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Rayleigh-Poisson Distribution: Properties and Estimation with Applications of COVID-19 Mortality Rate

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Abstract: In this paper, we introduce the Rayleigh-Poisson distribution, a generalization of the Rayleigh distribution with three parameters. The Rayleigh-Poisson distribution is more flexible than the Rayleigh distribution and can be used to model a wider range of lifetime data. The article derives closed-form expressions for several properties of the Rayleigh-Poisson distribution. The unknown model parameters are estimated and a simulation study is performed to study the behavior of the estimators. Three real datasets of COVID-19 are applied to demonstrate the Rayleigh-Poisson distribution superior performance compared to well-known lifetime distributions.

Keywords: COVID-19 mortality rate; Maximum product spacing estimation; Monte Carlo simulation; Probability weighted moments; Rayleigh–Poisson distribution.

1 Introduction

Statistical models play a crucial role in analyzing and modeling lifetime phenomena across various fields. Consequently, statisticians are continuously striving to develop flexible models capable of accommodating diverse datasets. In this pursuit, numerous generalizations and transformation techniques have been proposed for traditional distributions. These approaches involve introducing additional parameters or modifying the functional form of the distribution. Researchers have also established criteria for evaluating the best distribution, which include a minimal number of parameters, a flexible *probability density function* (pdf) and *hazard rate function* (hrf) allowing for a wide range of shapes, as well as superior performance compared to competing distributions in data modeling.

Among the distributions commonly used for modeling positive random phenomena, the Rayleigh distribution, introduced by [22], holds particular interest. It finds applications in diverse fields such as acoustics, communication engineering, and life-testing of electrovacuum devices. Extensive research has been conducted by several authors focusing on the estimation and prediction of the one-parameter Rayleigh distribution. For further insights into the Rayleigh distribution, refer to [14].

Various generalizations of the one-parameter Rayleigh distribution have been proposed by different authors. For instance, [11] presented the *Weibull Rayleigh* (WR) distribution, [19] proposed the *Rayleigh-geometric* (RG) distribution, [6] introduced the power Rayleigh distribution, [3] developed the *Marshall-Olkin alpha power Rayleigh* (MOAPR) distribution, [4] proposed the *extended odd Weibull Rayleigh* (EOWR) distribution, [2] presented the modified Kies Rayleigh distribution, and [26] introduced the exponential T-X generalized Rayleigh distribution. Additionally, researchers have explored the inclusion of an extra location parameter in the Rayleigh distribution.

The two-parameter Rayleigh (R) distribution, consisting of scale and location parameters, is characterized by the following pdf and *cumulative distribution* (cdf), respectively:

$$g(y;\alpha,\beta)=\frac{y-\beta}{\alpha^2}e^{\left[-\frac{(y-\beta)^2}{2\alpha^2}\right]},\qquad y>\beta;\ \alpha,\beta>0, \tag{1}$$

and

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$$G(y;\alpha,\beta) = 1 - e^{\left[-\frac{(y-\beta)^2}{2\alpha^2}\right]}, \qquad y > \beta;\ \alpha,\beta > 0, \tag{2}$$

where the location parameter is denoted as β and the scale parameter is represented by α.

[8] conducted a comprehensive study on various estimation methods for the R distribution. [9] focused on progressive Type-II censored samples and obtained *maximum likelihood* (ML) estimators of the R distribution using the Newton-Raphson algorithm and the expectation-maximization method. Building upon this, [24] proposed a novel estimation method for the R distribution parameters based on progressive Type-II censored samples. Recently, [17] developed confidence intervals for quantiles, mean, and survival probability of the R distribution using pivotal-based methods. They also established prediction intervals for the mean of future samples. However, despite these advancements, the two-parameter R distribution has not received much attention in the generalization theory, possibly due to its restricted domain.

This paper introduces the *Rayleigh-Poisson* (R-P) distribution, which consists of three parameters derived by compounding the Rayleigh and zero-truncated Poisson distributions. Most lifetime distributions, including exponential, gamma, and Weibull distributions, typically begin at zero. However, in certain applications such as medical, biological, and engineering sciences, researchers often only report positive values greater than zero. In such cases, the R-P distribution serves as an ideal alternative. While a few distributions with a location parameter have been defined in the literature, proposing a new distribution with three parameters (location, scale, and shape) can be a valuable addition.

The objective of this paper is to present the R-P distribution as a flexible model and derive some of its properties. Additionally, various methods of parameter estimation are employed. The structure of the paper is as follows: Section 2: Derivation of the R-P distribution, accompanied by plots of the pdf and hrf to demonstrate the model's flexibility and significance. Section 3: Derivation of several properties of the R-P distribution, including quantiles, *probability weighted moments* (PWM), ordinary and incomplete moments, inequality measures, uncertainty measures, moments of residual life and reversed residual life functions, order statistics, and their moments. Section 4: Estimation of the model parameters using different estimation methods such as *maximum likelihood estimation* (MLE), *maximum product spacing estimation* (MPSE), *least squares estimation* (LSE), *weighted least squares estimation* (WLSE), and *percentiles estimation* (PE). Section 5: Assessment of the estimation methods through simulation results. Additionally, three datasets related to COVID-19 are utilized to demonstrate the efficiency of the R-P distribution compared to competitor distributions. Section 6: Concluding remarks and suggestions for future research directions.

By presenting the R-P distribution and examining its properties, this paper aims to provide a new flexible model that can be effectively utilized in various fields.

2 The Rayleigh-Poisson Distribution

Let's consider that the failure of a device is dependent on the presence of an unknown number of initial defects, denoted as T.

If the failure times of the initial defects $Y_1, Y_2, \dots, Y_T$ have R distribution with pdf given by (1), where the Y_i's are independent of T and T has a zero truncated Poisson distribution with probability mass function given by

$$P(t;\ \lambda) = \frac{\lambda^t e^{-\lambda}}{(t!)(1-e^{-\lambda})}, \qquad t = 1,2,\cdots,$$

where λ is a shape parameter. Then the conditional density of $X|\ T = t$ is as follows:

$$f(x|t;\ \beta,\alpha)=t\frac{x-\beta}{\alpha^2}\left[e^{-\frac{(x-\beta)^2}{2\alpha^2}}\right]^t,$$

and the joint pdf of X and T is

$$f\ (x,t;\ \beta,\alpha,\lambda)=t\frac{x-\beta}{\alpha^2}\left[e^{-\frac{(x-\beta)^2}{2\alpha^2}}\right]^t\frac{\lambda^t e^{-\lambda}}{(t!)(1-e^{-\lambda})},$$

which leads directly, that the marginal pdf of X is given by

$$f(x;\ \beta,\alpha,\lambda)=\frac{(x-\beta)\lambda}{(1-e^{-\lambda})\alpha^2}e^{-\lambda-\frac{(x-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}},\qquad x>\beta;\ \alpha,\beta,\lambda>0, \tag{3}$$

where β,α and λ represent the location, scale, and shape parameters, respectively. The distribution of X is referred to as the R-P distribution, which arises from the compounding operation between the R and zero-truncated Poisson distributions. It can be shown that $f(x;\ \beta,\alpha,\lambda)$ represents the pdf of the R-P distribution. The cdf and *survival function* (sf) associated with (3) are given as below

$$F(x;\ \beta,\alpha,\lambda)=\frac{\left[1-e^{-\lambda+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}\right]}{(1-e^{-\lambda})}, \tag{4}$$

and

$$S(x;\ \beta,\alpha,\lambda)=\frac{e^{-\lambda}\left[e^{\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}-1\right]}{(1-e^{-\lambda})}. \tag{5}$$

From (3) and (5), the hrf and cumulative hrf can be expressed, respectively, as

$$h(x;\ \beta,\alpha,\lambda)=\frac{\lambda(x-\beta)e^{-\frac{(x-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}}{\alpha^2\left[e^{\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}-1\right]}, \tag{6}$$

and

$$H(x;\ \beta,\alpha,\lambda)=-\log\left\{\frac{e^{-\lambda}\left[e^{\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}-1\right]}{(1-e^{-\lambda})}\right\}. \tag{7}$$

The R-P distribution is highly flexible due to its scale, location, and shape parameters. This flexibility is evident when observing the plots of the pdf and hrf, as shown in Figures 1 and 2, respectively, for different parameter values. From these figures, it can be observed that the hrf exhibits various shapes, including increasing-decreasing-increasing, bathtub, reversed bathtub functions, as well as monotonic semi-linear functions. This indicates that the proposed R-P distribution involves a wide range of popular hrf shapes. Moreover, the pdf of the R-P distribution displays a right-skewed, J-shaped, and exponential shape, further highlighting its flexibility and usefulness.

3 Some Statistical Properties

This section focuses on deriving several important structural properties of the R-P distribution. These properties encompass a range of characteristics, including quantiles, PWM, ordinary and incomplete moments, inequality measures, uncertainty measures, moments of residual life and reversed residual life functions, order statistics, and moments associated with order statistics. By deriving these properties, the R-P distribution and its behavior in various statistical analyses can be understood deeper.

