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Solution of Conformable Volterra's Population Growth Model via Analytical and Numerical Approaches

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Abstract: In this paper, we aim to consider a conformable Volterra's population growth model which is a nonlinear integro-differential equation that represents population growth of a species in a closed system. We investigate an analytic solution in the form of rapidly convergent fractional power series whose coefficients are obtained depending on minimizing the residual function that related to the equation under study.
The approximate solution for the conformable Volterra's population growth model is presented by plotting its curves for different orders of the conformable derivative and for different values of the equation parameters. Numerical values for the residual function are tabulated to prove the efficiency and accuracy of our proposed algorithm. Moreover, the method of successive substitution is carried out to the same model in order to compare its results to those of the residual power series method, so we can show the validity and high accuracy of the proposed technique.

Keywords: Volterra's growth model, conformable derivative, fractional integro-differential equation, fractional residual power series, successive substitution method.

1 Introduction

The study of population growth models attracted the attention of many scientist since long time ago because of the limitation of resources. The first systematic description of population growth was by Malthus [1] who assumed that the time path of a quantity $P(\tau)$ and its growth rate $P'(\tau)$ are proportional and got the differential equation (DE):

$$P'(\tau) = \mu P(\tau), \tag{1}$$

where μ is a constant rate of growth. So, we obtain exponential growth $P(\tau) = p_0 e^{\mu\tau}$, where p_0 is the initial value of the growth. This equation may be reasonable for a young country as Malthus stated in 1789. But Alphonse Quetelet (1795-1874) didn't agree with this exponential growth since it eventually leads to impossible values. He asked his student Pierre-Francois Verhulst (1804-1849) to investigate in this problem. Verhulst [2] suggested to add a term that represents the increasing resistance to growth and got the logistic equation

$$P'(\tau) = \mu P(\tau) - \Psi(P(\tau)). \tag{2}$$

Many experiments on this equations revealed that Ψ can be chosen so that the logistic equation become quadratic as

$$P'(\tau) = \mu P(\tau) - \rho P^2(\tau), \tag{3}$$

where μ represents the birth rate and ρ is the crowding coefficient [3].
Although logistic model in (3) has been successfully applied in many situations, it is still not enough to describe accumulated toxicity. Consequently, an integral term was added to the logistic equation to indicate the "total

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metabolism" or total amount of toxins produced since time zero. So, the population growth can be modeled by the nonlinear Volterra integro-differential equation (VIDE):

$$P'(\tau) = \mu P(\tau) - \rho P^2(\tau) - \gamma P(\tau)\int_0^\tau P(\lambda)d\lambda. \quad (4)$$

Here, γ denotes relative size of the sensitivity to toxins and it specifies the manner in which the population thrives before its decay [4]. Clearly, setting $\gamma = 0$, the quadratic logistic equation (3) appears. Moreover, the individual death rate is corresponding to this integral, so the population death rate by virtue of toxicity must include a factor P.

On the other hand, throughout the last few decades, many scholars have focused their attention on fractional DEs and fractional integro-differential equations (IDEs) due to their importance in the context of mathematical modeling in a broad range of scientific domains such as in ecology [5], electrochemistry [6], biology [7], fluid mechanics [8], and many other fields. However, there is no unique definition for fractional operators. One can find many different approaches for fractional derivatives and fractional integrals. The most popular definitions are the Caputo derivative, Riemann-Liouville integral and differential operators that enjoy some features [9]. Anyhow, both have singular kernel functions which reduces their efficiency in modeling real-world problems. This disadvantage urged mathematicians to think of new fractional derivatives. Among the recent fractional operators is a simple fractional derivative called "the conformable derivative" [10] that depends on the basic limit definition of the derivative. It is characterized by its similarity with the classical integer order derivative in most properties such as product and chain rules which attracted the interest of many researchers who adopted it in their study of mathematical models. For example, tumor-immune system interaction was considered in conformable model in [11], the conformable derivative was generalized and its physical and geometrical interpretations were given which proved its potential in physics and engineering in [12]. In [13], a modified nonlinear conformable Schrödinger equation was investigated. Moreover, the solution of a conformable two-compartment pharmacokinetic model was discussed in [14].

In this study, we are interested with conformable Volterra's population growth model (CVPGM) of the form

$${}_\tau T_0^\sigma P(\tau) = \mu P(\tau) - \rho P^2(\tau) - \gamma P(\tau)(J_0^\sigma P)(\tau). \quad (5)$$

subject to the initial condition (IC)

$$P(0) = p_0, \quad p_0 > 0, \quad (6)$$

where ${}_\tau T_0^\sigma$ and J_0^σ in (5) represents the conformable derivative and integral of order $0 < \sigma \leq 1$, respectively. Notice that we change the classical integral in (4) to conformable integral in (5) so that we keep the symmetry aspect. That is, both derivative and integral have the same orders.

Now, if we introduce the non-dimensional variables into (5) and (6) as

$$t = \frac{\gamma\tau}{\rho}, \quad w = \frac{\rho P}{\mu},$$

then we get the non-dimensional CVPGM with non-dimensional parameter $\delta = \frac{\gamma}{\rho\mu}$ as

$$\delta({}_t T_0^\sigma w)(t) = w(t) - w^2(t) - w(t)(J_0^\sigma w)(t), \quad (7)$$

subject to

$$w(0) = w_0, \quad w_0 > 0. \quad (8)$$

Generally, studying the exact dynamical behavior for nonlinear systems is not an easy task especially with attempt to investigate the fractional effect of these systems. As a result, several powerful techniques have been modified to approximate their solutions. Among these methods are the differential transform method [15], the reproducing kernel Hilbert space method [16], Laplace Adam-Bashforth method [17], the variational iteration method and the Adomian decomposition method [18], Jacobi elliptic equation method [19] and series method [20]. In 2013, Abu Arqub developed a simple analytic technique to determine the coefficients of power series solutions for a class of fuzzy differential equation [21]. It is known as the residual power series method (RPSM) and have been proved its efficiency in many consequent researches. Some applications for this efficient approach are solving time-fractional Schrödinger equations in [22], solving time fractional reaction–diffusion equations in [23], and obtaining approximate solution for the fractional SIR epidemic model [24].

In this paper, we employ the RPSM for solving the nonlinear CVPGM in (7) and (8) then we apply the method of successive substitution (SS) to demonstrate the validity and accuracy of our method. This paper is structured as follows. In the following section, we give a quick review to some concepts of conformable derivative and fractional power series (FPS) which are necessary for establishing our results. Section 3 is devoted to present a description of the RPSM when applied to CVPGM, while the method of SS is employed in Section 4. The proposed techniques are carried out to obtain analytic and approximate solutions for different values of conformable orders through some graphical and numerical results for the population growth and the residual functions in Section 5. Finally, this paper is ended by a summarized conclusion.

