}}{\lambda})^{-\beta}}\right]\right\}^{\frac{-1}{\beta}}. \quad (28)$$

It is known that the parameters $\lambda$ and $\beta$ are unknown, so the CMP of Y can be obtained approximately by using the proposed estimators of such parameters. Accordingly, we use the MLEs of $\lambda$ and $\beta$ to approximate the CMP, yielding $CMP_{MLE}$. We also use the proposed percentile estimators of $\lambda$ and $\beta$ to obtain an approximation of the CMP. The results will be denoted by $CMP_{PE-I}$,$CMP_{PE-II}$ and $CMP_{PE-III}$.

## 5 Comparative Study

In this section, to clarify the accuracy and efficiency of the presented methods, we apply a Monte Carlo simulation for computing the estimates of the parameters $\lambda$ and $\beta$ as well as the prediction of future order statistics $Y = X_{s:n}(s \geq r+1)$.

### 5.1 Simulation Experiment

In this section, we use an extensive Monte Carlo simulation experiment to assess the proposed estimators and predictors. We gauge the performance of the examined estimators by evaluating their bias and mean square error (MSE). These measures are defined for an estimator $\widehat{\alpha}$ of a parameter $\alpha$ as

$$bias(\widehat{\alpha}) = \frac{1}{m}\Sigma_{j=1}^{m}(\widehat{\alpha}_j - \alpha),$$

$$MSE(\widehat{\alpha}) = \frac{1}{m}\Sigma_{j=1}^{m}(\widehat{\alpha}_j - \alpha)^2,$$

The performances of the proposed point predictors are measured in terms of the biases and their mean square prediction errors (MSPEs). The bias and PMSE of a predictor $\widehat{Y}$ of $Y = T_{s:n}(s \geq r+1)$, are defined as

$$bias(\widehat{Y}) = \frac{1}{m}\Sigma_{k=1}^{m}(\widehat{Y}_k - Y),$$

and

$$MSPE(\widehat{Y}) = \frac{1}{M}\Sigma_{k=1}^{M}(\widehat{Y}_k - Y)^2,$$

respectively. However, the Monte Carlo simulation is carried out based on different sample sizes and parameter values. The simulation process is replicated $M = 2000$ times via R- software. The estimates of the estimators are computed for effective sample sizes $r = 10, 20, 30, 50, 75$ and 100. The true values of the parameters used for simulating the Fréchet model are $\lambda = 0.5, \beta = 2, \lambda = 1, \beta = 0.5$ and $\lambda = 2, \beta = 1.5$.

For the prediction part, we consider particular values of $n, r$ and $s$, and then generate Type- II censored samples from the Fréchet distribution based on the following cases: $\lambda = 0.5, \beta = 2$ and $\lambda = 0.75, \beta = 3$. Based on these random samples, we obtain the prediction biases and MSPEs of the suggested predictors. Findings on biases, MSEs and MSPEs are displayed in Tables 1-5. The following observations can be drawn from these tables. The biases of the estimators are generally small with a preference for PE-III, indicating the advantageous performance of the estimator in the bias criterion. In the sense of the MSE criterion, it has been seen that PE-III is highly competitive and outperforms other estimators in most of the considered cases, while PE-II performs very well when r is large. It can be seen that as r increases, the MSEs of all estimators decrease.

For prediction problem, based on Table 4 and 5, it is observed that the biases of the $CMP_{MLE}$ are smaller than those of the predictors for most of the considered cases. In terms of the MSPE, it is notable that the $CMP_{MLE}$ and $CMP_{PE-III}$ perform better than the MLP. However, the ratio of MSPEs of MLPs to the MSPEs of CMPs approaches 1 in many cases, especially when $s$ is close to $r$. This result can be explained by observing that the MLEs and the PMLEs of the parameters are close to the each other in some of the considered cases.

Further, as expected, for fixed values of $n$ and $r$, the MSPEs of the MLPs and CMPs increase as s increases. Indeed, the fluctuation of the variable to be predicted, as $s$ gets large, leads to this observation.