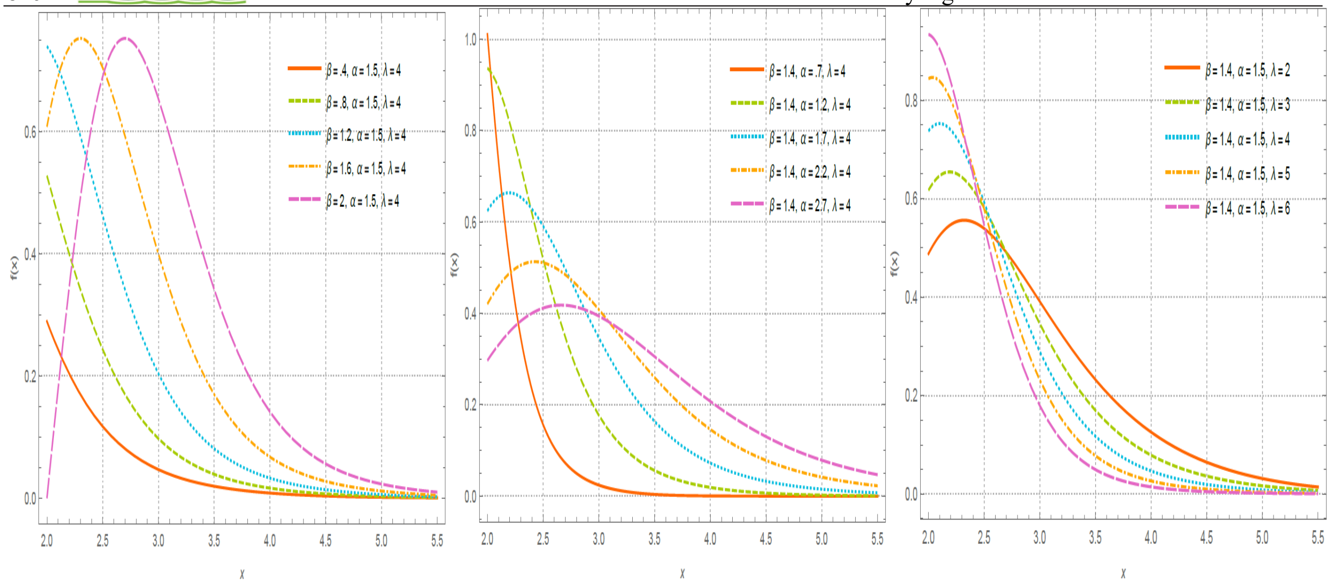

Fig.1. The plots of pdf for R-P distribution for various values of the parameters α, β and λ

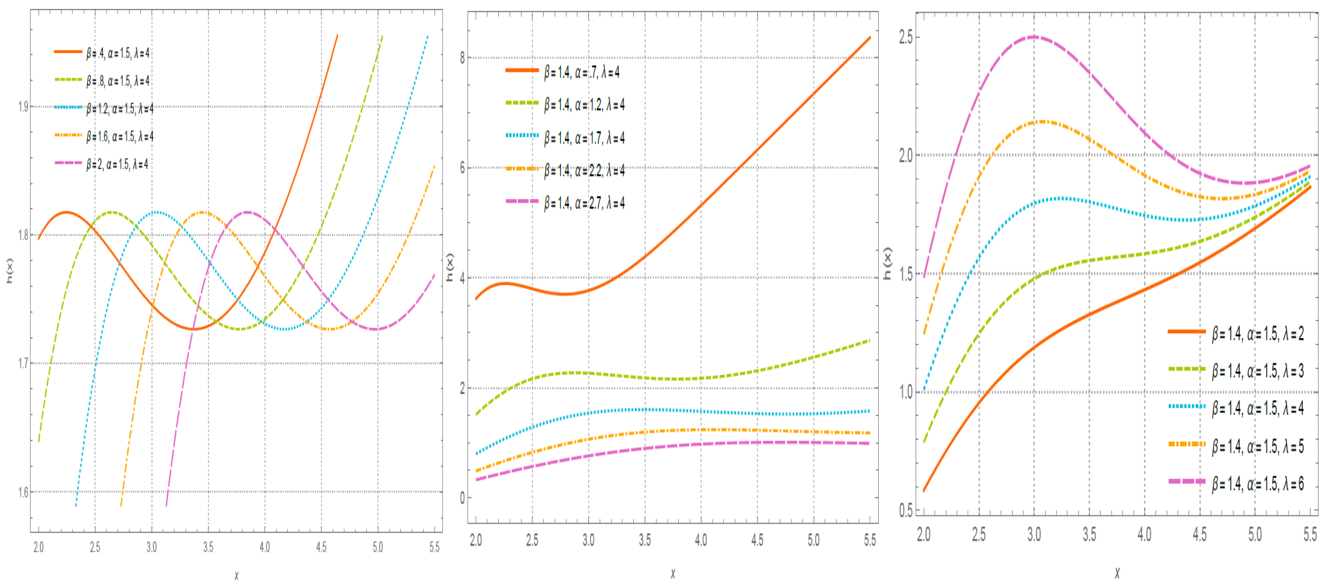

Fig.2. The plots of hrf for R-P distribution for various values of the parameters α, β and λ

3.1 Quantile and median

The quantile function of X, say $Q(p)$, is obtained by solving the following $F(Q(p)) = p$. From (4), one gets

$$\frac{\left[1 - e^{-\lambda + \lambda e^{-\frac{(Q(p)-\beta)^2}{2\alpha^2}}}\right]}{(1 - e^{-\lambda})} = p.$$

By simplifying the equation mentioned above, the quantile function can be obtained as follows:

$$Q(p) = \left(-2\alpha^2 \log\left\{\frac{\log\left[1 - \left(1 - e^{-\lambda}\right)p\right] + \lambda}{\lambda}\right\}\right)^{\frac{1}{2}} + \beta. \tag{9}$$

Hence, the median of the R-P distribution can be obtained by substituting $p = 1/2$ into (9), resulting in the following expression:

$$x_M = Q(1/2) = \left(-2\alpha^2 \log\left\{\frac{\log\left[1-\frac{(1-e^{-\lambda})}{2}\right]+\lambda}{\lambda}\right\}\right)^{\frac{1}{2}} + \beta.$$

3.2 Probability weighted moments

The PWM of a random variable X can be defined in terms of its cdf as follows:

$$M_{k,p,g} = E\left[X^k\left(F(X)\right)^p\left(1-F(X)\right)^g\right],$$

where p, g and k are any real numbers (see [12]). Note that: $M_{k,0,0}$ is the non-central moment of order k. For R-P distribution, the PWM can be expressed using (3) and (4) as

$$M_{k,p,g} = \frac{\lambda e^{-g\lambda-\lambda}}{\alpha^2(1-e^{-\lambda})^{p+g+1}} \times \int_\beta^\infty (x-\beta)x^k\left[1-e^{-\lambda+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}\right]^p\left[e^{\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}-1\right]^g e^{-\frac{(x-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}dx. \tag{10}$$

Let $u = e^{-\frac{(x-\beta)^2}{2\alpha^2}}$, thus (10) can be rewritten in terms of u as

$$M_{k,p,g} = \frac{\lambda e^{-(g+1)\lambda}}{(1-e^{-\lambda})^{p+g+1}}\int_0^1\left[(-2\alpha^2\log(u))^{\frac{1}{2}}+\beta\right]^k\left(1-e^{-\lambda+\lambda u}\right)^p\left(e^{\lambda u}-1\right)^g e^{\lambda u}du. \tag{11}$$

Using the binomial expansion to expand $\left[(-2\alpha^2\log(u))^{\frac{1}{2}}+\beta\right]^k$, $\left(1-e^{-\lambda+\lambda u}\right)^p$ and $\left(e^{\lambda u}-1\right)^g$, (11) reduces to

$$M_{k,p,g} = \sum_{w_1=0}^{p}\sum_{w_2=0}^{g}\sum_{w_3=0}^{k}\frac{\lambda e^{-(g+1+w_1)\lambda}\binom{k}{w_3}\binom{p}{w_1}\binom{g}{w_2}}{(1-e^{-\lambda})^{p+g+1}}(-1)^{w_1+g+w_2}(\beta)^{k-w_3}(2\alpha^2)^{\frac{w_3}{2}} \times \int_0^1(-\log u)^{\frac{w_3}{2}}e^{[(w_1+w_2+1)\lambda]u}du.$$

Substituting $e^{[(w_1+w_2+1)\lambda]u}$ in the above equation with its Maclaurin series expansion, one has

$$M_{k,p,g} = \sum_{w_1=0}^{p}\sum_{w_2=0}^{g}\sum_{w_3=0}^{k}\sum_{w_4=0}^{\infty}\frac{\lambda e^{-(g+1+w_1)\lambda}\binom{k}{w_3}\binom{p}{w_1}\binom{g}{w_2}}{(1-e^{-\lambda})^{p+g+1}} \times \frac{(-1)^{w_1+g+w_2}(\beta)^{k-w_3}(2\alpha^2)^{\frac{w_3}{2}}[(w_1+w_2+1)\lambda]^{w_4}}{w_4!}\int_0^1(-\log u)^{\frac{w_3}{2}}u^{w_4}du. \tag{12}$$

If $(-\log[u]) = q$ in (12) and after some simplification, the PWM are given by

$$M_{k,p,g} = \sum_{w_1=0}^{p}\sum_{w_2=0}^{g}\sum_{w_3=0}^{k}\sum_{w_4=0}^{\infty}D(\beta,\alpha,\lambda)_{w_1,w_2,w_3,w_4},$$

where

$$D(\beta,\alpha,\lambda)_{w_1,w_2,w_3,w_4} = \frac{\binom{k}{w_3}\binom{p}{w_1}\binom{g}{w_2}(-1)^{w_1+g+w_2}(w_1+w_2+1)^{w_4}\left(\frac{w_3}{2}!\right)(\lambda)^{w_4+1}(\beta)^{k-w_3}(2\alpha^2)^{\frac{w_3}{2}}e^{-(g+1+w_1)\lambda}}{w_4!\,(w_4+1)^{\frac{w_3}{2}+1}(1-e^{-\lambda})^{p+g+1}}.$$