2 Preliminaries

In this section, we provide definitions of conformable derivative and conformable integral with some of their characteristics that will be used in the rest of the paper. Then we present the FPS concept with some related results.

Definition 1.*[10] For a function $w:(0,\infty)\to\Re$, the conformable derivative of order $0<\sigma\leq 1$ of $w(t)$ at $t>0$ is defined by*

$$ {}_tT_0^{\sigma}w(t) = lim_{h\to 0}\frac{w(t+ht^{1-\sigma})-w(t)}{h}. \tag{9}$$

If the conformable derivative of w of order σ exists, then we say that w is σ-differentiable. Moreover, if w is σ-differentiable in some interval $(0,\varepsilon)$, $\varepsilon>0$ then we define $({}_tT_0^{\sigma}w)(0) = lim_{t\to 0^+}({}_tT_0^{\sigma}w)(t)$ provided that $lim_{t\to 0^+}({}_tT_0^{\sigma}w)(t)$ exists.

Theorem 1.*[10] Let $\sigma\in(0,1]$ and w be $\sigma-$differentiable function at $t>0$ then $({}_tT_0^{\sigma}w)(t)=t^{1-\sigma}w(t)$, where w is differentiable.*

Definition 2.*[10] For $\sigma\in(\aleph,\aleph+1]$ and $\aleph-$differentiable function w at t where $t>0$, the conformable derivative of w of order σ is defined by*

$$ {}_tT_0^{\sigma}(w)(t) = lim_{h\to 0}\frac{w^{(\lceil\sigma\rceil-1)}(t+ht^{(\lceil\sigma\rceil-\sigma)})-w^{(\lceil\sigma\rceil-1}(t)}{h} \tag{10}$$

where $\lceil\sigma\rceil$ is smallest integer such that $\lceil\sigma\rceil\geq\sigma$.

*Remark.*Let $\sigma\in(\aleph,\aleph+1]$ and w is $(\aleph+1)-$differentiable at $t>0$. Then

$$ {}_tT_0^{\sigma}(w)(t) = t^{(\lceil\sigma\rceil-\sigma)}w^{\lceil\sigma\rceil}(t).$$

Definition 3.*[10] The conformable integral of order $\sigma\in(0,1]$ of a function $w:(0,\infty)\to\Re$ is given by*

$$ J_0^{\sigma}(w)(t) = \int_0^t \frac{w(\lambda)}{\lambda^{1-\sigma}}d\lambda$$

where the integral in the right hand side is the classical Riemann integral.

Theorem 2.*[10]For any continuous function w in the domain of J_0^{σ}, we have*

$$ {}_tT_0^{\sigma}J_0^{\sigma}(w)(t) = w(t), \quad t\geq 0.$$

Definition 2 and consequently Theorem 2 were generalized in [25] for any fractional order $\sigma>0$ as follows.

Definition 4.*[25] For $\sigma\in(\aleph-1,\aleph]$, $\aleph\in\mathbb{N}$, the conformable integral of order $\sigma>0$ of $w(t):[a,\infty)\to\Re$ is*

$$ (J_a^{\sigma}w)(t) = \frac{1}{(\aleph-1)!}\int_a^t \frac{(t-\lambda)^{\aleph-1}w(\lambda)}{(\lambda-a)^{\aleph-\sigma}}d\lambda, \quad t>\lambda\geq a\geq 0. \tag{11}$$

Theorem 3.*[25] For $\sigma\in(\aleph-1,\aleph]$, $\aleph\in\mathbb{N}$, the following properties are satisfied for an $\aleph-$times differentiable function $w(t):[a,\infty)\to\Re$.*

1. ${}_tT_a^{\sigma}(J_a^{\sigma}w(t)) = w(t)$.
2. $J_a^{\sigma}({}_tT_a^{\sigma}w(t)) = w(t) - \sum_{k=0}^{\aleph-1}\frac{w^{(k)}(a)(t-a)^k}{k!}$.

Now, we give the definition of FPS about $t=0$ that is basic in our current work.

Definition 5.[26] *The FPS about $t=0$ has the form*

$$\sum_{k=0}^{\infty} a_k t^{k\sigma} = a_0 + a_1 t^{\sigma} + a_2 t^{2\sigma} + \cdots, \quad 0 \le \aleph - 1 < \sigma \le \aleph, \aleph \in \mathbb{N}, t \ge 0. \tag{12}$$

Theorem 4.[26] *For the FPS $\sum_{k=0}^{\infty} C_k t^{k\sigma}, \quad t \ge 0$, we have two cases:*
Case1: If the FPS $\sum_{k=0}^{\infty} C_k t^{k\sigma}$ converges when $t=\beta>0$, then it converges for $0 \le t < \beta$.
Case 2: If the FPS $\sum_{k=0}^{\infty} C_k t^{k\sigma}$ diverges when $t=\beta>0$, then it diverges for $t>\beta$.

Theorem 5.[26] *For the FPS $\sum_{k=0}^{\infty} C_k t^{k\sigma}, \quad t \ge 0$, there are only three possibilities:*

1. *The series converges only when $t=0$*
2. *The series converges for all $t \ge 0$*
3. *There is a number $\vartheta > 0$ such that the series converges if $0 \le t < \vartheta$ and diverges if $t > \vartheta$. The number ϑ is called the radius of convergence of the FPS.*

Theorem 6.[27] *Suppose that the FPS $w(t) = \sum_{k=0}^{\infty} C_k t^{k\sigma}$ has radius of convergence $\vartheta > 0$ for all $0 \le t < \vartheta$. Then $w(t)$ is infinitely $\sigma-$differentiable over the interval $[0,\vartheta)$ with $C_k = \frac{({}^{(k)}{}_tT_0^{\sigma} w)(0)}{\sigma^k k!}$, where ${}^{(k)}{}_tT_0^{\sigma} w = {}_tT_0^{\sigma}({}_tT_0^{\sigma}({}_tT_0^{\sigma} \cdots {}_tT_0^{\sigma}(w)))$ ($k-$times).*

Theorem 7.[25]*Let $w(t):[0,\infty) \to \Re$ has infinite conformable derivatives ${}^{(k)}{}_tT_0^{\sigma} w(t), \quad k=0,1,2,\cdots$ for $0<\sigma\le 1$ over a neighborhood of zero and $w(t)$ has the FPS form in (12) about $t=0$ with radius of convergence $\vartheta>0$. If $|{}^{(\aleph+1)}{}_tT_0^{\sigma} w(t)| \le \xi$ for some $\aleph \in \mathbb{N}$ and $\xi > 0$, then $\forall t \in (0,\vartheta)$, the error term of the FRP has the form*

$$|\varepsilon_{\aleph}^{\sigma}(t)| \le \frac{\xi}{(\aleph+1)!\sigma^{\aleph+1}} t^{(\aleph+1)\sigma}, \tag{13}$$

where $\varepsilon_{\aleph}^{\sigma}(t) = \sum_{k=\aleph+1}^{\infty} \frac{{}^{\aleph}_{t}T_0^{\sigma} w(0)}{k!\sigma^k} t^{k\sigma}$.

