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Existence Results on Mittag-Leffler Kernel of Fractional Integro-Differential Inclusion Problem Under Boundary Conditions

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Abstract: The main objective of this manuscript is to establish the sufficient conditions for the existence of solutions to the proposed inclusion problem under certain boundary conditions by applying fixed point results. Two cases of multivalued maps are explored with convex and non convex values. The results in the case of convex set-valued map are established using the Leray-Schauder theorem while the results in the case of non convex set-valued map are established through Nadler and Covitz set-valued theorem. The manuscript is concluded with apt examples to demonstrate the theoretical findings and inclusive innovation.

Keywords: Fixed point theorem, inclusive innovation, inclusion problem, boundary conditions.

1 Introduction

In a fractional integro-differential inclusion (FIDI), the unknown function is defined on a given interval, and the inclusion involves a set-valued operator that contains both fractional derivatives and integrals. The inclusion represents a family of differential equations that can have multiple solutions [1,2,3,4,5,6].

The combination of fractional calculus and integro-differential equations in an inclusion setting allows for the consideration of non-local effects, memory effects, and long-range interactions, making it a powerful tool for modeling complex phenomena in physics, engineering, and other fields [7,8,9,10,11,12,13,14].

FIDI finds its applications in various areas such as population dynamics, fractional diffusion processes, and so on. Analyzing and solving these inclusions typically involve specialized mathematical techniques such as fixed-point theory, semigroup theory and so forth [15,16,17,18,19]. The application of Atangana-Baleanu (AB) derivative in variational problems of mathematical physics and signal processing are aplenty [20].

Over the years, extensive research studies aimed at exploring the existence of solutions for various mathematical problems in fractional differential equations (FDEs) and FIDEs have been conducted [21,22,23,24,25,26]. Lachouri et al [27], in their work have established the sufficient conditions for the existence of solutions for a class of inclusion problems of fractional order involving the Atangana-Baleanu-Caputo (ABC) derivative under certain boundary conditions.

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Motivated by the aforementioned studies, this manuscript explores the existence results of the following FIDI problem with ABC derivative involving BCs as given below:

$$
\begin{aligned}
&{}^{ABC}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}z(\varpi) \in H\left(\varpi, z(\varpi), \int_0^b e(\varpi,v,z(v))dv, \int_0^{\varpi} h(\varpi,v,z(v))dv\right), \varpi \in I = [0,b] \\
&z(0) = z_0, z(b) = z. \qquad (1.1)
\end{aligned}
$$

where ${}^{ABC}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}$ is the ABC derivative of order $\varpi \in I = [0,b]$. Here $D = (\varpi, s) \in v \times v : s \leq \varpi$ and $e,h : D \times Y \to Y$ are continuous functions and $z_0, z_1 \in R$, $H : I \times R \to \mathscr{O}(R)$ is an SVM.
For our convenience, we assume that $E_1 w(\alpha) = \int_0^b e(\varpi,v,z(v))dv$ and $E_2 w(\alpha) = \int_0^{\varpi} h(\varpi,v,z(v))dv$.

$$
\begin{aligned}
&{}^{ABC}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}z(\varpi) \in H(\varpi, z(\varpi), E_1 w(\alpha), E_2 w(\alpha)), \varpi \in I = [0,b] \\
&z(0) = z_0, z(b) = z. \qquad (1.2)
\end{aligned}
$$

This manuscript is organized as follows: in Section 3 we determine the existence results for a class of FIDIs under certain BCs. In section 4 we validate the results with suitable examples and in section 5 we conclude the results.

2 Preliminaries

Let $\mathfrak{E} = \mathscr{C}[I,R]$ be a Banach space under the norm $||\mathscr{X}|| = sup_{\varpi \in I}|\mathscr{X}(\varpi)|$.