3.3 Moments

Moments are crucial properties that play a vital role in statistical analysis, especially in various applications. While non-central moments can be obtained directly from the PWM, they can also be derived using (3), as shown below:

$$E\left(\frac{X-\beta}{\alpha^2}\right)^k = \int_\beta^\infty \left(\frac{x-\beta}{\alpha^2}\right)^k \frac{(x-\beta)\lambda}{(1-e^{-\lambda})\alpha^2} e^{-\lambda-\frac{(x-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}} dx, \qquad k=1,2,3,\ldots. \tag{13}$$

Substituting $u = e^{-\frac{(x-\beta)^2}{2\alpha^2}}$ in (13) then using the Maclaurin series expansion, thus

$$E\left(\frac{X-\beta}{\alpha^2}\right)^k = \frac{(2)^{\frac{k}{2}}\lambda e^{-\lambda}\alpha^{-k}}{(1-e^{-\lambda})} \sum_{w=0}^{\infty} \frac{\lambda^w}{w!} \int_0^1 (-\log u)^{\frac{k}{2}} u^w du. \tag{14}$$

Let $q = -\log[u]$ in (14), one gets

$$E\left(\frac{X-\beta}{\alpha^2}\right)^k = \sum_{w=0}^{\infty} \frac{(2)^{\frac{k}{2}} e^{-\lambda}\alpha^{-k}\lambda^{w+1}\left(\frac{k}{2}!\right)}{w!\,(1-e^{-\lambda})(w+1)^{\frac{k}{2}+1}}. \tag{15}$$

Applying the binomial expansion for the left side bracket in (15), the k^{th} non-central moments can be expressed as

$$\acute{\mu}_k = E(X^k) = \sum_{w=0}^{\infty} \frac{(2)^{\frac{k}{2}} e^{-\lambda}\alpha^{k}\lambda^{w+1}\left[\left(\frac{k}{2}\right)!\right]}{w!\,(1-e^{-\lambda})(w+1)^{\frac{k}{2}+1}} - \sum_{j=0}^{k-1} \binom{k}{j} (-\beta)^{k-j} E(X^j), \qquad k=1,2,3,\ldots. \tag{16}$$

Equation (16) also establishes a recurrence relation among the single moments of the R-P distribution. To provide further insight into the impact of the distribution's parameters on statistical measures, Table 1 displays numerical values for the mean, variance, *skewness* (Sk), and *kurtosis* (Ku) across selected parameter values. From Table 1, one can observe that the shape parameter, λ, exerts a significant influence on the mean, variance, Sk, and Ku, while the scale parameter, α, primarily affects the mean and variance. On the other hand, the location parameter, β, solely impacts the mean. Additionally, Figure 3 illustrates the plots of Sk and Ku for the R-P distribution, demonstrating its ability to model various types of positively skewed data.

Table 1. Mean, variance, Sk and Ku of the R-P distribution for some values of α, β and λ

	$\lambda=2$				$\lambda=4$				$\lambda=8$			
α, β	**E(X)**	**Var(X)**	**Sk(X)**	**Ku(X)**	**E(X)**	**Var(X)**	**Sk(X)**	**Ku(X)**	**E(X)**	**Var(X)**	**Sk(X)**	**Ku(X)**
0.2 0.5	0.760	0.025	1.080	4.505	0.695	0.015	1.376	6.201	0.633	0.006	1.190	6.040
1.5	1.760	0.025	1.080	4.505	1.695	0.015	1.376	6.201	1.6323	0.006	1.190	6.040
3.5	3.760	0.025	1.080	4.505	3.69	0.015	1.376	6.201	3.633	0.006	1.190	6.040
1.2 0.5	2.060	0.888	1.080	4.505	1.669	0.531	1.376	6.201	1.295	0.213	1.190	6.040
1.5	5.060	0.888	1.080	4.505	2.669	0.531	1.376	6.201	2.295	0.213	1.190	6.040
3.5	5.060	0.888	1.080	4.505	4.669	0.531	1.376	6.201	4.296	0.213	1.190	6.040
2.5 0.5	3.750	3.856	1.080	4.505	2.936	2.305	1.376	6.201	2.157	0.926	1.190	6.040
1.5	4.750	3.856	1.080	4.505	3.936	2.305	1.376	6.201	3.157	0.926	1.190	6.040
3.5	6.750	3.856	1.080	4.505	5.936	2.305	1.376	6.201	5.157	0.926	1.190	6.040

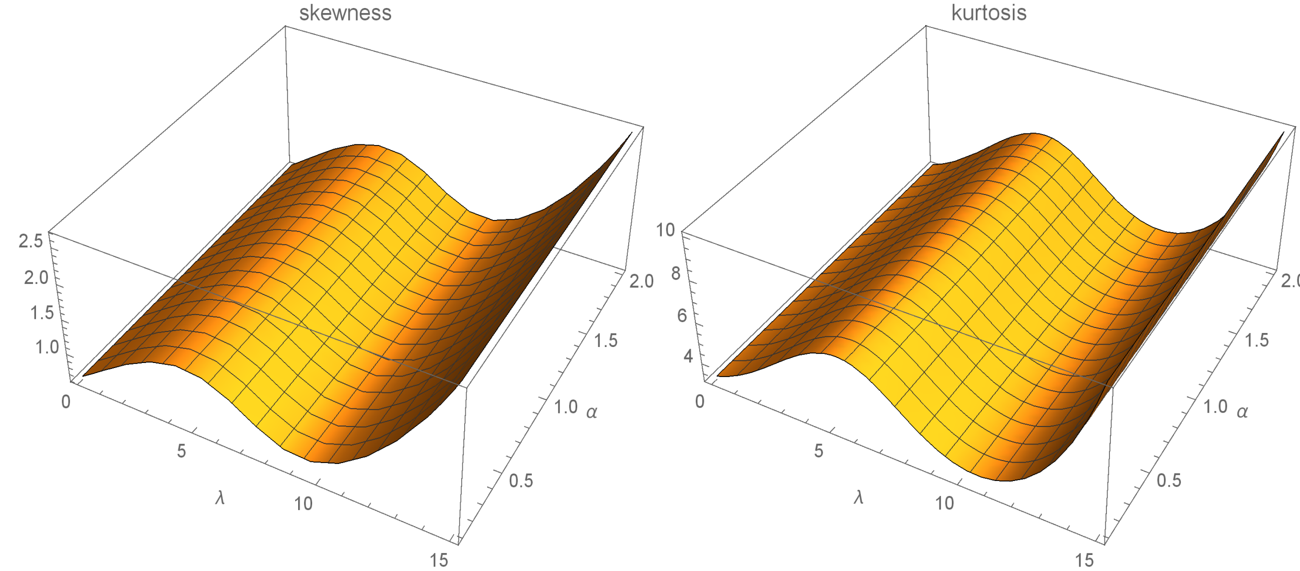

Fig.3. The plots of Sk and Ku for the R-P distribution

3.4 Incomplete moments

Incomplete moments are valuable tools for characterizing the shapes of various distributions, and they play a crucial role in deriving measures such as mean deviations, Lorenz curves, Bonferroni curves, and Zenga curves. These curves find wide applications in fields such as reliability, economics, insurance, demography, and medicine. For a random variable X following the R-P distribution defined in (3), the k^{th} incomplete moment can be expressed as follows:

$$\varphi_k(t) = \int_\beta^t (x)^k f(x;\,\beta,\alpha,\lambda)dx = \int_\beta^t (x)^k \frac{(x-\beta)\lambda}{(1-e^{-\lambda})\alpha^2}\, e^{-\lambda-\frac{(x-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}\,dx.$$

Following the same algorithm of the integration applied in the previous sections, the above equation can be rewritten as

$$\varphi_k(t) = \sum_{w_1=0}^{k}\sum_{w_2=0}^{\infty} \frac{\lambda e^{-\lambda}\binom{k}{w_1}(\beta)^{k-w_1}(-2\alpha^2)^{\frac{w_1}{2}}(\lambda)^{w_2}(-1)^{\frac{w_1}{2}+2}}{w_2!\,(1-e^{-\lambda})}\int_0^{\frac{(t-\beta)^2}{2\alpha^2}} (z)^{\frac{w_1}{2}} e^{-(w_2+1)z}dz\,.$$

Using the result $\frac{1}{n!}\int_x^\infty y^n\, e^{-y}dy = \sum_{s=0}^{n}\frac{x^s e^{-x}}{s!}$, the k^{th} incomplete moment is given by

$$\varphi_k(t) = \sum_{w_1=0}^{k}\sum_{w_2=0}^{\infty} W\left[1-\sum_{w_3=0}^{\frac{w_1}{2}}\frac{\left(\frac{(t-\beta)^2}{2\alpha^2}\right)^{w_3} e^{-\frac{(t-\beta)^2}{2\alpha^2}}}{w_3!}\right], \qquad (17)$$

where $W = \frac{\lambda e^{-\lambda}\binom{k}{w_1}(\beta)^{k-w_1}(2\alpha^2)^{\frac{w_1}{2}}(\lambda)^{w_2}(-1)^{w_1+2}\left(\frac{w_1}{2}!\right)}{w_2!(w_2+1)^{\frac{w_1}{2}+1}(1-e^{-\lambda})}$.

3.5 Mean deviations

The dispersion within a population can be partially assessed by examining deviations from the median and mean. These deviations are referred to as the mean deviation about the median and the mean deviation about the mean. They can be defined as follows:

$\delta_1(X) = 2\mu F(\mu) - 2\varphi_1(\mu),$ and $\delta_2(X) = \mu - 2\varphi_1(x_M),$

where $\mu = E(X)$, x_M =Median (X) and $\varphi_1(t)$ is the first incomplete moment. From (4), (9), (16) and (17), the above the mean deviation can be easily calculated.