Depending on the previous theorem, we may consider

$$w_{\aleph}(t) = \sum_{k=0}^{\aleph} \frac{({}^{(k)}{}_tT_0^{\sigma} w)(0)}{\sigma^k k!} t^{k\sigma}$$

as an approximation of $w(t)$ with truncation error $\varepsilon_{\aleph}^{\sigma}(t) = \sum_{k=\aleph+1}^{\infty} \frac{{}^{(\aleph)}{}_tT_0^{\sigma} w(0)}{k!\sigma^k} t^{k\sigma}$ and the upper bound of this truncated error can be estimated as

$$|\varepsilon_{\aleph}^{\sigma}(t)| \le \left|\frac{\xi \vartheta^{(\aleph+1)\sigma}}{(\aleph+1)!\sigma^{\aleph+1}}\right|$$

provided that

$$|{}^{(\aleph+1)}{}_tT_0^{\sigma} w(t)| \le \xi \quad \text{on} \quad [0,\vartheta).$$

3 Fractional Residual Power Series Method for Solving CVPGM

In this section, we give a summarized description for fractional residual power series method (FRPSM) in order to obtain analytic and approximate solutions for the CVPGM in (7) and (8). This will be done through replacing the unknown growth function by its FPS expansion and minimizing the residual function that corresponds to the equation. This approach is simply applied to investigate solutions of linear and nonlinear DEs and IDEs without linearization, perturbation, or discretization. Unlike the classical power series approach, the FRPSM neither requires comparing the identical terms' coefficients nor is a recursion relation needed as well. Besides that, the FRPSM calculates the power series' coefficients through a chain of equations. The FRPS approach provides the solution in terms of convergent FPS with easy exchangeable components. It is considered as an efficient, viable and easy technique.
To apply FRPSM to solve the CVPGM in (7) and (8), we begin by assuming that the solution has a FPS in the form

$$w(t) = \sum_{i=0}^{\infty} C_i t^{i\sigma} \tag{14}$$

Substituting the IC of (8) into (14), we get $C_0 = w_0$. So, the FPS solution can be rewritten as

$$w(t) = w_0 + \sum_{i=1}^{\infty} C_i t^{i\sigma} \tag{15}$$

Now, the solution can be approximated by the $\aleph$th-truncated series as follows

$$w_{\aleph}(t) = w_0 + \sum_{i=1}^{\aleph} C_i t^{i\sigma}. \tag{16}$$

The main idea of the FRPSM is to define the residual function as follows

$$Resid_w(t) = \delta({}_tT_0^{\sigma} w)(t) - w(t) + w^2(t) + w(t)(J_0^{\sigma} w)(t). \tag{17}$$

And the following $\aleph$th-residual function

$$Resid_{w,\aleph}(t) = \delta({}_tT_0^{\sigma} w_{\aleph})(t) - w_{\aleph}(t) + w_{\aleph}^2(t) + w_{\aleph}(t)(J_0^{\sigma} w_{\aleph})(t). \tag{18}$$

Clearly, $Resid_w(t) = lim_{\aleph\to\infty} Resid_{w,\aleph}(t) = 0$ for $t \geq 0$. Consequently, ${}_tT_0^{\sigma} Resid_w(t) = 0$ for $t \geq 0$. And more generally,

$$({}_t^{(j)}T_0^{\sigma} Resid_w)(0) = ({}_t^{(j)}T_0^{\sigma} Resid_{w,\aleph})(0) = 0 \quad \text{for each} \quad j = 0, 1, 2, \ldots, \aleph - 1.$$

In fact, these formulas give us a simple technique to compute the coefficients of the FPS in (15) manually. More precisely, we have the following equations

$$({}_t^{(j)}T_0^{\sigma} Resid_{w,\aleph})(0) = 0 \quad \text{for each} \quad j = 0, 1, 2, \ldots, \aleph - 1. \tag{19}$$

to determine the values of coefficient $C_{\aleph}$. Thus the analytic solution for the CVPGM has been completely constructed. To see this, let us apply this technique for obtaining some coefficients. To determine the value of first unknown coefficient, C_1, we substitute $\aleph = 1$ in equations (16), (18) and (19) and get

$$\begin{aligned}
w_1(t) &= w_0 + C_1 t^{\sigma}, \\
Resid_{w,1}(t) &= \delta({}_tT_0^{\sigma} w_1)(t) - w_1(t) + w_1^2(t) + w_1(t) \int_0^t \frac{w_1(\lambda)}{\lambda^{1-\sigma}} d\lambda, \\
&= \delta({}_tT_0^{\sigma}(w_0 + C_1 t^{\sigma})) - (w_0 + C_1 t^{\sigma}) + (w_0 + C_1 t^{\sigma})^2 + (w_0 + C_1 t^{\sigma}) \int_0^t \frac{(w_0 + C_1 \lambda^{\sigma})}{\lambda^{1-\sigma}} d\lambda \\
&= -t^{\sigma} C_1 + \delta\sigma C_1 + t^{2\sigma} C_1^2 + \frac{t^{3\sigma} C_1^2}{2\sigma} - w_0 + 2t^{\sigma} C_1 w_0 + \frac{3t^{2\sigma} C_1 w_0}{2\sigma} + w_0^2 + \frac{t^{\sigma} w_0^2}{\sigma}.
\end{aligned}$$