Definition 2.1[28] Let $\mathscr{X} \in H'(I)$ and $0 < \mathfrak{F} < 1$. For order $0 < \mathfrak{F} < 1$, the non-singular derivatives for the function $\mathscr{X}$ is given by,

$$
{}^{ABC}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}z(\varpi) = \frac{\mathscr{M}(\mathfrak{F})}{1-\mathfrak{F}} \int_0^{\varpi} x'(v) E_{\mathfrak{F}}\big(-\frac{\mathfrak{F}(\varpi - v)^{\mathfrak{F}}}{1-\mathfrak{F}}\big)dv,
$$

and

$$
{}^{ABR}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}z(\varpi) = \frac{\mathscr{M}(\mathfrak{F})}{1-\mathfrak{F}} \int_0^{\varpi} x(v) E_{\mathfrak{F}}\big(-\frac{\mathfrak{F}(\varpi - v)^{\mathfrak{F}}}{1-\mathfrak{F}}\big)dv,
$$

respectively. Here the normalizing function $\mathscr{M}(\mathfrak{F}) > 0$ satisfies $\mathscr{M}(0) = \mathscr{M}(1) = 1$ where $E_{\mathfrak{F}}$ is a Mittag-Liffler function.

Definition 2.2 [28] The fractional integral of AB of order $\varpi \in (0,1)$ and function $r : (d,\mathfrak{F}) \to R$ is,

$$
{}^{ABC}_{0}I^{\mathfrak{F}}_{\varpi}z(\varpi) = \frac{1-\mathfrak{F}}{\mathscr{M}(\mathfrak{F})}z(\varpi) + \frac{\mathfrak{F}}{\mathscr{M}(\mathfrak{F})\Gamma(\mathfrak{F})} \int_0^{\varpi} x(v)(\varpi - v)^{-1}dv, \mathfrak{F} \in (0,1).
$$

Definition 2.3 [29] Let $\mathfrak{F} \in (n-1,n]$ and let x be $x^{(n)} \in H^1(I)$,

$$
{}^{ABC}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}z(\varpi) = {}^{ABC}_{0}\mathscr{D}^{\tau}_{\varpi}z^{(n)}(\varpi).
$$

Lemma 1 [30] Let $h \in \mathfrak{E}$, the soln of problem,

$$
{}^{ABC}_{0}\mathscr{D}^{\mathfrak{F}}_{\varpi}z(\varpi) \in H[h(\varpi), E_1 w(\alpha), E_2 w(\alpha)], \varpi \in I
$$

with boundary conditions

$$
w(0) = w_0, x(b) = x,
$$

is obtained as,

$$
\begin{aligned}
z(\varpi) = & \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b h(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} h(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^\varpi h(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^\varpi (\varpi-v)^{\mathfrak{F}-1} h(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv. \qquad (2.1)
\end{aligned}
$$

Suppose $(\mathfrak{B}, ||.||)$ is a normed space and let,

$$\mathfrak{O}_{cl}(\mathfrak{B}) = \mathscr{M} \in \mathfrak{O}(\mathfrak{B}) : \mathscr{M} \text{ is closed},$$

$$\mathfrak{O}_{cp}(\mathfrak{B}) = \mathscr{M} \in \mathfrak{O}(\mathfrak{B}) : \mathscr{M} \text{ is compact},$$

and

$$\mathfrak{O}_{cp,c}(\mathfrak{B}) = \mathscr{M} \in \mathfrak{O}(\mathfrak{B}) : \mathscr{M} \text{ is compact and convex}.$$

We refer [31,32,33] and H at point $o \in \mathfrak{E}$ is $\mathscr{Q}_{H,o} = \{\mathfrak{I} \in L^1(I,R) : \mathfrak{I}(\varpi) \in H(\varpi, o)\,(\text{a.e})\varpi \in I\}$.

Definition 2.4 [27] Let $H : I \times R \to \mathscr{O}(R)$ is MVM. H can be called a Caratheodory, if the map $\varpi \to H(\varpi, x)$ is measurable for all $x \in R, \& x \to H(\varpi, x)$ is $\varpi \in I$ a.e. Similarly, a SVM H is L^1- Caratheodory, if for every $w > 0$, $\Phi \in L^1(I, R^+)$ so,

$$||H(\varpi, x)|| = sup\{|\delta| \in H(\varpi, x)\} \leq \Phi(\varpi),$$

$\forall ||x|| \leq w$ and $\varpi \in I$ a.e.

3 Existence results

Definition 3.1 The function $x \in \mathfrak{E}$ is a solution of (1), if $\mathfrak{I} \in L^1(I,R)$ with $\mathfrak{I}(\varpi) \in H(\varpi, x)$, for every $\varpi \in I$ satisfies the BCs as, $z(0) = z_0, z(b) = z$.
The solution is obtained as,

$$
\begin{aligned}
z(\varpi) = & \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \mathfrak{I}(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \mathfrak{I}(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^\varpi \mathfrak{I}(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^\varpi (\varpi-v)^{\mathfrak{F}-1} \mathfrak{I}(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv.
\end{aligned}
$$

For multi-valued maps, we apply Leray-Schauder-type theorem [34] to obtain the existence results related to the convex valued H map.