Inequality measures

The Lorenz, Bonferroni, and Zenga curves can also be constructed using the first incomplete moment. These curves are widely employed in measuring income and wealth inequality. The definitions of the Lorenz, Bonferroni, and Zenga curves, as given by [20], are as follows:

$$L_F(t)=\frac{\int_{-\infty}^{t} xf(x)dx}{E(X)}=\frac{\varphi_1(t)}{E(X)},\qquad B_F(t)=\frac{\int_{-\infty}^{t} xf(x)dx}{E(X)F(t)}=\frac{L_F(t)}{F(t)}\qquad \text{and}\qquad A_F(t)=1-\frac{M^-(t)}{M^+(t)},$$

where

$$M^-(t)=\frac{\int_{-\infty}^{t} xf(x)dx}{F(t)}=\frac{\varphi_1(t)}{F(t)},\qquad \text{and}\qquad M^+(t)=\frac{\int_{t}^{\infty} xf(x)dx}{1-F(t)}=\frac{E(X)-\varphi_1(t)}{1-F(t)}.$$

For the R-P distribution, these quantities can be derived using (4), (16), and (17). The plots of the Lorenz, Bonferroni, and Zenga curves corresponding to the R-P distribution are presented in Figure 4.

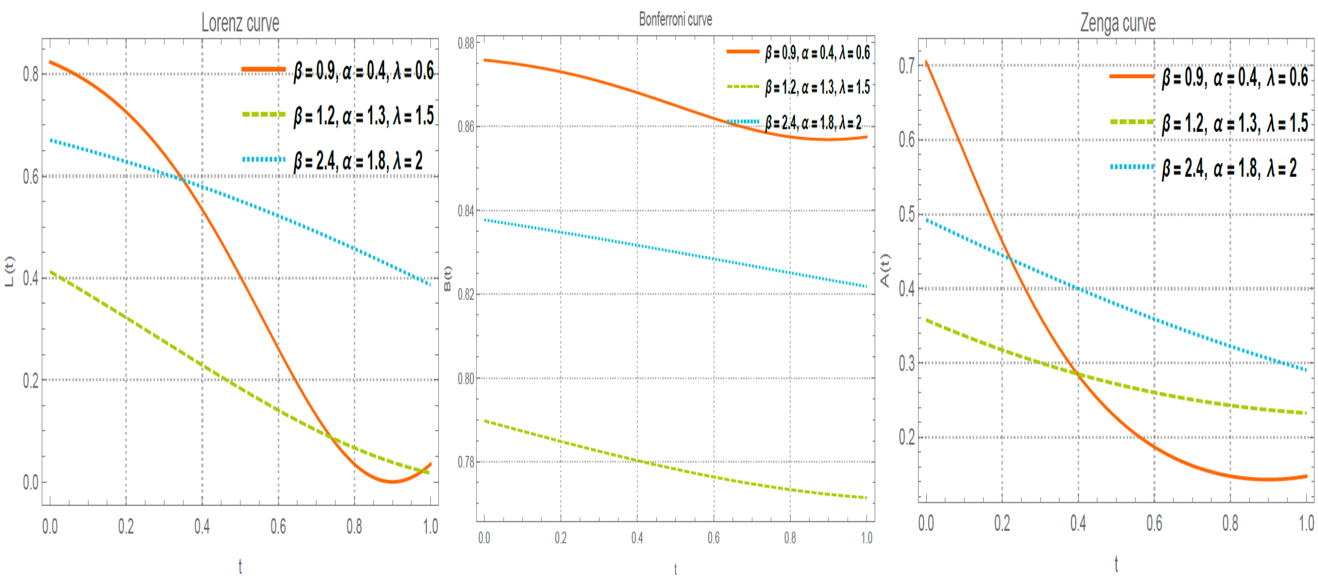

Fig. 4. Plots of Lorenz, Bonferroni and Zenga curves for R-P distribution

These curves have various applications and can serve as useful indicators of income inequality for econometricians, particularly when the data is modeled by R-P distributions.

3.6 Moments of residual life and reversed residual life functions

The residual life is a significant measure of aging that holds great importance in life-testing and reliability theory. The k^{th} moment of the residual life, say $m_k(t)=\mathrm{E}((X-t)^k|X>t), k=1,2,\ldots$ is defined by

$$m_k(t)=\frac{1}{\mathrm{S(t)}}\int_t^{\infty}(x-t)^k f(x)\,dx.$$

Then

$$m_k(t)=\frac{1}{\mathrm{S(t)}}\sum_{\mathrm{w4=0}}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{w4}}(-t)^{k-w_4}\int_t^{\infty}x^{w_4}f(x)\,dx.$$

Thus, the k^{th} moment of the residual life for R-P distribution can be represented in terms of the w_4^{th} moment and the w_4^{th} incomplete moment as

$$m_k(t)=\frac{1}{S(t;\,\beta,\alpha,\lambda)}\sum_{\mathrm{w4=0}}^{\mathrm{k}}\binom{\mathrm{k}}{\mathrm{w4}}(-t)^{k-w_4}\left[E(X^{w_4})-\varphi_{w_4}(t)\right],$$

where $S(t;\ \beta,\alpha,\lambda), E(X^{w_4})$ and $\varphi_{w_4}(t)$ can be obtained by substituting t, w_4 in (4), (16) and (17), respectively. Another interesting function is *the mean residual life* (MRL) function at a given time t, which measures the expected remaining lifetime of an individual of age t. The MRL can be obtained by setting $k = 1$ in the last equation. The k^{th} moment of the reversed residual life, say

$\Phi_k(t) = \mathrm{E}((t - X)^k | X \le t)$ for $t \ge 0$ and $k = 1,2, \ldots$ is

$$\Phi_k(t) = \frac{1}{\mathrm{F(t)}} \sum_{\mathrm{w4}=0}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{w4}} (t)^{k-w_4} (-1)^{w_4} \int_{\beta}^{t} x^{w_4} f(x)\, dx.$$

Using the incomplete moment given in (17), the k^{th} moment of the reversed residual life of R-P distribution can be expressed as

$$\Phi_k(t) = \frac{1}{\mathrm{F(t)}} \sum_{\mathrm{w4}=0}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{w4}} (t)^{k-w_4} (-1)^{w_4} \varphi_{w_4}(t).$$

The mean inactivity time (MIT) or mean waiting time refers to the average duration of time that has elapsed since the failure of an item, given that the failure occurred within the interval $(0, t)$. The MIT for the R-P distribution can be computed by substituting $k = 1$ into the preceding equation.

3.7 Measure of uncertainty

Entropy is a fundamental concept in information theory as it quantifies the degree of uncertainty and disorder in a distribution or system. There are two widely used measures of entropy: Shannon entropy, introduced by [25], and Rényi entropy, presented by [23]. Rényi entropy can be viewed as an extension of Shannon entropy, as it converges to Shannon entropy as the parameter τ approaches 1.

$$I_\tau(X) = \frac{1}{1-\tau} \log\left[\int_{-\infty}^{\infty} \left(f(x)\right)^\tau dx\right], \quad \tau > 0\,, \tau \neq 1. \tag{18}$$

From the pdf of R-P distribution in (3), one obtains

$$\int_{-\infty}^{\infty} [f(x)]^\tau dx = \left[\frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})\alpha^2}\right]^\tau \int_{\beta}^{\infty} (x-\beta)^\tau e^{-\tau\frac{(x-\beta)^2}{2\alpha^2} + \tau\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}} dx.$$

Using the transformation $u = e^{-\frac{(x-\beta)^2}{2\alpha^2}}$, then Maclaurin series expansion and some simplification, the above integral will be

$$\int_{-\infty}^{\infty} [f(x)]^\tau dx = \sum_{j=0}^{\infty} \frac{\left(\frac{\tau-1}{2}\right)!\,(\tau\lambda)^j}{j!\,(\tau+j)^{\frac{\tau+1}{2}}} \left(\frac{2}{\alpha^2}\right)^{\frac{\tau-1}{2}} \left(\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right)^\tau. \tag{19}$$

Substituting (19) in (18), the Rényi entropy for R-P distribution is obtained.

Shannon entropy for R-P distribution is

$$E\{-\log[f(X)]\} = -\log\left[\frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})\alpha^2}\right] - E[\log(X-\beta)] + E\left[\frac{(X-\beta)^2}{2\alpha^2}\right] - \lambda E\left[e^{-\frac{(X-\beta)^2}{2\alpha^2}}\right]. \tag{20}$$

The above equation has three expectations on the right-hand side, the first expectation can be derived as

$$E[\log(X-\beta)] = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})\alpha^2} \int_{\beta}^{\infty} [\log(x-\beta)](x-\beta) e^{\left[-\frac{(x-\beta)^2}{2\alpha^2} + \lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}\right]} dx$$

$$= \sum_{w=0}^{\infty} \frac{(\lambda)^{w+1} e^{-\lambda} \left[\log\left(\frac{2\alpha^2}{w+1}\right) - \gamma\right]}{(w+1)!\,2(1-e^{-\lambda})}, \tag{21}$$

where γ is the Euler constant. The second expectation can be obtained from (15) as

$$E\left[\frac{(X-\beta)^2}{2\alpha^2}\right]=\sum_{w=0}^{\infty}\frac{e^{-\lambda}\lambda^{w+1}}{w!\,(1-e^{-\lambda})(w+1)^2},\tag{22}$$

and the third expectation can be derived as follows:

$$E\left[e^{-\frac{(X-\beta)^2}{2\alpha^2}}\right]=\frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})\alpha^2}\int_{\beta}^{\infty}(x-\beta)e^{-2\frac{(x-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x-\beta)^2}{2\alpha^2}}}=\frac{e^{-\lambda}+\lambda-1}{\lambda(1-e^{-\lambda})}\,.\tag{23}$$

Consequently, Shannon entropy is obtained by substituting (21)-(23) in (20).