**Table 1** Biases and MSEs for the estimators with true values:$\lambda = 0.5, \beta = 2$

| r | | PE-I | | PE-II | | PE-III | |
|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE |
| 10 | $\lambda$ | 0.029 | 0.0108 | 0.0274 | 0.0101 | 0.0064 | 0.0088 |
| | $\beta$ | 0.7236 | 2.5274 | 0.8231 | 4.6047 | 0.0708 | 0.9323 |
| 20 | $\lambda$ | 0.0119 | 0.0047 | 0.0116 | 0.0045 | 0.0007 | 0.0047 |
| | $\beta$ | 0.3088 | 0.6554 | 0.3021 | 0.6381 | -0.0291 | 0.3142 |
| 30 | $\lambda$ | 0.0094 | 0.0031 | 0.0092 | 0.0029 | -0.0007 | 0.0031 |
| | $\beta$ | 0.1941 | 0.3175 | 0.1900 | 0.3162 | -0.0508 | 0.2049 |
| 50 | $\lambda$ | 0.0057 | 0.0017 | 0.0055 | 0.0016 | -0.0015 | 0.0018 |
| | $\beta$ | 0.1152 | 0.1609 | 0.1131 | 0.1562 | -0.0557 | 0.1248 |
| 75 | $\lambda$ | 0.0033 | 0.0012 | 0.0032 | 0.0011 | -0.0025 | 0.0013 |
| | $\beta$ | 0.0840 | 0.1033 | 0.0818 | 0.0987 | -0.0536 | 0.0884 |
| 100 | $\lambda$ | 0.0027 | 0.0009 | 0.0026 | 0.0008 | -0.0028 | 0.0010 |
| | $\beta$ | 0.0657 | 0.0768 | 0.0653 | 0.0727 | -0.0539 | 0.0737 |

**Table 2** Biases and MSEs for the estimators with true values:$\lambda = 1, \beta = 0.5$

| r | | PE-I | | PE-II | | PE-III | |
|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE |
| 10 | $\lambda$ | 0.6737 | 2.9733 | 0.5925 | 2.479 | 0.3994 | 2.0151 |
| | $\beta$ | 0.1677 | 0.1381 | 0.2102 | 0.2145 | 0.0162 | 0.0624 |
| 20 | $\lambda$ | 0.2728 | 0.6834 | 0.2426 | 0.5814 | 0.1367 | 0.5455 |
| | $\beta$ | 0.0738 | 0.0347 | 0.0885 | 0.0469 | -0.0112 | 0.0196 |
| 30 | $\lambda$ | 0.1638 | 0.3248 | 0.1505 | 0.2877 | 0.0882 | 0.2808 |
| | $\beta$ | 0.0474 | 0.0193 | 0.0522 | 0.0214 | -0.0113 | 0.0128 |
| 50 | $\lambda$ | 0.0922 | 0.1472 | 0.0875 | 0.1314 | 0.0401 | 0.1484 |
| | $\beta$ | 0.0324 | 0.0104 | 0.0317 | 0.0104 | -0.0014 | 0.0075 |
| 75 | $\lambda$ | 0.0609 | 0.0921 | 0.0536 | 0.0814 | 0.0087 | 0.0911 |
| | $\beta$ | 0.0199 | 0.0066 | 0.0219 | 0.0063 | -0.0125 | 0.0056 |
| 100 | $\lambda$ | 0.0291 | 0.0657 | 0.0261 | 0.0595 | -0.0135 | 0.0651 |
| | $\beta$ | 0.0149 | 0.0048 | 0.0149 | 0.0046 | -0.0138 | 0.0047 |

**Table 3** Biases and MSEs for the estimators with true values:$\lambda = 2, \beta = 1.5$

| r | | PE-I | | PE-II | | PE-III | |
|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE |
| 10 | $\lambda$ | 0.1704 | 0.3373 | 0.1621 | 0.3127 | 0.0493 | 0.2719 |
| | $\beta$ | 0.5637 | 1.3161 | 0.5819 | 2.2983 | 0.0838 | 0.6132 |
| 20 | $\lambda$ | 0.0968 | 0.1516 | 0.0942 | 0.1414 | 0.0443 | 0.1393 |
| | $\beta$ | 0.2213 | 0.3324 | 0.2347 | 0.4026 | -0.0257 | 0.1705 |
| 30 | $\lambda$ | 0.0575 | 0.0963 | 0.0564 | 0.0898 | 0.0074 | 0.0907 |
| | $\beta$ | 0.1495 | 0.1841 | 0.1465 | 0.1940 | -0.024 | 0.1221 |
| 50 | $\lambda$ | 0.0385 | 0.0533 | 0.0347 | 0.0486 | -0.0013 | 0.0522 |
| | $\beta$ | 0.0904 | 0.0953 | 0.0974 | 0.0936 | -0.0427 | 0.0674 |
| 75 | $\lambda$ | 0.0245 | 0.034 | 0.0234 | 0.0319 | -0.0085 | 0.0405 |
| | $\beta$ | 0.0468 | 0.0535 | 0.0475 | 0.0510 | -0.0469 | 0.0505 |
| 100 | $\lambda$ | 0.0161 | 0.0271 | 0.0156 | 0.0248 | -0.011 | 0.0299 |
| | $\beta$ | 0.0392 | 0.0401 | 0.0373 | 0.0380 | -0.0429 | 0.0421 |