Theorem 3.2 Let

$$\eta = \frac{1}{\mathscr{M}(\mathfrak{F}-1)} \left(2b + \frac{2b^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)} \right), \qquad (3.1)$$

and

(H1)$H : I \times R \to \mathscr{O}_{cp,c}(R)$ is a L^1- Caratheodory MVM.

(H2)$\exists \quad \psi_1 \in C(I,[0,\infty))$ and a non-decreasing $\psi_2 \in C([0,\infty),[0,\infty))$ so that

$$||H(\varpi,x)||_{\mathfrak{O}} = sup|\odot| : \odot \in H(\varpi,x) \leq \psi_1(\varpi)\psi_2(||x||), \forall (\varpi,x) \in I \times R.$$

(H3)Let a constant $\mathscr{N} > 0$,

$$\frac{\mathscr{N}}{|x_1|+|x_0|+\eta||\psi_1||\psi_2(\mathscr{N})} > 1.$$

(H4)There exist $e,h \in \mathscr{C}(\mathscr{J},R_+)$,
$||\int_0^b e(\varpi,v,z(v))dv|| \leq e(\varpi)||\mathfrak{z}||$, for each $\varpi \in \mathscr{J}, \mathfrak{z} \in E$ and
$||\int_0^{\varpi} h(\varpi,v,z(v))dv|| \leq h(\varpi)||\mathfrak{z}||$, for each $\varpi \in \mathscr{J}, \mathfrak{z} \in E$.

Then (1) has a solution on I.
Proof: Firstly, let the problem (1) be converted to a fixed point problem and for this purpose let us define $\mathscr{Z} : \mathfrak{E} \to \mathfrak{O}(\mathfrak{E})$ as

$$\begin{aligned} z(x) = \zeta \in \mathfrak{E} : \zeta(\varpi) = &\frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &- \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &+ \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^{\varpi} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &+ \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv, \end{aligned}$$

for $\psi \in \mathscr{Q}_{H,x}$.
$\therefore \mathscr{Z}$ is a fixed point solution of (1).

Case 1: $\mathscr{Z}(x)$ is convex for any $\zeta \in \mathfrak{E}$.
Let $\zeta_1, \zeta_2 \in \mathscr{Z}(x)$. Then there exist $\psi_1, \psi_2 \in \mathscr{Q}_{H,x}$, so that for every $\varpi \in I$,

$$\begin{aligned} \zeta_j(\varpi) = &\frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \Im_j(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &- \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \Im_j(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &+ \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^{\varpi} \Im_j(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &+ \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1} \Im_j(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv, \end{aligned}$$

and letting $\delta \in [0,1]$, we have for all $\varpi \in I$

$$\begin{aligned} &[\delta\zeta_1 + (1-\delta)\zeta_2](\varpi) \\ &= \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \delta\Im_1(v) + (1-\delta)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &- \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \delta\Im_1(v) + (1-\delta)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &+ \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^{\varpi} \delta\Im_1(v) + (1-\delta)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\ &+ \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1} \delta\Im_1(v) + (1-\delta)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv. \end{aligned}$$

Hence, H and $\mathscr{Q}_{H,x}$ has a convex values, and $[\delta\Im_1(v) + (1-\delta)\Im_2(v)] \in \mathscr{Q}_{H,x}$.
Thus $\delta\zeta_1 + (1-\delta)\zeta_2 \in \mathscr{Z}(x)$.

Case 2: $\mathscr{Z}$ be bounded on bounded sets of $\mathfrak{E}$. Let for $\varsigma \in R^+$,

$$B_\varsigma = \{x \in \mathfrak{E} : ||x|| \leq \varsigma\},$$

be a bounded set in $\mathfrak{E}$,

$$\zeta \in \mathscr{Z}(x), x \in B_\varsigma, \text{and } \psi \in \mathscr{Q}_{H,x}$$

then,

$$\begin{aligned}
\zeta(\varpi) = &\frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&- \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&+ \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^\varpi \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&+ \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^\varpi (\varpi-v)^{\mathfrak{F}-1} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv.
\end{aligned}$$