3.8 Order statistics

Let $X_{(1,n)}<X_{(2,n)},\ldots<X_{(n,n)}$ denote the order statistics for a sample of size n from the R-P distribution, where the pdf and cdf are given by (3) and (4) respectively. The pdf of the i^{th} order statistic, denoted as $X_{i:n}$ or simply $X_{(i)}$, is given by the following expression:

$$f_{(i,n)}(x_{(i)})=\frac{n!}{(i-1)!\,(n-i)!}f(x_{(i)})\left[F(x_{(i)})\right]^{i-1}\left[1-F(x_{(i)})\right]^{n-i}$$

$$=\frac{n!}{(i-1)!\,(n-i)!}\frac{(x_{(i)}-\beta)\lambda(-e^{-\lambda})^{n-i}}{(1-e^{-\lambda})^n\alpha^2}e^{\left[-\lambda-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}+\lambda e^{-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}}\right]}$$

$$\times\left[1-e^{-\lambda+\lambda e^{-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}}}\right]^{i-1}\left[1-e^{\lambda e^{-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}}}\right]^{n-i},\qquad i=1,2,\ldots,n.$$

(for more details see [5]).

By applying the binomial expansion to the equation above, one can obtain the pdf of the i^{th} order statistic from the R-P distribution as follows:

$$f_{(i,n)}(x_{(i)})=\sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}D_1\,(x_{(i)}-\beta)e^{\left[-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}+\lambda(w_1+w_2+1)e^{-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}}\right]},\qquad x_{(i)}>\beta,\tag{24}$$

where $D_1=\frac{\binom{i-1}{w_1}\binom{n-i}{w_2}n!(-1)^{w_1+w_2+n-i}\lambda e^{-(n-i+1+w_1+w_2)\lambda}}{(i-1)!\,(n-i)!(1-e^{-\lambda})^n\alpha^2}$.

The pdf of the smallest order statistic, $X_{(1,n)}$, can be obtained by substituting $i=1$ into (24), resulting in the following expression:

$$f_{(1,n)}(x_{(1)})=\sum_{w_2=0}^{\infty}D_2(x_{(1)}-\beta)e^{\left[-\frac{(x_{(1)}-\beta)^2}{2\alpha^2}+(w_2+1)\lambda e^{-\frac{(x_{(1)}-\beta)^2}{2\alpha^2}}\right]},$$

where $D_2=\frac{n\binom{n-1}{w_2}(-1)^{w_2+n-1}\lambda e^{-(n+w_2)\lambda}}{(1-e^{-\lambda})^n\alpha^2}$.

The pdf of the largest order statistic, $X_{(n,n)}$, can be obtained by substituting $i=n$ into (24), yielding the following expression:

$$f_{(n,n)}(x_{(n)})=\sum_{w_1=0}^{\infty}D_3(x_{(n)}-\beta)e^{-\frac{(x_{(n)}-\beta)^2}{2\alpha^2}+(w_1+1)\lambda e^{-\frac{(x_{(n)}-\beta)^2}{2\alpha^2}}},$$

where $D_3=\frac{n\binom{n-1}{w_1}(-1)^{w_1}\lambda e^{-(w_1+1)\lambda}}{(1-e^{-\lambda})^n\alpha^2}$.

The k^{th} moment of the i $^{\text{th}}$order statistic $X_{(i)}$ is derived using (24), as

$$\mathrm{E}(X_{(i)}-\beta)^k = \sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty} D_1 \int_{\beta}^{\infty} (X_{(i)}-\beta)^{k+1} e^{-\frac{(x_{(i)}-\beta)^2}{2\alpha^2}+\lambda(w_1+w_2+1)e^{-\frac{\left(x_{(i)}-\beta\right)^2}{2\alpha^2}}} dx.$$

Using the same algorithm employed to obtain the ordinary moments, one can derive the moments of the order statistics as follows:

$$\mathrm{E}(X_{(i)}-\beta)^k = \sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\sum_{w_3=0}^{\infty} \frac{\alpha^2 D_1 (2\alpha^2)^{\frac{k}{2}} [\lambda(w_1+w_2+1)]^{w_3} \left(\frac{k}{2}!\right)}{w_3!}. \tag{25}$$

Applying the binomial expansion, the k^{th} moment of the i th order statistic $X_{(i)}$ can be expressed as

$$\mathrm{E}(X_{(i)})^{\mathrm{k}} = \sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\sum_{w_3=0}^{\infty} \frac{\alpha^2 D_1 (2\alpha^2)^{\frac{k}{2}} [\lambda(w_1+w_2+1)]^{w_3} \left(\frac{k}{2}!\right)}{w_3!} - \sum_{\mathrm{w}_4=0}^{\mathrm{k}-1} \binom{\mathrm{k}}{\mathrm{w}_4} (-\beta)^{k-\mathrm{w}_4}\, \mathrm{E}(X_{(i)})^{\mathrm{w}_4},$$

where D_1 is given in (24). These moments are commonly used in several areas such as insurance, reliability and quality control for the prediction of future failure times based on a set of previous or past failures.

4 Estimation Methods

In this section, several estimation methods are presented to obtain the estimators for the parameters α, β and λ of the R-P distribution such methods, namely MLE, MPSE, LSE, WLSE and PE.

4.1 Maximum likelihood method

The MLE is a vastly applied method due to its coveted properties including asymptotic efficiency, consistency and invariance. Let $x_1, \cdots, x_n$ be a random sample from the R-P distribution defined by (3) with parameters α, β and λ, the log-likelihood function can be written without the additive constant for

$\beta < X_{(1,n)} = \min(X_{(1,n)} < X_{(2,n)}, \dots < X_{(n,n)})$ (see [8]) as

$$\ell(\Theta) = n\log\left(\frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})}\right) - 2n\log(\alpha) + \sum_{i=1}^{n}\log(x_i-\beta) - \sum_{i=1}^{n}\frac{(x_i-\beta)^2}{2\alpha^2} + \lambda\sum_{i=1}^{n} e^{-\frac{(x_i-\beta)^2}{2\alpha^2}},$$

where $X_{(1,n)} < X_{(2,n)}, \dots < X_{(n,n)}$ denotes the corresponding order statistics for a sample of size n from the R-P distribution and $\Theta = (\alpha, \beta, \lambda)$ is a vector of the R-P parameters. The partial derivatives of $\ell(\Theta)$ with respect to the model parameters α, β and λ are given by

$$\frac{\partial\ell(\Theta)}{\partial\alpha} = -\frac{2n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n}(x_i-\beta)^2 + \frac{\lambda}{\alpha^3}\sum_{i=1}^{n}(x_i-\beta)^2 e^{-\frac{(x_i-\beta)^2}{2\alpha^2}}, \tag{26}$$

$$\frac{\partial\ell(\Theta)}{\partial\beta} = \sum_{i=1}^{n}\frac{x_i-\beta}{\alpha^2} + \lambda\sum_{i=1}^{n}\frac{x_i-\beta}{\alpha^2} e^{-\frac{(x_i-\beta)^2}{2\alpha^2}} - \sum_{i=1}^{n}\frac{1}{x_i-\beta}, \tag{27}$$

and

$$\frac{\partial\ell(\Theta)}{\partial\lambda} = \frac{n(1-e^{-\lambda}-\lambda)}{\lambda(1-e^{-\lambda})} + \sum_{i=1}^{n} e^{-\frac{(x_i-\beta)^2}{2\alpha^2}}. \tag{28}$$

The MLE of α and λ can be calculated by solving (26)-(28) after equating to zero. Numerical methods are necessary to find the solution to the nonlinear system.

4.2 Maximum product spacing method

According to [7], the MPSE is an alternative to MLE for estimating parameters of continuous univariate distributions. [21] independently derived the same method as an approximation to the Kullback-Leibler measure of information. So, the MPSE is used in this subsection to have the point estimation of the unknown parameters of R-P distribution. Let

$$D_i(\Theta) = F\left(x_{(i)}\middle|\Theta\right) - F\left(x_{(i-1)}\middle|\Theta\right) = \frac{\left(e^{\lambda e^{-\frac{\left(x_{(i-1)}-\beta\right)^2}{2\alpha^2}}} - e^{\lambda e^{-\frac{\left(x_{(i)}-\beta\right)^2}{2\alpha^2}}}\right)e^{-\lambda}}{(1-e^{-\lambda})}, i = 1,2,\dots,n+1,$$

be the uniform spacing of a random sample from the R-P distribution with cdf given by (4), where