**Table 4** Biases and MSPEs for the point predictors for the censored lifetimes

True values: $\lambda = 0.5, \beta = 2$

| (n,r) | s | MLP | | $CMP_{MLE}$ | | $CMP_{PE-I}$ | | $CMP_{PE-I}$ | | $CMP_{PE-I}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| (30,20) | 21 | -0.0496 | 0.0191 | -0.0184 | 0.0176 | -0.033 | 0.0190 | -0.0363 | 0.0191 | -0.0251 | 0.0187 |
| | 22 | -0.0613 | 0.0228 | -0.0255 | 0.0183 | -0.0638 | 0.0231 | -0.0724 | 0.0239 | -0.044 | 0.0213 |
| | 23 | -0.0803 | 0.0294 | -0.0398 | 0.0201 | -0.1049 | 0.0323 | -0.1197 | 0.0349 | -0.0701 | 0.0278 |
| | 24 | -0.0928 | 0.0386 | -0.0436 | 0.0235 | -0.1432 | 0.0466 | -0.1657 | 0.0522 | -0.0893 | 0.0364 |
| | 25 | -0.1210 | 0.0567 | -0.0614 | 0.0294 | -0.2045 | 0.0756 | -0.2362 | 0.0866 | -0.1247 | 0.0564 |
| (40,30) | 31 | -0.0444 | 0.0263 | -0.0082 | 0.0257 | -0.0215 | 0.0269 | -0.0256 | 0.0269 | -0.0131 | 0.0265 |
| | 32 | -0.0607 | 0.0275 | -0.0196 | 0.0233 | -0.0548 | 0.0278 | -0.0655 | 0.0286 | -0.033 | 0.0261 |
| | 33 | -0.0784 | 0.0383 | -0.0304 | 0.0292 | -0.0919 | 0.0401 | -0.1107 | 0.0426 | -0.0515 | 0.0351 |
| | 34 | 0.1016 | 0.0471 | 0.1585 | 0.0546 | 0.0667 | 0.0388 | 0.0394 | 0.0342 | 0.1257 | 0.0527 |
| | 35 | 0.1671 | 0.0763 | 0.2407 | 0.0926 | 0.1062 | 0.0542 | 0.0659 | 0.0441 | 0.1934 | 0.0867 |
| (50,40) | 41 | -0.0087 | 0.0254 | 0.031 | 0.0278 | 0.0188 | 0.0279 | 0.0142 | 0.0276 | 0.0272 | 0.0282 |
| | 42 | -0.5107 | 0.2948 | -0.465 | 0.2497 | -0.4974 | 0.2827 | -0.5091 | 0.2940 | -0.4756 | 0.2622 |
| | 43 | -0.0027 | 0.0421 | 0.0510 | 0.0412 | -0.0068 | 0.0421 | -0.0273 | 0.0417 | 0.0355 | 0.0457 |
| | 44 | -0.1285 | 0.0677 | -0.0638 | 0.0474 | -0.1477 | 0.0711 | -0.1797 | 0.0804 | -0.088 | 0.0613 |
| | 45 | -0.0127 | 0.0591 | 0.0929 | 0.0535 | -0.0268 | 0.0592 | -0.0726 | 0.0587 | 0.0589 | 0.0674 |