From (H_2) and for every $\varpi \in I$, we have

$$\begin{aligned}
|\zeta(\varpi)| = &\frac{\varpi|x_1| + |x_0|(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&- \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&+ \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^\varpi |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&+ \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^\varpi (\varpi-v)^{\mathfrak{F}-1} |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv. \\
&\leq |x_1| + |x_2| + \frac{||\psi_1||\psi_2(\varsigma)}{\mathscr{M}(\mathfrak{F}-1)} \left(2b + \frac{2b^{\mathfrak{F}}}{\Gamma(\mathfrak{F}+1)}\right).
\end{aligned}$$

Thus,

$$||\zeta|| \leq |x_1| + |X_0| + \eta||\psi_1||\psi_2(\varsigma).$$

Case 3: To prove that $\mathscr{Z}(B_\zeta)$ is equi-continuous.
Let $x \in B_\zeta$ and $\zeta \in \mathscr{Z}(x)$ and a function $\psi \in \mathscr{Q}_{H,x}$, then

$$\begin{aligned}
\zeta(\varpi) = &\frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&- \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&+ \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^\varpi \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&+ \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^\varpi (\varpi-v)^{\mathfrak{F}-1} \Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv.
\end{aligned}$$

Let $\varpi_1, \varpi_2 \in I < \varpi_2$. Then

$$\begin{aligned}
&|\zeta(\varpi_2) - \zeta(\varpi_1)| \\
&\quad \leq \frac{(\varpi_2 - \varpi_1)(|x_1| + |x_0|)}{b} - \frac{(\varpi_2 - \varpi_1)(2 - \mathfrak{F})}{b\mathscr{M}(\mathfrak{F} - 1)} \int_0^b |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&\quad + \frac{(\varpi_2 - \varpi_1)(\mathfrak{F} - 1)}{b\mathscr{M}(\mathfrak{F} - 1)\Gamma(\mathfrak{F})} \int_0^b (b - v)^{(\mathfrak{F}-1)} |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&\quad + \frac{2 - \mathfrak{F}}{\mathscr{M}(\mathfrak{F} - 1)} \int_{\varpi_1}^{\varpi_2} |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&\quad + \frac{(\mathfrak{F} - 1)}{\mathscr{M}(\mathfrak{F} - 1)\Gamma(\mathfrak{F})} \int_0^{\varpi_1} \left((\varpi_2 - v)^{\mathfrak{F}-1} - (\varpi_1 - v)^{\mathfrak{F}-1}\right) |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
&\quad + \frac{(\mathfrak{F} - 1)}{\mathscr{M}(\mathfrak{F} - 1)\Gamma(\mathfrak{F})} \int_{\varpi_1}^{\varpi_2} (\varpi_2 - v)^{\mathfrak{F}-1} |\Im(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv.
\end{aligned}$$

Similarly by using $(H_2) - (H_4)$, we get

$$\begin{aligned}
&|\zeta(\varpi_2) - \zeta(\varpi_1)| \\
&\leq \frac{(\varpi_2 - \varpi_1)(|x_1| + |x_0|)}{b} + \frac{(\varpi_2 - \varpi_1)||\psi_1||\psi_2(\varsigma)||\mathfrak{z}||}{\mathscr{M}(\mathfrak{F} - 1)} + \frac{b^{\mathfrak{F}}(\varpi_2 - \varpi_1)||\psi_1||\psi_2(\varsigma)||\mathfrak{z}||}{b\mathscr{M}(\mathfrak{F} - 1)\Gamma(\mathfrak{F} + 1)} \\
&\quad + \frac{(\varpi_2 - \varpi_1)||\psi_1||\psi_2(\varsigma)||\mathfrak{z}||}{\mathscr{M}(\mathfrak{F} - 1)} + \frac{(\varpi_2^{\mathfrak{F}} - \varpi_1^{\mathfrak{F}})||\psi_1||\psi_2(\varsigma)||\mathfrak{z}||}{\mathscr{M}(\mathfrak{F} - 1)\Gamma(\mathfrak{F} + 1)}.
\end{aligned}$$

As $\varpi_1 \to \varpi_2$, we obtain

$$|\zeta(\varpi_2) - \zeta(\varpi_1)| \to 0.$$

By Arzela-Ascoli theorem, $\mathfrak{Z}$ is completely continuous and $\mathfrak{Z}(B_\zeta)$ is equi-continuous. From [[31], Proposition 1.2], $\mathfrak{Z}$ has a closed graph, then consequently $\mathfrak{Z}$ is SVM.