Now, at $t = 0$, $Resid_{w,1}(0) = \delta\sigma C_1 + (-1 + w_0)w_0 = 0$ which produces

$$C_1 = \frac{w_0 - w_0^2}{\delta\sigma}.$$

To find the value of second coefficient in the FPS, we substitute again in equations (16), (18) and (19)but with $\aleph = 2$ to get second truncated series:

$$w_2(t) = w_0 + \frac{w_0 - w_0^2}{\sigma\delta} t^{\sigma} + C_2 t^{2\sigma}$$

and the second residual function will be

$$\begin{aligned}
Resid_{w,2}(t) &= \delta({}_tT_0^{\sigma} w_2)(t) - w_2(t) + w_2^2(t) + w_2(t) \int_0^t \frac{w_2(\lambda)}{\lambda^{1-\sigma}} d\lambda, \\
&= -t^{2\sigma} C_2 + 2t^{\sigma}\delta\sigma C_2 + t^{4\sigma} C_2^2 + \frac{t^{5\sigma} C_2^2}{3\sigma} - \frac{t^{\sigma} w_0}{\sigma\delta} + 2t^{2\sigma} C_2 w_0 + \frac{5t^{4\sigma} C_2 w_0}{6\delta\sigma^2} + \frac{4t^{3\sigma} C_2 w_0}{3\sigma} + \frac{2t^{3\sigma} C_2 w_0}{\delta\sigma} + \frac{t^{3\sigma} w_0^2}{2\delta^2\sigma^3} \\
&+ \frac{t^{2\sigma} w_0^2}{\delta^2\sigma^2} + \frac{3t^{2\sigma} w_0^2}{2\delta\sigma^2} + \frac{t^{\sigma} w_0^2}{\sigma} + \frac{3t^{\sigma} w_0^2}{\delta\sigma} - \frac{5t^{4\sigma} C_2 w_0^2}{6\delta\sigma^2} - \frac{2t^{3\sigma} C_2 w_0^2}{\delta\sigma} - \frac{t^{3\sigma} w_0^3}{\delta^2\sigma^3} - \frac{2t^{2\sigma} w_0^3}{\delta^2\sigma^2} - \frac{3t^{2\sigma} w_0^3}{2\delta\sigma^2} - \frac{2t^{\sigma} w_0^3}{\delta\sigma} + \frac{t^{3\sigma} w_0^4}{2\delta^2\sigma^3} \\
&+ \frac{t^{2\sigma} w_0^4}{\delta^2\sigma^2}.
\end{aligned}$$

To use $(T_0^\sigma Resid_{w,2})(0)=0$, we first compute the conformable derivative of the second residual function as

$$
\begin{aligned}
(T_0^\sigma Resid_{w,2})(t) &= t^{1-\sigma}\Big(\frac{d}{dt}\Big(-t^{2\sigma}C_2+2t^{\sigma}\delta\sigma C_2+t^{4\sigma}C_2^2+\frac{t^{5\sigma}C_2^2}{3\sigma}-\frac{t^{\sigma}w_0}{\sigma\delta}+2t^{2\sigma}C_2w_0+\frac{5t^{4\sigma}C_2w_0}{6\delta\sigma^2}+\frac{4t^{3\sigma}C_2w_0}{3\sigma}\\
&+\frac{2t^{3\sigma}C_2w_0}{\delta\sigma}+\frac{t^{3\sigma}w_0^2}{2\delta^2\sigma^3}+\frac{t^{2\sigma}w_0^2}{\delta^2\sigma^2}+\frac{3t^{2\sigma}w_0^2}{2\delta\sigma^2}+\frac{t^{\sigma}w_0^2}{\sigma}+\frac{3t^{\sigma}w_0^2}{\delta\sigma}-\frac{5t^{4\sigma}C_2w_0^2}{6\delta\sigma^2}-\frac{2t^{3\sigma}C_2w_0^2}{\delta\sigma}-\frac{t^{3\sigma}w_0^3}{\delta^2\sigma^3}\\
&-\frac{2t^{2\sigma}w_0^3}{\delta^2\sigma^2}-\frac{3t^{2\sigma}w_0^3}{2\delta\sigma^2}-\frac{2t^{\sigma}w_0^3}{\delta\sigma}+\frac{t^{3\sigma}w_0^4}{2\delta^2\sigma^3}+\frac{t^{2\sigma}w_0^4}{\delta^2\sigma^2}\Big)\Big)\\
&=-2t^{\sigma}\sigma C_2+2\delta\sigma^2C_2+\frac{5}{3}t^{4\sigma}C_2^2+4t^{3\sigma}\sigma C_2^2-\frac{w_0}{\delta}+4t^{2\sigma}C_2w_0+\frac{6t^{2\sigma}C_2w_0}{\delta}+\frac{10t^{3\sigma}C_2w_0}{3\delta\sigma}+4t^{\sigma}\sigma C_2w_0\\
&+w_0^2+\frac{3w_0^2}{\delta}+\frac{3t^{2\sigma}w_0^2}{2\delta^2\sigma^2}+\frac{2t^{\sigma}w_0^2}{\delta^2\sigma}+\frac{3t^{\sigma}w_0^2}{\delta\sigma}-\frac{6t^{2\sigma}C_2w_0^2}{\delta}-\frac{10t^{3\sigma}C_2w_0^2}{3\delta\sigma}-\frac{2w_0^3}{\delta}-\frac{3t^{2\sigma}w_0^3}{\delta^2\sigma^2}-\frac{4t^{\sigma}w_0^3}{\delta^2\sigma}\\
&-\frac{3t^{\sigma}w_0^3}{\delta\sigma}+\frac{3t^{2\sigma}w_0^4}{2\delta^2\sigma^2}+\frac{2t^{\sigma}w_0^4}{\delta^2\sigma}.
\end{aligned}
$$

Consequently,

$$(T_0^\sigma Resid_{w,2})(0)=2\delta\sigma^2C_2+w_0^2+\frac{3w_0^2}{\delta}-\frac{2w_0^3}{\delta}=0.$$

Hence,

$$C_2=\frac{w_0-3w_0^2-\delta w_0^2+2w_0^3}{2\delta^2\sigma^2}.$$

Again, we can compute C_3 through similar successive steps for $\aleph=3$.

$$w_3(t)=w_0+\frac{w_0-w_0^2}{\sigma\delta}t^{\sigma}+\frac{w_0-3w_0^2-\delta w_0^2+2w_0^3}{2\delta^2\sigma^2}t^{2\sigma}+C_3t^{3\sigma},$$

$$Resid_{w,3}(t)=\delta({}_tT_0^\sigma w_3)(t)-w_3(t)+w_3^2(t)+w_3(t)\int_0^t\frac{w_3(\lambda)}{\lambda^{1-\sigma}}d\lambda,$$