**Table 5** Biases and MSPEs for the point predictors for the censored lifetimes

True values: $\lambda = 0.75, \beta = 3$

| (n,r) | s | MLP | | $CMP_{MLE}$ | | $CMP_{PE-I}$ | | $CMP_{PE-I}$ | | $CMP_{PE-I}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| (30,20) | 21 | -0.0448 | 0.0125 | -0.0183 | 0.0110 | -0.0307 | 0.0121 | -0.0336 | 0.0123 | -0.0245 | 0.0118 |
| | 22 | -0.0292 | 0.0145 | 0.0580 | 0.0155 | 0.0255 | 0.0142 | 0.0181 | 0.0136 | 0.043 | 0.0157 |
| | 23 | -0.0653 | 0.0203 | -0.0328 | 0.0136 | -0.0877 | 0.0224 | -0.1003 | 0.0243 | 0.0579 | 0.0191 |
| | 24 | -0.0750 | 0.0246 | -0.0372 | 0.0143 | -0.1187 | 0.0311 | -0.1376 | 0.0351 | -0.0765 | 0.0238 |
| | 25 | -0.0978 | 0.0334 | -0.0533 | 0.0169 | -0.1693 | 0.0482 | -0.1957 | 0.0564 | -0.1045 | 0.0337 |
| (40,30) | 31 | -0.0451 | 0.0171 | -0.0161 | 0.0157 | -0.027 | 0.0167 | -0.0302 | 0.0168 | -0.02 | 0.0164 |
| | 32 | -0.0466 | 0.0193 | -0.0151 | 0.0162 | -0.0424 | 0.0193 | -0.0512 | 0.0198 | -0.0256 | 0.0183 |
| | 33 | -0.0670 | 0.0215 | -0.0313 | 0.0154 | -0.0792 | 0.0229 | -0.093 | 0.0249 | -0.049 | 0.0195 |
| | 34 | -0.0758 | 0.0286 | -0.0343 | 0.0185 | -0.1071 | 0.0325 | -0.1280 | 0.0368 | -0.0590 | 0.0264 |
| | 35 | -0.0999 | 0.0362 | -0.0499 | 0.0205 | -0.1502 | 0.0466 | -0.1800 | 0.0557 | -0.0854 | 0.0330 |
| (50,40) | 41 | -0.0488 | 0.0178 | -0.0181 | 0.0163 | -0.0276 | 0.0172 | -0.0312 | 0.0173 | -0.0212 | 0.0168 |
| | 42 | -0.0521 | 0.0210 | -0.0179 | 0.0179 | -0.0437 | 0.0206 | -0.0526 | 0.0214 | -0.0253 | 0.0194 |
| | 43 | -0.0718 | 0.0230 | -0.0331 | 0.0168 | -0.0764 | 0.0238 | -0.0916 | 0.0261 | -0.0455 | 0.0201 |
| | 44 | -0.0808 | 0.0302 | -0.0358 | 0.0204 | -0.0996 | 0.0336 | -0.1227 | 0.0380 | -0.0553 | 0.0264 |
| | 45 | -0.098 | 0.0385 | -0.044 | 0.0231 | -0.1350 | 0.0452 | -0.1676 | 0.0541 | -0.0688 | 0.0345 |

## *5.2 Real-Life Example*

In this subsection, a real dataset is analyzed to show the effectiveness of the estimators and predictors proposed in Sections 3 and 4. This dataset has been considered by Ramos et al. [5]. The dataset contains the minimum flows of water in August from 1960 to 2014 to the Piracicaba River in Brazil. This study can be useful to protect and maintain aquatic resources for Sao Paulo state in Brazil. It was shown by Ramos et al. [5] that the Fréchet distribution is a suitable model for analyzing this data set. The data are recorded as follows:

| | |
|---|---|
| Flows | 16.00 , 9.52 , 9.43 , 53.72 , 17.10 , 8.52 , 10.00 , 15.23 , 8.78 , 28.97 , 28.06 , 18.26 , 9.69 , 51.43 , 10.96 , 13.74 , 20.01 , 10.00 , 12.46 , 10.40 , 26.99 , 7.72 , 11.84 , 18.39 , 11.22 , 13.10 , 16.58 , 12.46 , 58.98 , 7.11 , 11.63 , 8.24 , 9.80 , 15.51 , 37.86 , 30.20 , 8.93 , 14.29 , 12.98 , 12.01 , 6.80. |

As true parameters are unknown in real-life dataset, the biases and the MSEs cannot be used to assess estimators' performance in such cases. Thus, we use the Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) as performance measures for comparing the existing estimators. These measures are given by

$$MAE = \frac{\sum_{i=1}^{n} |S(x_i) - \widehat{F}(x_i)|}{n},$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n} \{S(x_i) - \widehat{F}(x_i)\}^2}{n}},$$

where,

$$S(x_i) = \frac{\textit{number of elements in the sample} \leq x}{n} = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \leq x},$$

and

$$\widehat{F}(x_i) = e^{\left(\frac{x_i}{\lambda}\right)^{-\widehat{\beta}}},$$

with parameter estimates ($\widehat{\lambda}$ and $\widehat{\beta}$) based on the presented methods. The outcomes gained by applying the suggested estimation techniques are displayed in Table 6. Here, we have also considered the MLEs of the parameters $\lambda$ and $\beta$ for comparison purposes. To show the efficiency of the Fréchet distribution taken with the estimation methods, Fig. 1 represents the histogram of the data points and the estimated pdfs based on the presented methods in this study. Moreover the true CDF of the data is plotted in Fig. 2, along with the estimated CDFs. Table 6 indicates the efficiency of PE-III in comparison to other modified percentile estimators and the MLE. In fact, all performance measures for the PE-III were observed to be smaller than for the other estimators.