Case 4: The graph of $\mathfrak{Z}$ is closed.
Let $x_n \to x_*, \zeta_n \in \mathfrak{Z}(x_n)$ and ζ_n converges to ζ_*.
To prove $\zeta_* \in \mathfrak{Z}(x_*)$.
Because $h_n \in \mathfrak{Z}(x_n)$, and $\psi_n \in \mathscr{Q}_{H,x_n}$,

$$\begin{aligned}
\zeta_n(\varpi) &= \frac{\varpi x_1 + x_0(b - \varpi)}{b} - \frac{\varpi(2 - \mathfrak{F})}{b\mathscr{M}(\mathfrak{F} - 1)} \int_0^b \Im_n(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&\quad - \frac{\varpi(\mathfrak{F} - 1)}{b\mathscr{M}(\mathfrak{F} - 1)(\Gamma(\mathfrak{F}))} \int_0^b (b - v)^{(\mathfrak{F}-1)} \Im_n(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&\quad + \frac{2 - \mathfrak{F}}{\mathscr{M}(\mathfrak{F} - 1)} \int_0^{\varpi} \Im_n(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&\quad + \frac{(\mathfrak{F} - 1)}{\mathscr{M}(\mathfrak{F} - 1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi - v)^{\mathfrak{F}-1} \Im_n(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv.
\end{aligned}$$

Thus, we need to show that $\psi_* \in \mathscr{Q}_{H,x_*}, \varpi \in I$,

$$\begin{aligned}
\zeta_*(\varpi) &= \frac{\varpi x_1 + x_0(b - \varpi)}{b} - \frac{\varpi(2 - \mathfrak{F})}{b\mathscr{M}(\mathfrak{F} - 1)} \int_0^b \Im_*(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&\quad - \frac{\varpi(\mathfrak{F} - 1)}{b\mathscr{M}(\mathfrak{F} - 1)(\Gamma(\mathfrak{F}))} \int_0^b (b - v)^{(\mathfrak{F}-1)} \Im_*(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&\quad + \frac{2 - \mathfrak{F}}{\mathscr{M}(\mathfrak{F} - 1)} \int_0^{\varpi} \Im_*(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
&\quad + \frac{(\mathfrak{F} - 1)}{\mathscr{M}(\mathfrak{F} - 1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi - v)^{\mathfrak{F}-1} \Im_*(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv.
\end{aligned}$$

Define the continuous linear operator $\mathscr{P} : L^1(I,(-\infty,\infty)) \to C(I(-\infty,\infty))$ by,

$$\begin{aligned}\psi \to \mathscr{P}(\psi)(\varpi) = \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)}\int_0^b \Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^b (b-v)^{(\mathfrak{F}-1)}\Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)}\int_0^{\varpi} \Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1}\Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv.\end{aligned}$$

Observe that,

$$\begin{aligned}||\zeta_n-\zeta_*|| = ||-\frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)}\int_0^b \Im_n(v)+\Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^b (b-v)^{(\mathfrak{F}-1)}\Im_n(v)-\Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)}\int_0^{\varpi} \Im_n(v)-\Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1}\Im_n(v)-\Im_2(*)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \to 0||,\end{aligned}$$

From Lazota-Opial result [35],when $n\to\infty$ and $\mathscr{P}\circ\mathscr{Q}_{H,x}$ ia a closed graph operator,
we get

$$\zeta_n \in \mathscr{P}\left(\mathscr{Q}_{H,x_n}\right).$$

Because $x_n \to x_*$,

$$\begin{aligned}\zeta_*(\varpi) = \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)}\int_0^b \Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^b (b-v)^{(\mathfrak{F}-1)}\Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)}\int_0^{\varpi} \Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1}\Im_*(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv,\end{aligned}$$

for some $\Im_* \in \mathscr{Q}_{H,x_*}$.