$\Theta = (\alpha,\beta,\lambda)$, $x_{(i)}$ is the i^{th} order statistics, $F\left(x_{(0)}\middle|\Theta\right) = 0$, $F\left(x_{(n+1)}\middle|\Theta\right) = 1$ and $\sum_{i=1}^{n} D_i(\Theta) = 1$. The MPSE for $\Theta = (\alpha,\beta,\lambda)$ can be obtained by maximizing the geometric mean of the spacing defined by

$$G(\Theta) = \left[\prod_{i=1}^{n+1} D_i(\Theta)\right]^{\frac{1}{(n+1)}} = \frac{e^{-\lambda}}{(1-e^{-\lambda})}\left[\prod_{i=1}^{n+1}\left(e^{\lambda e^{-\frac{(x_{i-1}-\beta)^2}{2\alpha^2}}} - e^{\lambda e^{-\frac{(x_i-\beta)^2}{2\alpha^2}}}\right)\right]^{\frac{1}{(n+1)}},$$

with respect to α,β and λ or equivalently, by maximizing the logarithm of the geometric mean of sample spacings given by

$$H(\Theta) = \frac{1}{(n+1)}\sum_{i=1}^{n+1} log[D_i(\Theta)] = log\left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right) + \frac{1}{(n+1)}\sum_{i=1}^{n+1} log\left(e^{\lambda e^{-\frac{(x_{i-1}-\beta)^2}{2\alpha^2}}} - e^{\lambda e^{-\frac{(x_i-\beta)^2}{2\alpha^2}}}\right), \quad (29)$$

with respect to α,β and λ. Consequently by differentiating (29) with respect to R-P parameters as

$$\frac{\partial H(\Theta)}{\partial\alpha} = \frac{1}{(n+1)}\sum_{i=1}^{n+1}\frac{1}{D_i(\Theta)}\left[\Delta_1\left(x_{(i)}\middle|\Theta\right) - \Delta_1\left(x_{(i-1)}\middle|\Theta\right)\right], \quad (30)$$

$$\frac{\partial H(\Theta)}{\partial\beta} = \frac{1}{(n+1)}\sum_{i=1}^{n+1}\frac{1}{D_i(\Theta)}\left[\Delta_2\left(x_{(i)}\middle|\Theta\right) - \Delta_2\left(x_{(i-1)}\middle|\Theta\right)\right], \quad (31)$$

and

$$\frac{\partial H(\Theta)}{\partial\lambda} = \frac{1}{(n+1)}\sum_{i=1}^{n+1}\frac{1}{D_i(\Theta)}\left[\Delta_s\left(x_{(i)}\middle|\Theta\right) - \Delta_s\left(x_{(i-1)}\middle|\Theta\right)\right], \quad (32)$$

where $\Delta_1(.|\Theta) = \frac{\partial}{\partial\alpha}F(.|\Theta), \Delta_2(.|\Theta) = \frac{\partial}{\partial\beta}F(.|\Theta)$ and $\Delta_3(.|\Theta) = \frac{\partial}{\partial\lambda}F(.|\Theta)$, the MPSE for the R-P parameters can be obtained by equating the system of (30)-(32), to zero and solving numerically.

4.3 Least squares method

The LSE was originally proposed by [27] to estimate the parameters of Beta distribution. The method can be described as follows:

Suppose $x_1, x_2, \dots, x_n$ is a random sample of size n from a distribution function $F(.)$ and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics of the observed sample. It is well known that

$$E\left[F(X_{(i)})\right] = \frac{i}{n+1}, \quad \text{and} \quad V\left[F(X_{(i)})\right] = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \quad (33)$$

Using the expectations and variances, one obtains two variants of the least squares method.

4.3.1 Ordinary least squares

The ordinary LSE of the unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}\right) - \frac{i}{n+1}\right)^2,$$

with respect to the unknown parameters. Therefore, in this case, the least squares estimators of the R-P parameters, α,β and λ, say $\hat{\alpha}_{LS}, \hat{\beta}_{LS}$ and $\hat{\lambda}_{LS}$ respectively, can be derived by minimizing

$$\sum_{i=1}^{n}\left[\frac{1-e^{-\lambda+\lambda e^{-\frac{\left(x_{(i)}-\beta\right)^{2}}{2\alpha^{2}}}}}{1-e^{-\lambda}}-\frac{i}{n+1}\right]^{2}, \tag{34}$$

with respect to α, β and λ. Or equivalently, the ordinary LSE can be obtained by solving the system of equations resulting from the partial derivative of (34) with respect to α, β and λ after equating to zero.

4.3.2 Weighted least squares method

The WLSE of the unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n}\frac{1}{V\left[F(X_{(i)})\right]}\left[F\left(x_{(i)}\right)-\frac{i}{n+1}\right]^{2},$$

with respect to the unknown parameters. Therefore, in this case, the WLSE of the R-P parameters, α, β and λ, say $\hat{\alpha}_{WLS}, \hat{\beta}_{WLS}$ and $\hat{\lambda}_{WLS}$, respectively, can be derived by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(\frac{1-e^{-\lambda+\lambda e^{-\frac{\left(x_{(i)}-\beta\right)^{2}}{2\alpha^{2}}}}}{1-e^{-\lambda}}-\frac{i}{n+1}\right)^{2}, \tag{35}$$

with respect to α, β and λ. Or equivalently, the WLS can be obtained by solving the system of equations resulting from the partial derivative of (35) with respect to α, β and λ after equating to zero.

4.4 Percentiles method

The PE method was originally suggested by [15] and [16] and applied to the Weibull distribution. It is a statistical method used for any distribution which has an explicit distribution function to estimate the unknown parameters by comparing the sample points with the theoretical points. It was used quite successfully as in [13] and [18] since they used the PE and compared it with the other estimators for generalized exponential and generalized Rayleigh distributions, respectively.

For R-P distribution with quantile function defined in (9), the PE of α, β and λ, say $\hat{\alpha}_P, \hat{\beta}_P$ and $\hat{\lambda}_P$, can be obtained by minimizing

$$\sum_{i=1}^{n}\left[x_{(i)}-\beta-\left(-2\alpha^{2}\log\left\{\frac{\log\left[1-\left(1-e^{-\lambda}\right)p_{i}\right]+\lambda}{\lambda}\right\}\right)^{\frac{1}{2}}\right]^{2}, \tag{36}$$

with respect to α, β and λ. Here, $X_{(1)} < \cdots < X_{(n)}$ is the sample order statistics, and p_i denotes the estimate of $F\left(x_{(i)};\ \alpha, \beta, \lambda\right)$. Although several estimates of $p's$ are available in the literature, $p_i = \frac{i}{(n+1)}$ seems to be an acceptable choice. In this article, $p_i = \frac{i}{(n+1)}$. The PE of α, β and λ can also be obtained equivalently by solving the system of equations resulting from the partial derivative of (36) with respect to α, β and λ after equating them to zero. The PE for β can be derived from (36) in closed form as

$$\hat{\beta}_P = \frac{1}{n}\sum_{i=1}^{n}\left[x_{(i)}-\left(-2\alpha^{2}\log\left\{\frac{\log\left[1-\left(1-e^{-\lambda}\right)p_{i}\right]+\lambda}{\lambda}\right\}\right)^{\frac{1}{2}}\right].$$

5 Numerical Results

In this section, a Monte Carlo simulation is implemented to evaluate the performance of the estimation methods. Additionally, we utilized three real datasets of COVID-19 from the Netherlands, Canada, and Mexico to showcase the significance and flexibility of the new model. The goodness of fit of the proposed distribution was compared to other distributions using standard statistical criteria and plots, further validating the importance of the proposed model.

5.1 Simulation study

In this subsection, a simulation study is conducted to compare the efficiency of different estimation methods discussed in the previous section. Different samples of sizes $n = 30, 60$ and 100 are generated from the R-P distribution with various combinations of population parameter values. For each sample size, we performed $N = 10000$ replications based on the quantile function given in (9). The performance of different estimators was evaluated using the *relative error* (RE). For each estimate, the average of the estimates and the corresponding RE, RE $= \frac{\sqrt{mean\ square\ error\ (\text{MSE})}}{\text{population parameter}}$, are calculated. The simulation results were obtained using Mathematica 11 software. Tables 2 and 3 provide a summary of the simulation results for the point estimation methods proposed in this paper.

Based on the findings in Tables 2 and 3, the concluding remarks are:

- As the sample size increases, the averages of the estimates generally perform better in most cases.
- It is observed that as the sample size increases, the REs decrease. This verifies the consistency properties of all the estimators.
- In most cases, the MPSE method provides better estimation values compared to the MLE method, as indicated by the lower RE values. This demonstrates that the MPSE method is a favorable alternative to the MLE method.

5.2 Applications to COVID-19 mortality rate

In this subsection, three real datasets of COVID-19 mortality rates from the Netherlands, Canada, and Mexico are presented. The first dataset represents COVID-19 data from the Netherlands spanning 30 days, recorded from March 31 to April 30, 2020. This data comprises rough mortality rates. The second dataset represents COVID-19 data from Canada, covering 36 days from April 10 to May 15, 2020. These data also consist of rough mortality rates. The third dataset represents COVID-19 mortality rates from Mexico, encompassing 108 days recorded from March 4 to July 20, 2020. Similar to the previous datasets, this data is formed of rough mortality rates. The sources for the first and third datasets are [4], while the second dataset is sourced from [3].

To determine an appropriate model, the hazard shape information for the applications is considered. *The total time on test* (TTT) transform, a graphical technique explained in [1], is employed. If the TTT plot follows a straight diagonal line, the hazard is constant. A convex shape indicates decreasing hazards, while a concave shape suggests increasing hazards. The hazard shape resembles a bathtub when it starts convex and transitions to concave. For the bimodal shape hazard, the TTT plot is initially concave and then becomes convex. Figure 5 illustrates the TTT plots for all three datasets, with all of them exhibiting convex curves, indicating decreasing hazard rates.

Table 4 presents selected values of goodness-of-fit tests, such as *Crammer-von Mises* (W^*), *Anderson–Darling*(A^*), *Kolmogorov-Smirnov* (KS) statistics, and their corresponding P-values for the R-P distribution based on the three real datasets. Additionally, empirical PP-plots, QQ-plots, and histogram plots are provided in Figures 6-8. Both Table 4 and Figures 6-8 demonstrate that the R-P distribution provides a better fit to the observed datasets.

To validate the suitability of the R-P distribution for fitting the datasets, a comparison is made between the R-P distribution and other distributions. These compared distributions include generalizations of the Rayleigh distribution and other models previously used to fit COVID-19 data, such as R, RG, MOAPR, EOWR, WR distributions, and the *exponentiated Weibull* (EW) distribution, which was employed as a model by [10] for COVID-19 data.

Various accuracy measures, such as *Akaike's Information Criterion* (AIC), *Consistent Akaike's Information Criterion* (CAIC), *Bayesian Information Criterion* (BIC), and *Hannan-Quinn Information Criterion* (HQIC), are used for model comparisons, as shown in Tables 5-7. In general, smaller values of these statistics indicate a better fit to the datasets. Tables 5-7 conclude that the R-P distribution consistently outperforms its competitors, providing the best fit for all applications. This demonstrates the superiority of the R-P distribution in fitting this type of COVID-19 mortality rate data.