In the context of prediction, we assume that the data ended when the 25-th flow is observed, i.e., we observe a Type-II censored sample with $n = 41, r = 25$. So, we aim to obtain point predictors of the unobserved temperatures $Y = X_{s:n\prime}$ $s = 26, ..., 32$. Point predictors are displayed in Table 7. It can be observed that all point predictors are close to each other's, specially, the CMPs. It can be observed also that the point predictors underestimated the new order statistics when s gets large, the reason is that the variation of $Y = X_{s:n}$ tends to be high as Y moves away from the observed flows. It can be seen that all the point predictors are close to the true values, but the MLP and $CMP_{PE-III}$ obtained have the best performance.

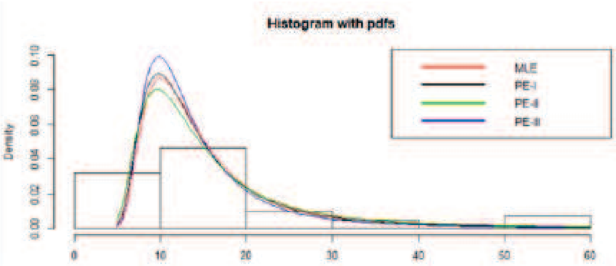


**Fig. 1** Plots of the histogram and the estimated pdfs.

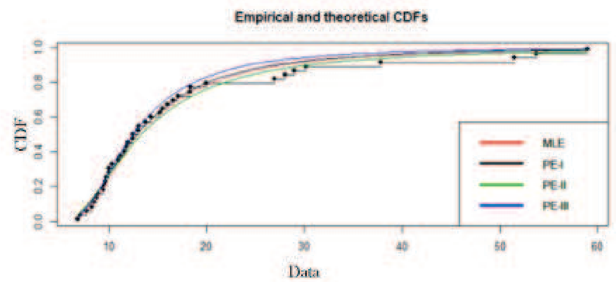


**Fig. 2** The empirical CDF(dots)and the estimated CDFs based on the suggested methods.

**Table 6** The values of the estimates for the real data example, in addition to MAE and RMSE

| Method | $\widehat{\lambda}$ | $\widehat{\beta}$ | MAE | RMSE |
|---|---|---|---|---|
| MLE | 11.456 | 2.516 | 0.0321 | 0.0014 |
| PE-I | 11.152 | 2.527 | 0.0211 | 0.0009 |
| PE-II | 11.322 | 2.263 | 0.0329 | 0.0014 |
| PE-III | 10.922 | 2.782 | 0.0207 | 0.0008 |

**Table 7** Point predictors for new $Y = X_{s:n}$.

| s | True value | MLP | $CMP_{MLE}$ | $CMP_{PE-I}$ | $CMP_{PE-II}$ | $CMP_{PE-III}$ |
|---|---|---|---|---|---|---|
| 26 | 15.23 | 14.290 | 14.475 | 14.431 | 14.405 | 14.520 |
| 27 | 15.51 | 14.761 | 14.756 | 14.643 | 14.579 | 14.869 |
| 28 | 16.00 | 15.265 | 15.063 | 14.875 | 14.769 | 15.253 |
| 29 | 16.58 | 15.81 | 15.400 | 15.128 | 14.975 | 15.672 |
| 30 | 17.1 | 16.603 | 15.766 | 15.405 | 15.201 | 16.134 |
| 31 | 18.26 | 17.054 | 16.173 | 15.710 | 15541 | 16.648 |
| 32 | 18.39 | 17.778 | 16.629 | 16.309 | 15.727 | 17.226 |

## 6 Conclusion

In this study, we have explored percentile estimation techniques for Fréchet distribution parameters, introducing two alternative percentile estimation approaches. These modifications are founded on the geometric mean and the empirical cumulative distribution function estimation for the first order statistic of the Fréchet distribution. Additionally, we have delved into the prediction of future order statistics following the Fréchet distribution, based on the observed data. Through comprehensive comparisons involving simulation studies and real-life example, we have discerned notable advantages. Specifically, the modified percentile estimator relying on the empirical cumulative distribution function for the first-order statistic demonstrates superior performance over traditional percentile estimators in terms of mean squared error and bias criteria. Furthermore, our investigation reveals that the conditional median predictor, as employed in the second modified percentile estimator, outperforms not only other percentile-based estimators but also the maximum-likelihood prediction method. Taking into account these compelling simulation results and practical applications, we recommend adopting the modified percentile estimation techniques for parameter estimation in the context of the Fréchet distribution. While our findings have primarily focused on Type-II censored data, it is worth noting that the techniques presented here can readily extend to other censoring scenarios, including Type-I, hybrid, or progressive censoring.

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