Case 5: Let $\mathscr{V} \subseteq \mathfrak{E}$ be an open set with $x \notin \vee\mathfrak{Z}(x)$ for every $\vee \in (0,1)$ and $\forall x \in \iota\mathscr{V}$.
Let $\vee \in (0,1)$ and $x \in \vee\mathfrak{Z}(x)$. Then for $\Im \in \mathscr{Q}_{H,x}$,

$$\begin{aligned}|z(\varpi)| = |\frac{\vee\varpi x_1 + \vee x_0(b-\varpi)}{b} - \frac{\vee\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)}\int_0^b \Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ - \frac{\vee\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^b (b-v)^{(\mathfrak{F}-1)}\Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{\vee(2-\mathfrak{F})}{\mathscr{M}(\mathfrak{F}-1)}\int_0^{\varpi} \Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv \\ + \frac{\vee(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^{\varpi} (\varpi-v)^{\mathfrak{F}-1}\Im(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv| \\ \le |x_1| + |x_0| + \eta||\psi_1||\psi_2(\varsigma)||\mathfrak{z}||.\end{aligned}$$

Thus, we have

$$|x(\varpi)| \le |x_1| + |x_0| + \eta||\psi_1||\psi_2(x)||\mathfrak{z}||, \varpi \in I$$

and we obtain

$$\frac{||x||}{|x_1|+|x_0|+\eta||\psi_1||\psi_2(x)||\mathfrak{z}||}\leq 1.$$

By $[H_3]$, we have $\mathfrak{N}>0$ is a constant, $||x||\neq\mathfrak{N}$. Let the set $\mathscr{V}$ be defined as,

$$\mathscr{V}=\{x\in\mathfrak{E}:||x||<\mathfrak{N}\}.$$

By Case $1-4$, $\mathfrak{Z}\mathscr{V}\to\mathfrak{O}((E))$ is completely continuous.
$\therefore$ There is at least one solution to problem (1).

Next using a theorem of Covitz-Nadler [21], we derive an existence result for the inclusion problem (1), when H is a nonconvex-valued mapping.
Define $H_d:\mathfrak{O}(\varepsilon)\times\mathfrak{O}(\varepsilon)\to[0,\infty)\cup\infty$ by,

$$H_d(\widetilde{G},\widetilde{J})=max\left\{\sup_{\widetilde{g}\in\widetilde{G}}d(\widetilde{G},\widetilde{J}),\sup_{\widetilde{j}\in\widetilde{J}}d(\widetilde{g},\widetilde{j})\right\},$$

where $d(\widetilde{G},\widetilde{J})=inf_{\widetilde{g},\widetilde{G}}d(\widetilde{G},\widetilde{J})$ and $d(\widetilde{g},\widetilde{J})=inf_{\widetilde{j},\widetilde{J}}d(\widetilde{G},\widetilde{J})$. Then $\big((O)_{c,cl}(\varepsilon),H_d\big)$ is a metric space.

Definition 3.3 $\mathfrak{Z}:\varepsilon\to\mathfrak{O}_{cl}(\varepsilon)$ is $\tau-$ Lipschitz iff there exists $\tau>0$, so that

$$H_d(\mathfrak{Z}(\beta),\mathfrak{Z}(U_2))\leq\tau d(\beta,U_2)\text{for any}\,\beta,U_2\in\varepsilon.$$

In particular if, $\tau<1$, we have that $\mathfrak{Z}$ is a contraction.

Theorem 3.4 Let,

(H5)$H:I\times R\to\mathscr{O}_{cp}(R)$ is a $H(x):I\to\mathscr{O}_{cp}(R)$ is measurablefor any $x\in R$.
(H6)$H_d(H(\varpi,x),H(\varpi,\bar{x}))\leq\rho(\varpi)|x-\bar{x}|\forall\varpi\in I$ and $x,\bar{x}\in R$ with $\rho\in C(I,[0,\infty))$ and $d(0.H(\varpi,0))\leq\rho(\varpi)\forall\varpi\in I$.

The FIDI of (1) has one solution on I, for when

$$\eta||\rho||<1,$$

η is given in equation (3.1).

Proof: From [H4] and [36], Theorem III H is measurable $\mathfrak{I}:I\to R,\mathfrak{I}\in L^1((I),R)$. $\mathscr{Q}_{H,x}\neq\oslash$. Let $\{\omega_n\}_n\geq 0\in\mathfrak{Z}(x)$ be a $\omega_n\to\omega(n\to\infty)\,in\,\mathfrak{E}.\omega\in\mathfrak{E}\&\mathfrak{I}_n\omega\in\mathscr{Q}_{H,x_n}$,

$$\begin{aligned}\omega_n(\varpi)=&\frac{\varpi x_1+x_0(b-\varpi)}{b}-\frac{\varpi(2-\mathfrak{F})}{b(\mathfrak{F}-1)}\int_0^b\mathfrak{I}_n(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv\\&-\frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^b(b-v)^{(\mathfrak{F}-1)}\mathfrak{I}_n(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv\\&+\frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)}\int_0^{\varpi}\mathfrak{I}_n(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv\\&+\frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^{\varpi}(\varpi-v)^{\mathfrak{F}-1}\mathfrak{I}_n(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv,\forall\varpi\in I.\end{aligned}$$