$$
\begin{aligned}
(T_0^\sigma Resid_{w,3})(t) &= -t^{3\sigma}C_3+3t^{2\sigma}\delta\sigma C_3+t^{6\sigma}C_3^2+\frac{t^{7\sigma}C_3^2}{4\sigma}-\frac{t^{2\sigma}w_0}{2\delta^2\sigma^2}+2t^{3\sigma}C_3w_0+\frac{7t^{6\sigma}C_3w_0}{24\delta^2\sigma^3}+\frac{t^{5\sigma}C_3w_0}{\delta^2\sigma^2}\\
&+\frac{3t^{5\sigma}C_3w_0}{4\delta\sigma^2}+\frac{5t^{4\sigma}C_3w_0}{4\sigma}+\frac{2t^{4\sigma}C_3w_0}{\delta\sigma}+\frac{t^{5\sigma}w_0^2}{12\delta^4\sigma^5}+\frac{t^{4\sigma}w_0^2}{4\delta^4\sigma^4}+\frac{5t^{4\sigma}w_0^2}{12\delta^3\sigma^4}+\frac{t^{3\sigma}w_0^2}{\delta^3\sigma^3}+\frac{7t^{3\sigma}w_0^2}{6\delta^2\sigma^3}\\
&+\frac{7t^{2\sigma}w_0^2}{2\delta^2\sigma^2}+\frac{2t^{2\sigma}w_0^2}{\delta\sigma^2}-\frac{7t^{6\sigma}C_3w_0^2}{8\delta^2\sigma^3}-\frac{7t^{6\sigma}C_3w_0^2}{24\delta\sigma^3}-\frac{3t^{5\sigma}C_3w_0^2}{\delta^2\sigma^2}-\frac{7t^{5\sigma}C_3w_0^2}{4\delta\sigma^2}-\frac{2t^{4\sigma}C_3w_0^2}{\delta\sigma}\\
&-\frac{t^{5\sigma}w_0^3}{2\delta^4\sigma^5}-\frac{t^{5\sigma}w_0^3}{6\delta^3\sigma^5}-\frac{3t^{4\sigma}w_0^3}{2\delta^4\sigma^4}-\frac{13t^{4\sigma}w_0^3}{6\delta^3\sigma^4}-\frac{5t^{4\sigma}w_0^3}{12\delta^2\sigma^4}-\frac{4t^{3\sigma}w_0^3}{\delta^3\sigma^3}-\frac{4t^{3\sigma}w_0^3}{\delta^2\sigma^3}-\frac{2t^{3\sigma}w_0^3}{3\delta\sigma^3}\\
&-\frac{6t^{2\sigma}w_0^3}{\delta^2\sigma^2}-\frac{5t^{2\sigma}w_0^3}{2\delta\sigma^2}+\frac{7t^{6\sigma}C_3w_0^3}{12\delta^2\sigma^3}+\frac{2t^{5\sigma}C_3w_0^3}{\delta^2\sigma^2}+\frac{13t^{5\sigma}w_0^4}{12\delta^4\sigma^5}+\frac{t^{5\sigma}w_0^4}{2\delta^3\sigma^5}+\frac{t^{5\sigma}w_0^4}{12\delta^2\sigma^5}\\
&+\frac{13t^{4\sigma}w_0^4}{4\delta^4\sigma^4}+\frac{43t^{4\sigma}w_0^4}{12\delta^3\sigma^4}+\frac{2t^{4\sigma}w_0^4}{3\delta^2\sigma^4}+\frac{5t^{3\sigma}w_0^4}{\delta^3\sigma^3}+\frac{17t^{3\sigma}w_0^4}{6\delta^2\sigma^3}+\frac{3t^{2\sigma}w_0^4}{\delta^2\sigma^2}-\frac{t^{5\sigma}w_0^5}{\delta^4\sigma^5}-\frac{t^{5\sigma}w_0^5}{3\delta^3\sigma^5}\\
&-\frac{3t^{4\sigma}w_0^5}{\delta^4\sigma^4}-\frac{11t^{4\sigma}w_0^5}{6\delta^3\sigma^4}-\frac{2t^{3\sigma}w_0^5}{\delta^3\sigma^3}+\frac{t^{5\sigma}w_0^6}{3\delta^4\sigma^5}+\frac{t^{4\sigma}w_0^6}{\delta^4\sigma^4},
\end{aligned}
$$

$$
\begin{aligned}
(T_0^{2\sigma} Resid_{w,3})(t) = & -6t^{\sigma}\sigma^2 C_3 + 6\delta\sigma^3 C_3 + \frac{21}{2}t^{5\sigma}\sigma C_3^2 + 30t^{4\sigma}\sigma^2 C_3^2 - \frac{w_0}{\delta^2} + \frac{20t^{3\sigma}C_3 w_0}{\delta^2} + \frac{15t^{3\sigma}C_3 w_0}{\delta} + \frac{35t^{4\sigma}C_3 w_0}{4\delta^2\sigma} \\
& + 15t^{2\sigma}\sigma C_3 w_0 + \frac{24t^{2\sigma}\sigma C_3 w_0}{\delta} + 12t^{\sigma}\sigma^2 C_3 w_0 + \frac{7w_0^2}{\delta^2} + \frac{4w_0^2}{\delta} + \frac{5t^{3\sigma}w_0^2}{3\delta^4\sigma^3} + \frac{3t^{2\sigma}w_0^2}{\delta^4\sigma^2} + \frac{5t^{2\sigma}w_0^2}{\delta^3\sigma^2} \\
& + \frac{6t^{\sigma}w_0^2}{\delta^3\sigma} + \frac{7t^{\sigma}w_0^2}{\delta^2\sigma} - \frac{60t^{3\sigma}C_3 w_0^2}{\delta^2} - \frac{35t^{3\sigma}C_3 w_0^2}{\delta} - \frac{105t^{4\sigma}C_3 w_0^2}{4\delta^2\sigma} - \frac{35t^{4\sigma}C_3 w_0^2}{4\delta\sigma} - \frac{24t^{2\sigma}\sigma C_3 w_0^2}{\delta} - \frac{12w_0^3}{\delta^2} \\
& - \frac{5w_0^3}{\delta} - \frac{10t^{3\sigma}w_0^3}{\delta^4\sigma^3} - \frac{10t^{3\sigma}w_0^3}{3\delta^3\sigma^3} - \frac{18t^{2\sigma}w_0^3}{\delta^4\sigma^2} - \frac{26t^{2\sigma}w_0^3}{\delta^3\sigma^2} - \frac{5t^{2\sigma}w_0^3}{\delta^2\sigma^2} - \frac{24t^{\sigma}w_0^3}{\delta^3\sigma} - \frac{24t^{\sigma}w_0^3}{\delta^2\sigma} - \frac{4t^{\sigma}w_0^3}{\delta\sigma} \\
& + \frac{40t^{3\sigma}C_3 w_0^3}{\delta^2} + \frac{35t^{4\sigma}C_3 w_0^3}{2\delta^2\sigma} + \frac{6w_0^4}{\delta^2} + \frac{65t^{3\sigma}w_0^4}{3\delta^4\sigma^3} + \frac{10t^{3\sigma}w_0^4}{\delta^3\sigma^3} + \frac{5t^{3\sigma}w_0^4}{3\delta^2\sigma^3} + \frac{39t^{2\sigma}w_0^4}{\delta^4\sigma^2} + \frac{43t^{2\sigma}w_0^4}{\delta^3\sigma^2} + \frac{8t^{2\sigma}w_0^4}{\delta^2\sigma^2} \\
& + \frac{30t^{\sigma}w_0^4}{\delta^3\sigma} + \frac{17t^{\sigma}w_0^4}{\delta^2\sigma} - \frac{20t^{3\sigma}w_0^5}{\delta^4\sigma^3} - \frac{20t^{3\sigma}w_0^5}{3\delta^3\sigma^3} - \frac{36t^{2\sigma}w_0^5}{\delta^4\sigma^2} - \frac{22t^{2\sigma}w_0^5}{\delta^3\sigma^2} - \frac{12t^{\sigma}w_0^5}{\delta^3\sigma} + \frac{20t^{3\sigma}w_0^6}{3\delta^4\sigma^3} + \frac{12t^{2\sigma}w_0^6}{\delta^4\sigma^2}.
\end{aligned}
$$