6 Conclusion

The development of flexible and easily applicable lifetime models has become an essential topic in statistical theory. While various distributions can fit the same dataset, there is still competition in finding a groundbreaking distribution. In this study, the authors introduced a novel distribution called R-P, which serves as a valuable addition to the existing statistical literature. The pdf of the R-P distribution and several corresponding reliability functions were derived, and extensive plots of the pdf and hrf were presented. These plots highlight the exceptional flexibility of the R-P distribution, as they encompass nearly all known shapes of pdf and hrf within a single distribution.

Moreover, the study extensively explored the statistical properties and potential shapes of the R-P model, showcasing its versatility. Various estimation methods, including MLE, Mean MPSE, LSE, WLSE, and PE, were successfully employed to estimate the model parameters. Furthermore, simulation studies were conducted to evaluate the performance of these estimation methods.

The practical significance of the R-P model was demonstrated through its application to three scenarios involving COVID-19 mortality rates. By implementing the MLE method, the R-P distribution outperformed other competing models, demonstrating its superior goodness of fit for the given datasets.

Considering the transparent flexibility of the R-P distribution, it holds promise for future studies in reliability theory and related fields. Additionally, the R-P distribution, with its three distinct parameters (shape, scale, and location), can serve as a valuable alternative regression model.

Table 2. Averages, REs of R-P distribution parameters for MLE, MPS, LSE, WLSE and PE

$(\alpha = 2.6, \beta = 1.5 \text{ and } \lambda = 0.3)$

n	Θ	MLE		MPSE		LSE		WLSE		PE	
		Average	RE	Average	RE	Average	RE	Average	RE	Average	RE
100	α	2.4930	**0.0733**	2.5918	**0.0437**	2.5843	**0.0517**	2.6066	**0.0680**	2.5986	**0.0506**
	β	1.6553	**0.1519**	1.4482	**0.0797**	1.4309	**0.1070**	1.5136	**0.1285**	1.4272	**0.1439**
	λ	0.3012	**0.5116**	0.1754	**0.7212**	0.1226	**0.8050**	0.2947	**0.1799**	0.2827	**0.4954**
60	α	2.4524	**0.0927**	2.6048	**0.0457**	2.5847	**0.0568**	2.6125	**0.0925**	2.5267	**0.0624**
	β	1.7029	**0.1811**	1.3829	**0.1229**	1.3874	**0.1427**	1.4586	**0.1757**	1.3936	**0.1940**
	λ	0.2447	**0.6231**	0.1685	**0.7354**	0.1228	**0.8260**	0.2983	**0.2494**	0.1672	**0.7762**
30	α	2.3847	**0.1348**	2.6143	**0.0509**	2.5881	**0.0625**	2.6153	**0.1531**	2.3702	**0.1573**
	β	1.8406	**0.2883**	1.3497	**0.1617**	1.3907	**0.1580**	1.4764	**0.2554**	1.7674	**0.2569**
	λ	0.2220	**0.6820**	0.1210	**0.8067**	0.1234	**0.8369**	0.3309	**0.7138**	0.1443	**0.7981**

Table 3. Averages, REs of R-P distribution parameters for MLE, MPS, LSE, WLSE and PE

$(\alpha = 1.2, \beta = 0.9 \text{ and } \lambda = 2.2)$

n	Θ	MLE		MPSE		LSE		WLSE		PE	
		Average	RE	Average	RE	Average	RE	Average	RE	Average	RE
100	α	1.1202	**0.1176**	1.1711	**0.1053**	1.1240	**0.1357**	1.2042	**0.0729**	1.1676	**0.1010**
	β	0.9592	**0.0842**	0.8835	**0.0535**	0.8550	**0.0978**	0.8990	**0.0656**	0.8729	**0.0794**
	λ	1.9637	**0.2605**	1.9193	**0.2876**	1.5401	**0.4917**	2.2011	**0.0066**	1.7979	**0.3467**

60	α	1.0514	**0.1804**	1.1351	**0.1356**	1.0882	**0.1625**	1.2171	**0.0920**	1.1536	**0.1329**
	β	0.9773	**0.1125**	0.8516	**0.0819**	0.8355	**0.1251**	0.8902	**0.0993**	0.8379	**0.1219**
	λ	1.6479	**0.4341**	1.6136	**0.4623**	1.2185	**0.6143**	2.1981	**0.0115**	1.5527	**0.4818**
30	α	1.0138	**0.2148**	1.1351	**0.1457**	1.0882	**0.2114**	1.2171	**0.1718**	1.1536	**0.1677**
	β	1.0003	**0.1416**	0.8582	**0.1141**	0.8336	**0.1759**	0.8882	**0.1449**	0.8507	**0.2092**
	λ	1.3950	**0.5397**	1.5467	**0.4975**	1.0297	**0.6860**	2.2170	**0.0502**	1.3662	**0.5465**

Table 4. Goodness of fit tests for COVID-19 data to R-P distribution

Test / COVID-19 data	A^*		W^*		KS	
	statistic	P-value	statistic	P-value	statistic	P-value
Netherlands	0.2145	**0.9859**	0.0217	**0.9952**	0.1000	**0.9988**
Canada	0.3624	**0.8847**	0.0496	**0.8789**	0.1389	**0.8849**
Mexico	0.3925	**0.8560**	0.0557	**0.8409**	0.0833	**0.8499**

Table 5. ML estimates, AIC, BIC, $CAIC$ and $HQIC$ for COVID-19 data of Netherlands

Distribution	MLE			AIC	BIC	$CAIC$	$HQIC$
	α	β	λ				
R-P	5.7559	0.8112	1.7638	**152.38**	**156.58**	**153.30**	**153.72**
EOWR	1.3170	2.7617	0.0335	158.99	163.19	159.91	160.33
EW	2.5073	1.1870	4.0837	159.35	163.56	160.28	160.70
MOAPR	4.4086	0.2659	5.8284	159.60	163.80	160.52	160.95
WR	1.3250	13.811	0.9400	160.07	164.27	160.99	161.41
R	4.3101	1.2730	–	155.19	157.99	155.64	156.08
RG	5.6756	0.3867	–	157.63	160.43	158.07	158.53

Table 6. ML estimates, AIC, BIC, $CAIC$ and $HQIC$ for COVID-19 data of Canada

Distribution	MLE			AIC	BIC	$CAIC$	$HQIC$
	α	β	λ				
R-P	4.5378	1.4769	10.624	**97.497**	**102.25**	**98.247**	**99.155**
EOWR	2.6348	1.5889	0.0727	100.54	105.29	101.29	102.198
EW	8.1788	1.5114	1.7233	102.19	106.94	102.94	103.85
MOAPR	10301.2	0.0211	2.4140	99.643	104.39	100.39	101.30
WR	2.9400	0.7652	1.6568	108.95	113.70	109.70	110.61
R	1.6510	1.200	----	98.275	101.45	98.642	99.383

RG	2.42261	0.000002	----	120.93	124.10	121.30	122.04

Table 7. ML estimates, AIC, BIC, CAIC and HQIC for COVID-19 data of Mexico

Distribution	MLE			AIC	BIC	$CAIC$	$HQIC$
	α	β	λ				
R-P	5.6083	0.7835	2.1707	**530.21**	**538.23**	**530.45**	**533.46**
EOWR	1.9750	6.6784	0.0635	535.73	543.78	535.96	538.99
EW	4.5368	0.9486	2.4934	538.33	546.38	538.96	541.59
MOAPR	7.7795	0.1764	5.6663	541.93	549.98	542.16	545.19
WR	1.3395	11.850	0.9484	543.91	551.96	544.14	547.18
R	4.4477	0.3998	----	535.048	540.39	535.16	537.22
RG	5.3176	0.3892	----	540.22	545.58	540.33	542.40