Thus $\mathfrak{I}\in\mathscr{Q}_H,x$ and

$$\begin{aligned}\omega_n(\varpi)\to\omega(\varpi)=&\frac{\varpi x_1+x_0(b-\varpi)}{b}-\frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)}\int_0^b\mathfrak{I}(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv\\&-\frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^b(b-v)^{(\mathfrak{F}-1)}\mathfrak{I}(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv\\&+\frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)}\int_0^{\varpi}\mathfrak{I}(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv\\&+\frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))}\int_0^{\varpi}(\varpi-v)^{\mathfrak{F}-1}\mathfrak{I}(v)\,[E_1w(\alpha),E_2w(\alpha)]\,dv,\forall\varpi\in I.\end{aligned}$$

Hence $\omega \in \mathfrak{Z}(x)$.
Now to show that, $0 < \varpi < 1, (\varpi = \eta ||\rho||)$,

$$H_d(\mathfrak{Z}(x), \mathfrak{Z}(\bar{x}) \leq \varpi ||x - \bar{x}|| \text{for each} x, \bar{x} \in \mathscr{I}.$$

Here $x, \bar{x}$ and $\zeta_1 \in \mathfrak{Z}(x)$, and $\Im_1(\varpi) \in H(\varpi, x(\varpi))$, $\varpi \in I$

$$\begin{aligned}
\zeta_1(\varpi) = & \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \Im_1(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \Im_1(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^{\varpi} \Im_1(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi - v)^{\mathfrak{F}-1} \Im_1(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv, \forall \varpi \in I.
\end{aligned}$$

Using $[H_6]$

$$H_d(H(\varpi,x), H(\varpi,\bar{x})) \leq \rho(\varpi)|x(\varpi) - \bar{x}(\varpi)|.$$

Thus, $\xi(\varpi) \in H(\varpi, \bar{x})$,

$$|\Im_1(\varpi) - \xi| \leq \rho(\varpi)|x(\varpi) - \bar{x}(\varpi)|, \varpi \in I.$$

Let the operator $\mathscr{F} : I \to \mathfrak{O}(R)$ be defined as

$$\mathscr{F}(\varpi) = \xi \in R : |\Im_1(\varpi) - \xi| \leq \rho(\varpi)|x(\varpi) - \bar{x}(\varpi)|.$$

We find that $\Im_1$ and $\Lambda = \rho|x - \bar{x}|$ are measurable, so MVM $\mathscr{F}(\varpi) \cap H(\varpi, \bar{x})$ is measurable. Choosing $\Im_2(\varpi) \in H(\varpi, \bar{x})$, we have

$$|\Im_1(\varpi) - \Im_2(\varpi)| \leq \rho(\varpi)|x(\varpi) - \bar{x}(\varpi)|, \forall \varpi \in I.$$

Using

$$\begin{aligned}
\zeta_2(\varpi) = & \frac{\varpi x_1 + x_0(b-\varpi)}{b} - \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b \Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} \Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^{\varpi} \Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv \\
& + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi - v)^{\mathfrak{F}-1} \Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,dv, \forall \varpi \in I.
\end{aligned}$$

Also,

$$\begin{aligned}
|\zeta_1(\varpi) - \zeta_2(\varpi)| = & \frac{\varpi(2-\mathfrak{F})}{b\mathscr{M}(\mathfrak{F}-1)} \int_0^b |\Im_1(v) - \Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
& - \frac{\varpi(\mathfrak{F}-1)}{b\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^b (b-v)^{(\mathfrak{F}-1)} |\Im_1(v)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
& + \frac{2-\mathfrak{F}}{\mathscr{M}(\mathfrak{F}-1)} \int_0^{\varpi} |\Im_1(v)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
& + \frac{(\mathfrak{F}-1)}{\mathscr{M}(\mathfrak{F}-1)(\Gamma(\mathfrak{F}))} \int_0^{\varpi} (\varpi - v)^{\mathfrak{F}-1} |\Im_1(v)\Im_2(v)\,[E_1 w(\alpha), E_2 w(\alpha)]\,|dv \\
& \leq ||\alpha||||x - \bar{x}|| \left(\frac{2b}{\mathscr{M}(\mathfrak{F}-1)} + \frac{2b^B}{\mathscr{M}(\mathfrak{F}-1)\Gamma(\mathfrak{F}+1)} \right).
\end{aligned}$$