$$(T_0^{2\sigma} Resid_{w,3})(0) = 6\delta\sigma^3 C_3 - \frac{w_0}{\delta^2} + \frac{7w_0^2}{\delta^2} + \frac{4w_0^2}{\delta} - \frac{12w_0^3}{\delta^2} - \frac{5w_0^3}{\delta} + \frac{6w_0^4}{\delta^2}.$$

Which leads to

$$C_3 = \frac{w_0 - 7w_0^2 - 4\delta w_0^2 + 12w_0^3 + 5\delta w_0^3 - 6w_0^4}{6\delta^3\sigma^3}.$$

Repeating this process up to $\aleph = 6$, we get the following values:

$$C_4 = \frac{w_0 - 15w_0^2 - 11\delta w_0^2 + 50w_0^3 + 37\delta w_0^3 + 4\delta^2 w_0^3 - 60w_0^4 - 27\delta w_0^4 + 24w_0^5}{24\delta^4\sigma^4},$$

$$
\begin{aligned}
C_5 = & \frac{1}{120\delta^5\sigma^5}(w_0 - 31w_0^2 - 26\delta w_0^2 + 180w_0^3 + 178\delta w_0^3 + 34\delta^2 w_0^3 - 390w_0^4 - 319\delta w_0^4 \\
& - 49\delta^2 w_0^4 + 360w_0^5 + 168\delta w_0^5 - 120w_0^6),
\end{aligned}
$$

$$
\begin{aligned}
C_6 = & \frac{1}{720\delta^6\sigma^6}(w_0 - 63w_0^2 - 57\delta w_0^2 + 602w_0^3 + 710\delta w_0^3 + 180\delta^2 w_0^3 - 2100w_0^4 - 2350\delta w_0^4 \\
& - 654\delta^2 w_0^4 - 34\delta^3 w_0^4 + 3360w_0^5 + 2896\delta w_0^5 + 515\delta^2 w_0^5 - 2520w_0^6 - 1200\delta w_0^6 + 720w_0^7).
\end{aligned}
$$

This procedure can be repeated many times to get the required accuracy. Of course, higher accuracy can be achieved by evaluating more components.
To summarize the basic idea of the FRPSM for solving CVPGM, we have the following algorithm.

Algorithm 1 To get analytic and approximate solutions for the CVPGM in (7) and (8), do the following steps:
Step 1: Assume the solution has the FPS form:

$$w(t) = \sum_{i=0}^{\infty} C_i t^{i\sigma}$$

Step 2: Use the IC in (8) to get the zeroth coefficient of the FPS. Hence, $C_0 = w_0$ and

$$w(t) = w_0 + \sum_{i=1}^{\infty} C_i t^{i\sigma}$$

Step 3: Define the truncated $\aleph$-th FPS as

$$w_{\aleph}(t) = w_0 + \sum_{i=1}^{\aleph} C_i t^{i\sigma}$$

Step 4: Define the $\aleph$th residual function as follows:

$$Resid_{w,\aleph}(t) = \delta_t T_0^{\sigma} w_{\aleph}(t) - w_{\aleph}(t) + w_{\aleph}^2(t) + w_{\aleph}(t)(J_0^{\sigma} w_{\aleph})(t)$$

Step 5: In a successive manner, solve the equations:

$$\left({}^{(\aleph-1)}{}_{t}T_0^{\sigma} Resid_{w,\aleph}\right)(0) = 0 \quad \text{to obtain} \quad C_{\aleph}.$$

Step 6: Repeat steps 3-5 to get the required accuracy for approximating the solution of the CVPGM.

4 The Successive Substitutions Technique

This section summarizes the idea of a simple iterative scheme, namely, the SS method when it is applied to solve the CVPGM in (7) and (8). As a starting step, we apply the conformable integral in Definition 4 to both sides of equation (7). Hence, a Volterra integral equation (VIE) whose unknown is w(t) is produced as follows.

$$w(t) = w(0) + \frac{1}{\delta}\int_0^t \frac{w(\xi)}{\xi^{1-\sigma}}(1 - w(\xi) - \int_0^\eta \frac{w(\eta)}{\eta^{1-\sigma}} d\eta) d\xi$$

As a second step, we use the IC in (8) as a zeroth approximation of the VIE and apply a SS technique as in the following successive formulas.

$$w_0(t) = w_0 = w(0)$$

$$w_m(t) = w_0 + \frac{1}{\delta}\int_0^t \frac{w_{m-1}(\xi)}{\xi^{1-\sigma}}(1 - w_{m-1}(\xi) - \int_0^\eta \frac{w_{m-1}(\eta)}{\eta^{1-\sigma}} d\eta) d\xi, \quad m = 1,2,3,\cdots$$

So, the first approximation is:

$$w_1(t) = w_0 - \frac{t^\sigma w_0(-2\sigma + (t^\sigma + 2\sigma)w_0)}{2\delta\sigma^2},$$

and the second approximation is:

$$\begin{aligned} w_2(t) = w_0 + \frac{t^\sigma w_0}{360\delta^3\sigma^6}(&-360\delta^2\sigma^5(-1+w_0) - 5t^{5\sigma}w_0^3 - 6t^{4\sigma}\sigma w_0^2(-5+8w_0) - 180t^\sigma\delta\sigma^4(-1+(3+\delta-2w_0)w_0) \\ &- 15t^{3\sigma}\sigma^2 w_0(3 - 4(3+\delta)w_0 + 9w_0^2) - 60t^{2\sigma}\sigma^3 w_0(2 + 4\delta + w_0(-4 - 5\delta + 2w_0))). \end{aligned}$$

Continuing this process, we obtain the exact solution as $w(t) = lim_{m\to\infty} w_m(t)$. A brief description for the SS approach is given in the following algorithm.