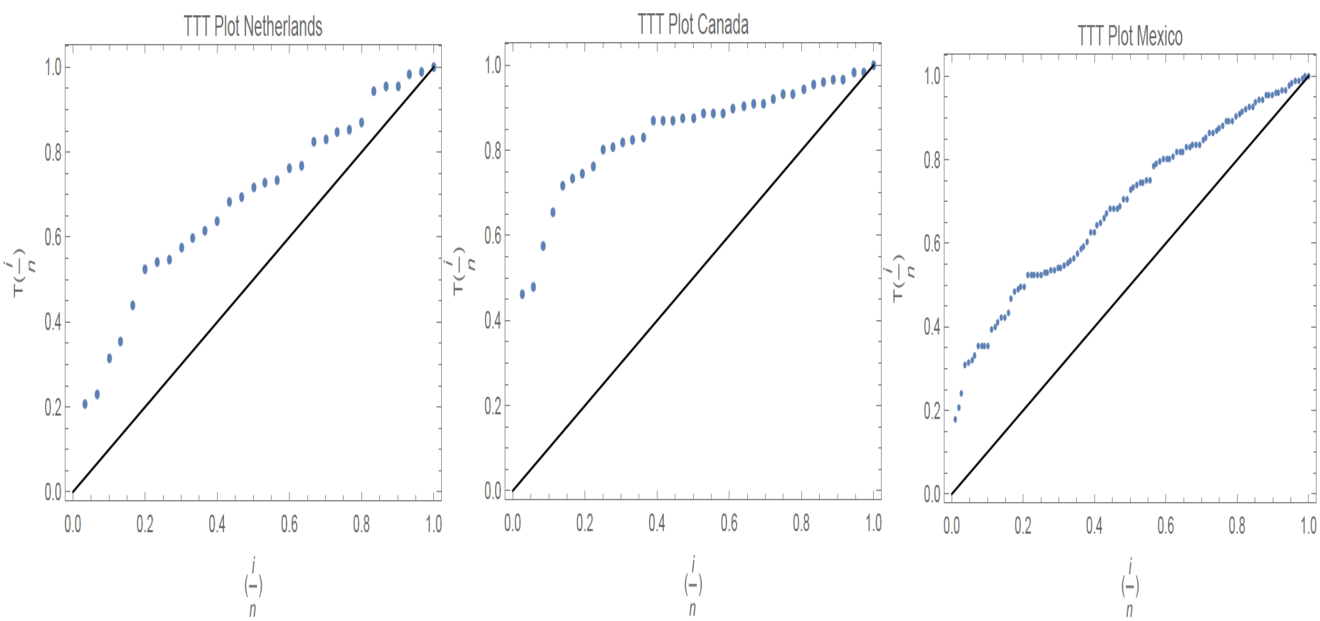

Fig.5. *The TTT plots for COVID-19 data sets*

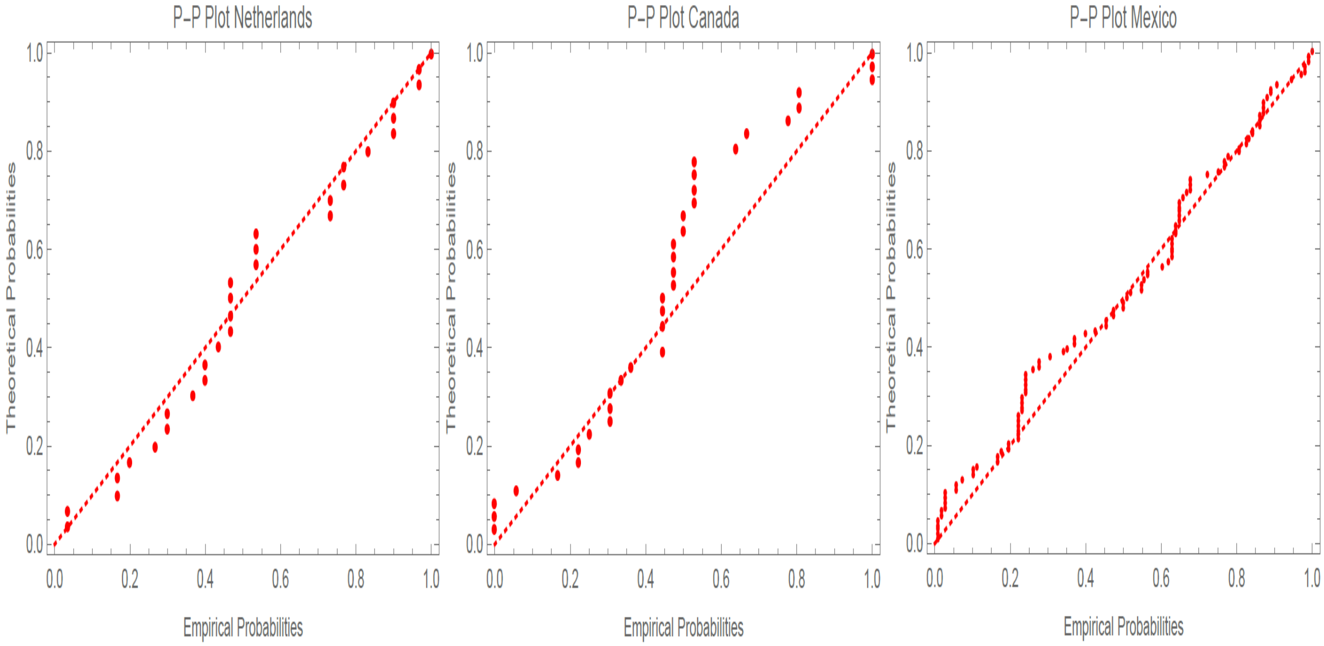

Fig. 6. *The PP plots for COVID-19 data sets*

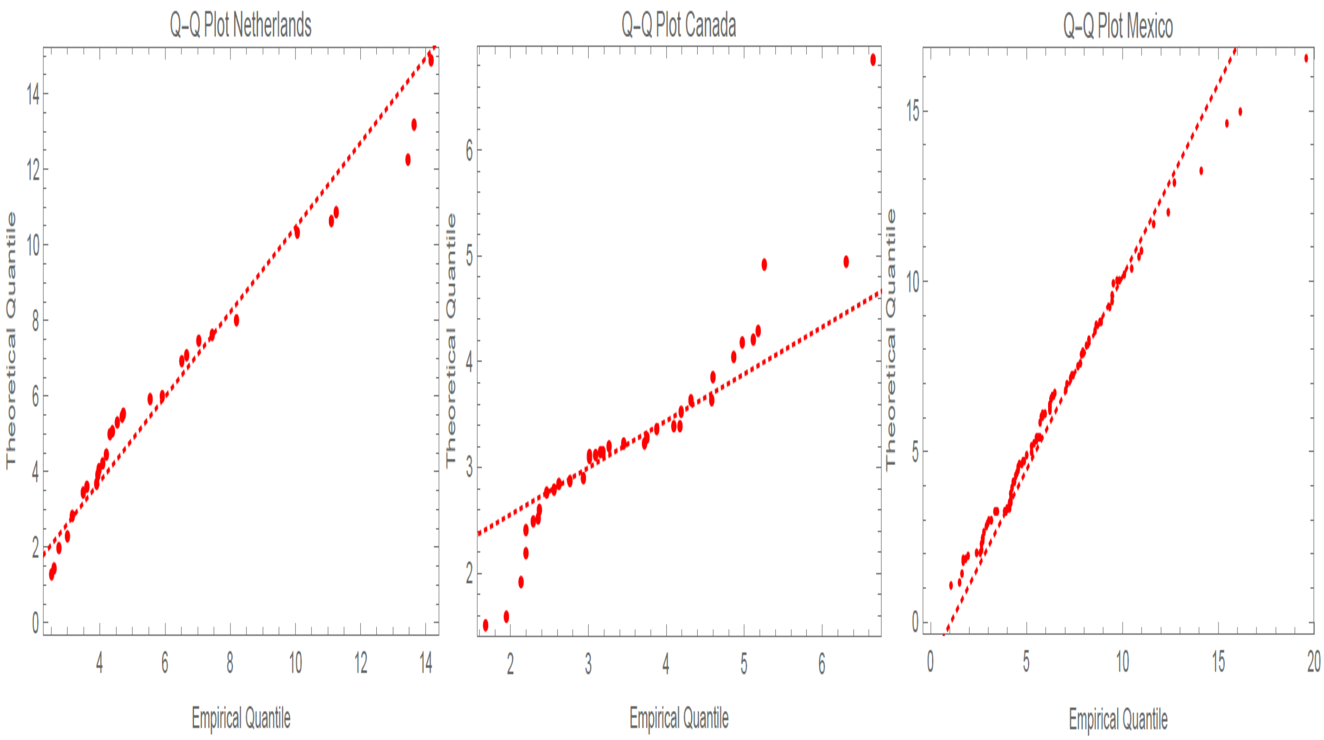

Fig.7. *The QQ plots for COVID-19 data sets*

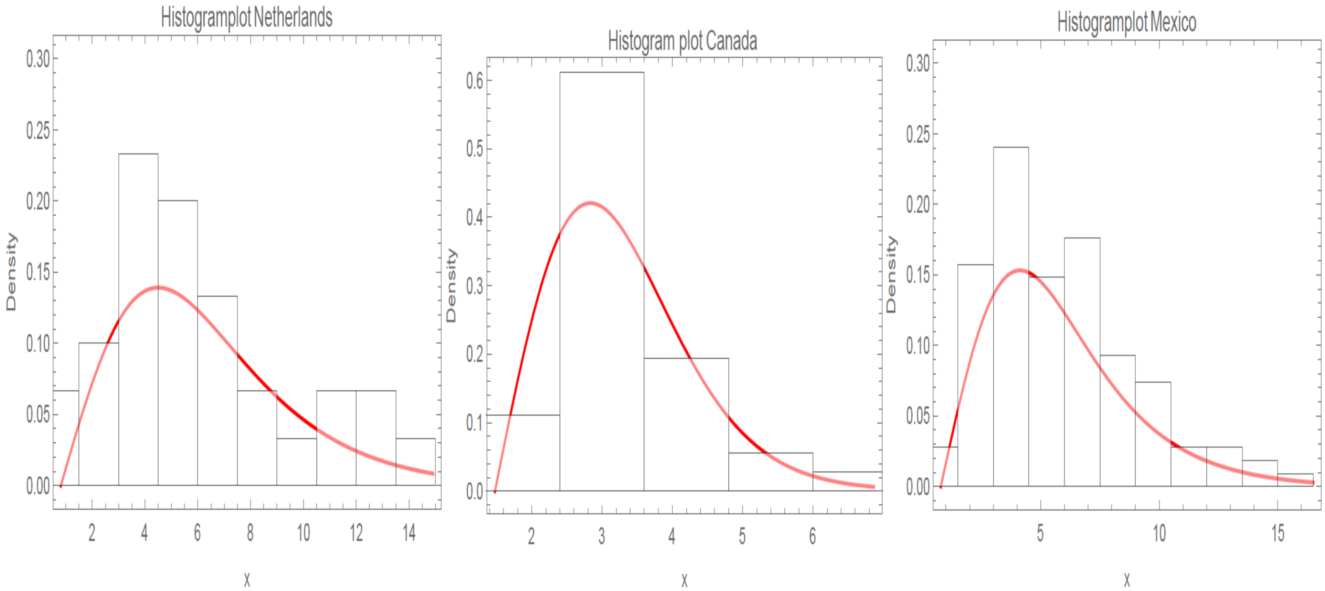

Fig. 8. *Histogram plots for COVID-19 data sets*

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