Hence

$$||\zeta_1 - \zeta_2|| \leq \eta ||\rho|| \, ||x - \bar{x}||.$$

Now, swapping x and $\bar{x}$, one has

$$H_d(\mathfrak{Z}(x),\mathfrak{Z}(\bar{x})) \leq \eta||\rho||\,||x-\bar{x}||,$$

As I is a contraction, we can conclude that it has FP and hence according to Covitz-Nadler theorem problem (1) has a solution.

4 Examples

To demonstrate the validation of the results obtained in the previous section, we consider the following two examples.
Example 1: Suppose we have the following FIDI

$$\begin{aligned}
&{}_0^{ABC}\mathscr{D}_{\varpi}^{\frac{3}{2}}z(\varpi) \in H\left(\varpi, z(\varpi), \int_0^b e(\varpi,s,z(s))ds, \int_0^{\varpi} h(\varpi,s,z(s))ds\right), \varpi \in I=(0,1)\\
&z(0)=z_0, z(1)=1, \mathscr{M}(\mathfrak{F}-1)=1
\end{aligned} \tag{4.1}$$

Here, $\mathfrak{F}=\frac{3}{2}, b=1 \& H:[0,1]\times R \to \mathscr{O}(R)$ is a MVM,

$$x \to H(\varpi,x) = \left[\frac{1}{10(\varpi^3+3\,exp(\varpi))}\frac{x^4}{(x^4+1)}, \frac{1}{\sqrt{\varpi+16}}\frac{|x|}{|x|+1}\right].$$

Obviously H fulfills $[H_1]$, and

$$\begin{aligned}
||H(\varpi,x)||_{\mathfrak{O}} &= sup\{|\odot| :\in H(\varpi,x)\}\\
&\leq \frac{1}{\sqrt{\varpi+16}} = \psi_1(\varpi)\psi_2(||x||)||\mathfrak{z}||,
\end{aligned}$$

which implies $||\psi_1|| = \frac{1}{4}$ and $\psi_2(||x||)=1$.
By $[H_3]$ and in view of theorem 3.2, we get $\mathfrak{N} > 1.8761$. Thus $\exists$ a soln for (1) on $[0,1]$.

Example 2: Suppose we have the following FIDI

$$\begin{aligned}
&{}_0^{ABC}\mathscr{D}_{\varpi}^{\frac{5}{4}}z(\varpi) \in H\left(\varpi, z(\varpi), \int_0^b e(\varpi,s,z(s))ds, \int_0^{\varpi} h(\varpi,s,z(s))ds\right), \varpi \in I=(0,1)\\
&z(0)=z_0, z(1)=1, \mathscr{M}(\mathfrak{F}-1)=1
\end{aligned} \tag{4.2}$$

Here, $\mathfrak{F}=\frac{5}{4}, b=1$ and $H:[0,1]\times R \to \mathscr{O}(R)$ is a MVM,

$$x \to H(\varpi,x) = \left[0, \frac{2sin(x^2)}{(\varpi^2+10)} + \frac{1}{14}\right].$$

$H_d(H(\varpi,x),H(\varpi,\bar{x})) \leq \rho(\varpi)|x-\bar{x}|$, where $\rho(\varpi)=\frac{2}{\varpi^2+10}$ as well $d(0,H(\varpi,0)) = \frac{1}{14} \leq \rho(\varpi), \varpi \in [0,1]$.
$\therefore ||\rho|| = \frac{1}{5}$ and $\eta||\rho|| \approx 0.75 < 1$.
By $[H_5]$ and in view of theorem 3.4, $\exists$ at least a solution for the problem (1) on $[0,1]$.

5 Conclusion

In this study we have analysed a class of FIDIs using the ABC derivative. Two cases of MVM with convex and non convex values were discussed wherein Leray-Schauder theorem was applied to study the former case and Covitz-Nadler theorem was applied to study the latter. The results obtained were then validated with the help of suitable examples.

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