Algorithm 2: To get an mth SS approximation $w_m(t)$ for the solution of the CVPGM in (7) and (8), do the following four steps:
Step 1: Apply the conformable integral to both sides of the CVPGM in (7).
Step 2: Use the IC $w_0(t) = w(0)$ as the zeroth SS approximate solution of $w(t)$.
Step 3: Obtain the m-approximate solution using the iterative formula:

$$w_m(t) = w_0 + \frac{1}{\delta}\int_0^t \frac{w_{m-1}(\xi)}{\xi^{1-\sigma}}(1 - w_{m-1}(\xi) - \int_0^\eta \frac{w_{m-1}(\eta)}{\eta^{1-\sigma}} d\eta) d\xi, \quad m = 1,2,3,\cdots$$

Step 4: Repeat step 3 to get the required accuracy.

5 Numerical Results

In this section, we compute approximate values for the solutions of the growth model in (7) using both FRPSM (with $\aleph = 15$) and the SS approach (with $m = 5$). We employ algorithms 1 and 2 for different values of the conformable derivative and display some numerical values and graphical results for the population growth so we can see how the results of the proposed techniques are in good agreement. In order to reveal the accuracy of our results, we carry out a comparison between the values of the residual functions for different values of the fractional order σ and different values of the parameter δ. Obviously, both techniques are convenient for controlling the convergence of the solution. The graphical results show that the variety of choice of conformable orders leads to a variety in predicted curves for the population growth. Computations in this section are performed using Mathematica software.

Assuming the IC in (8) is $w_0 = 0.4$, Table 1 presents some numerical results of both methods when $\delta = 0.65$ and $\sigma \in \{0.9, 0.8, 0.7\}$.

Table 2 presents some values of the residual errors using the FRPSM with 15 iterations for $\delta \in \{0.85, 0.65\}$ and $\sigma \in \{0.9, 0.8, 0.7\}$, while residual errors are shown in Table 3 for the same values of δ and σ when applying the SS approach with 5 iterations.

Table 1: Approximate values of the population growth in closed system using FRPS and SS methods when $\delta = 0.65$ and $w_0 = 0.4$.

	FRPSM			SS Method		
t	$\sigma = 0.9$	$\sigma = 0.8$	$\sigma = 0.7$	$\sigma = 0.9$	$\sigma = 0.8$	$\sigma = 0.7$
0.0	0.4	0.4	0.4	0.4	0.4	0.4
0.1	0.449832	0.469079	0.495544	0.449832	0.469079	0.495544
0.2	0.488534	0.511925	0.539107	0.488534	0.511925	0.539107
0.3	0.519754	0.541855	0.563028	0.519754	0.541855	0.563027
0.4	0.543745	0.56131	0.573527	0.543745	0.561309	0.573504
0.5	0.560671	0.571878	0.574267	0.56067	0.571865	0.574133
0.6	0.570867	0.574953	0.567946	0.570857	0.574883	0.567402
0.7	0.57485	0.57186	0.556896	0.574799	0.571583	0.55509
0.8	0.573295	0.563928	0.543683	0.573102	0.563028	0.538438
0.9	0.567027	0.552681	0.532026	0.56641	0.550093	0.51827
1.0	0.557102	0.540267	0.528075	0.55533	0.533443	0.495081

Table 2: The 15−th residual errors $|Resid_{w,15}(t)|$ by the FRPSM

	$\delta = 0.85$			$\delta = 0.65$		
t	$\sigma = 1$	$\sigma = 0.95$	$\sigma = 0.85$	$\sigma = 1$	$\sigma = 0.95$	$\sigma = 0.85$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	7.782×10^{-18}	7.82×10^{-18}	2.836×10^{-16}	6.864×10^{-18}	3.374×10^{-17}	7.639×10^{-15}
0.2	4.674×10^{-15}	3.319×10^{-14}	1.869×10^{-12}	1.208×10^{-13}	8.574×10^{-13}	4.85×10^{-11}
0.3	1.911×10^{-12}	9.899×10^{-12}	2.969×10^{-10}	4.962×10^{-11}	2.576×10^{-10}	7.774×10^{-9}
0.4	1.319×10^{-10}	5.459×10^{-10}	1.042×10^{-8}	3.447×10^{-9}	1.431×10^{-8}	2.75×10^{-7}
0.5	3.422×10^{-9}	1.189×10^{-8}	1.59×10^{-7}	9.008×10^{-8}	3.139×10^{-7}	4.234×10^{-6}
0.6	4.766×10^{-8}	1.432×10^{-7}	1.427×10^{-6}	1.264×10^{-6}	3.812×10^{-6}	3.84×10^{-5}
0.7	4.302×10^{-7}	1.142×10^{-6}	8.836×10^{-6}	1.15×10^{-5}	3.069×10^{-5}	2.41×10^{-4}
0.8	2.815×10^{-6}	6.702×10^{-6}	4.148×10^{-5}	7.608×10^{-5}	1.824×10^{-4}	1.153×10^{-3}
0.9	1.435×10^{-5}	3.097×10^{-5}	1.567×10^{-4}	3.933×10^{-4}	8.572×10^{-4}	4.468×10^{-3}
1.0	5.97×10^{-5}	1.179×10^{-4}	4.945×10^{-4}	1.669×10^{-3}	3.34×10^{-3}	1.463×10^{-2}

Table 3: The 5−th residual errors $|Resid_{w,5}(t)|$ by SS approach

	$\delta = 0.85$			$\delta = 0.65$		
t	$\sigma = 1$	$\sigma = 0.95$	$\sigma = 0.85$	$\sigma = 1$	$\sigma = 0.95$	$\sigma = 0.85$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	4.508×10^{-10}	5.223×10^{-10}	1.577×10^{-10}	8.46×10^{-10}	7.866×10^{-10}	4.557×10^{-10}
0.2	4.131×10^{-10}	7.297×10^{-10}	8.517×10^{-10}	1.172×10^{-9}	1.103×10^{-9}	1.163×10^{-8}
0.3	8.632×10^{-10}	3.443×10^{-9}	2.932×10^{-8}	1.185×10^{-8}	3.897×10^{-8}	5.833×10^{-8}
0.4	2.155×10^{-8}	3.135×10^{-8}	4.215×10^{-7}	9.914×10^{-8}	4.955×10^{-8}	5.066×10^{-6}
0.5	7.320×10^{-8}	4.953×10^{-7}	6.357×10^{-6}	1.611×10^{-6}	5.734×10^{-6}	4.940×10^{-5}
0.6	2.152×10^{-6}	5.812×10^{-6}	3.644×10^{-5}	1.916×10^{-5}	4.562×10^{-5}	2.392×10^{-4}
0.7	1.443×10^{-5}	3.080×10^{-5}	1.361×10^{-4}	1.03×10^{-4}	2.051×10^{-4}	8.019×10^{-4}
0.8	6.020×10^{-5}	1.121×10^{-4}	3.918×10^{-4}	3.788×10^{-4}	6.706×10^{-4}	2.123×10^{-3}
0.9	1.905×10^{-4}	3.22×10^{-4}	9.421×10^{-4}	1.093×10^{-4}	1.772×10^{-3}	4.755×10^{-3}
1.0	4.994×10^{-4}	7.823×10^{-4}	1.984×10^{-3}	2.654×10^{-4}	4.01×10^{-3}	9.403×10^{-3}

The behavior of the solution curves for the approximate CVPGM (7) for $\delta \in \{0.7, 0.9, 1, 2\}$ and $\sigma \in \{0.85, 1\}$ as presented in Figure 1 when applying the FRPSM, and in Figure 2 when applying the SS method. Clearly, an increase in the values of δ leads to a decrease in the exponential decay in both figures.

In Figure 3, the behavior of the solution curves for the conformable population growth model (7) for $\sigma \in \{0.9, 0.8, 0.7\}$ and $\delta = 0.85$ is presented. The key finding of this graph is that when the conformable derivative order decreases, the amplitude of $w(t)$ decreases, whereas the exponential decay increases. Moreover, the curves of the approximate solutions using FRPSM and SS method are in good agreement. In addition, to see the effect of the toxic term in the population

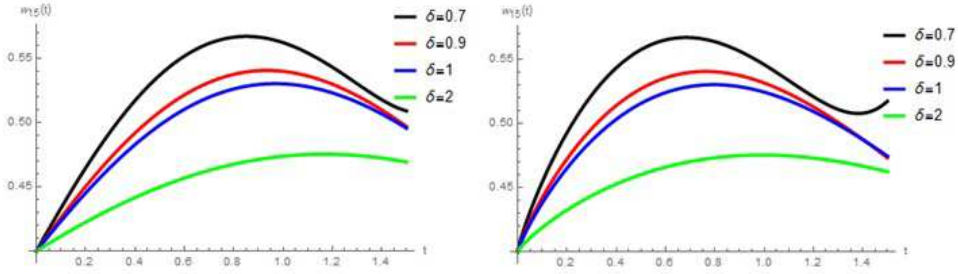

Fig. 1: Approximate solutions for the CVPGM in (7) with $\sigma = 1$ (left) and $\sigma = 0.85$ (right) using the FRPSM.

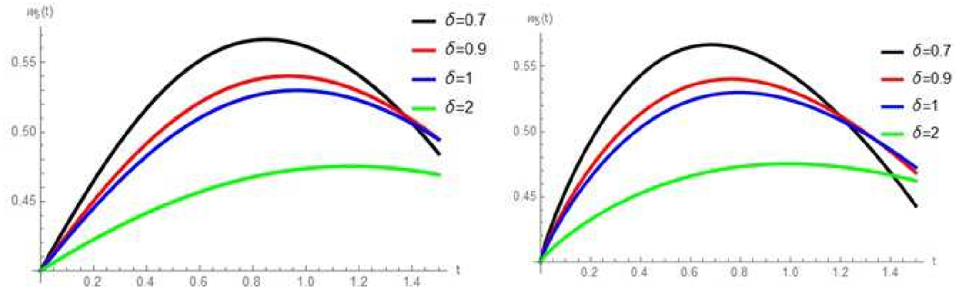

Fig. 2: Approximate solutions for the CVPGM in (7) with $\sigma = 1$ (left) and $\sigma = 0.85$ (right) using the SS approach.

growth model, we apply the FRPSM to the conformable logistic model:

$$\delta_t T_0^{\sigma} w(t) = w(t) - w^2(t), \tag{20}$$

$$w(0) = 0.4,$$

with $\delta = 0.85$. The growth curves in this case are shown in Figure 3. Comparing Figure 4 with Figure 3, one can simply deduce that the existence of the toxic term in a closed system causes the population level to fall to zero in the long run while it still increases exponentially if the toxic term is neglected.

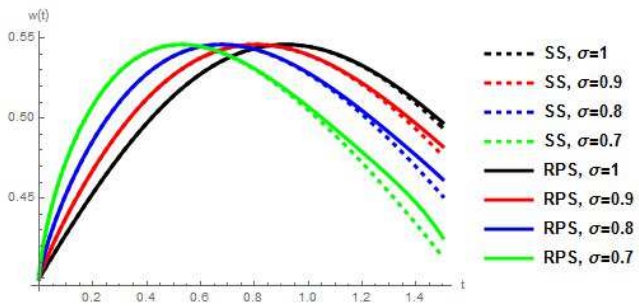

Fig. 3: Approximate solutions for the CVPGM in (7) with $\delta = 0.85$.

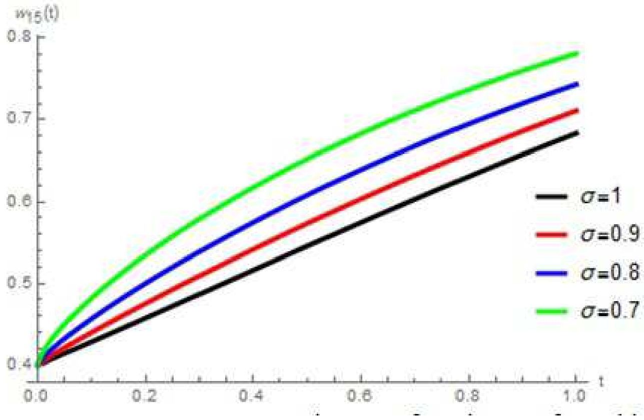

Fig. 4: Approximate solutions for the conformable quadratic logistic model (20) with $\delta = 0.85$.

6 Conclusion

In this paper, we carried out two simple and effective techniques to investigate the solution of a nonlinear conformable VIDE that represents the population growth model in closed system. Numerical results were given to show the efficiency and accuracy of the proposed approaches. Different values of the conformable orders were tested and it was clear that any increase of the fractional parameter changes the behavior of the CVPGM curves. The existence of the toxic term in closed system causes the population level to fall to zero on the contrary of the quadratic logistic model in which the population increases by time. Finally, an increase in non-dimensional parameter δ leads to a decrease in the exponential decay of the population growth.

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