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Generalized q-integral Inequalities using $(\hbar\text{-}m)$ and $(\alpha\text{-}m)$ convexities

Ghulam Farid[1] , *Waseela Bibi*[1], *Mawahib Elamin*[2,*], *Huda Uones Mohamd Ahamd*[3], *Jihan Alahmadi*[4], *Neama Yahia*[5], *and Mortada S. Ali*[6]

[1] Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan
[2] Department of Mathematics, College of Science and Arts, Qassim University, Riyadh Alkhbra, Saudi Arabia
[3] Mathematics department, Faculty of arts and science, King Khalid University, Sarat Abida, Saudi Arabia
[4] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al- Kharj, 11942, Saudi Arabia
[5] Department of Mathematics, College of Science, Tabuk University, Tabuk, Saudi Arabia
[6] Department of Mathematics, College of Science and Arts, Al-Baha University, Al-Makhwah, P.O.Box 1988, Saudi Arabia

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Abstract: In this paper certain inequalities are established for q-h-integrals by applying definitions of two types of convex functions. The Hermite-Hadamard inequality in different variants for q-h-integrals is given by using $(\hbar\text{-}m)$- and $(\alpha\text{-}m)$-convex functions. Some well-known q-integral inequalities for several types of convex functions are deduced.

Keywords: Differential equations, convex function, fractional derivatives, Hermite–Hadamard inequality

1 Introduction and Preliminaries

Inequalities have an important role in mathematical modeling of almost all kinds of real world problems. Differential equations along with initial and boundary conditions, are vital tools in expressing various problems of science and engineering. Then these problems are solved by applying different integral transformations and techniques. Nowadays, in place of usual derivatives and integrals some new and generalized notions are utilized in representing classical problems in general form. As a result difference equations occur which are dealt with new and generalized methods. In this context one can consider fractional derivatives, q-derivatives, h-derivatives and q-h-derivatives etc. For a detailed study we refer the readers to [1,2,3].
New and generalize inequalities established in recent years are due to fractional integrals, q-integrals and (p,q)-integrals [4,5]. Various new classes of real valued functions are also explored for the sake of generalizations, refinements and extensions of classical results. For example convex functions and related notions are very frequently analyzed to get diverse variants of classical inequalities for different kinds of integrals, see [6,7,8,9].

The aim of this paper is to utilize classes of $(\hbar\text{-}m)$- and $(\alpha\text{-}m)$-convex functions in deriving Hermite-Hadamard type inequalities. By using a new and generalize definition of derivatives and integrals on finite intervals these inequalities are obtained. Results of some recent articles are also reproduced from findings of this paper. Let we start by defining convex function and stating the Hermite-Hadamard inequality as follows:

Definition 1. *A real valued function f is said to be convex on an interval I, if the following inequality holds:*

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b),$$

for $t \in [0,1]$, $a,b \in I$.

Theorem 1. *The following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \qquad (1)$$

for a convex function f defined on an interval $I \subset \mathbb{R}$ and $a,b \in I$ where $a<b$.

The inequality (1) has been published in several variants for different types of functions and new kinds of

* Corresponding author e-mail: ma.elhag@qu.edu.sa

integrals. For instance, it is studied for fractional order derivatives/integrals in[6,10], for quantum derivatives / integrals one can see [11,12]. Next, we define q-definite integrals and state the Hadamard inequality for convex functions.

Definition 2.*[13] Let $f: I \to \mathbb{R}$ be a continuous function. Then the q-definite integral on $[a,b]$ is defined as*

$$\int_a^x f(x)d_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{2}$$

for $x \in [a,b]$, $a,b \in I$, $a<b$.

In [10], by applying q-definite integrals the following q-Hadamard inequality for convex functions is proved:

Theorem 2. *Let $f:[a,b] \to \mathbb{R}$ be a differentiable convex function. Then for q-integrals the following inequality holds:*

$$f\left(\frac{b+aq}{1+q}\right) \le \frac{\int_a^b f(x)d_qx}{b-a} \le \frac{qf(a)+f(b)}{1+q}. \tag{3}$$

Further, we give definition of q-h-integrals and the q-h-Hadamard inequality for convex functions.

Definition 3.*[14] Let $0<q<1$ and function $f: I=[a,b] \to \mathbb{R}$ be a continuous function. Then the left q-h-integral and the right q-h-integral on I denoted by $I^{a^+}_{q-h}f$ and $I^{b^-}_{q-h}f$ are defined as follows:*

$$I^{a^+}_{q-h}f(x) := \int_a^x f(t)_h d_qt. = ((1-q)(x-a)+qh)$$
$$\times \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)x + nq^n h),\ x>a, \tag{4}$$

$$I^{b^-}_{q-h}f(x) := \int_x^b f(t)_h d_qt. = ((1-q)(b-x)+qh)$$
$$\times \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b + nq^n h),\ x<b. \tag{5}$$

Theorem 3.*[15] Let $\sum_{k=0}^{\infty} kq^{2k} = S$ and $q \in (0,1)$. Also, let $f: I \to \mathbb{R}$ be a convex function, for $a,b \in I$, $a<b$, the following inequality holds for q-h-integrals:*

$$f\left(\frac{a+qx}{1+q} + (1-q)hS\right) + f\left(\frac{x+qb}{1+q} + (1-q)hS\right)$$
$$\le \frac{1-q}{(1-q)(x-a)+qh}\int_a^x f(x)_h d_qx$$
$$+ \frac{1-q}{(1-q)(b-x)+qh}\int_x^b f(x)_h d_qx \le$$
$$\frac{(f(a)+qf(b))(b-a)+(1+q)(f(a)(b-x)+f(b)(x-a))}{(1+q)(b-a)}$$
$$+ \frac{2(f(b)-f(a))}{b-a} \times hS(1-q). \tag{6}$$

In the following we define some extended definitions of convex functions. These definitions will be applied for establishing the results of this paper.

Definition 4.*[16] Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $\bar{h}: J \to \mathbb{R}$ be a non negative function. We say $f:[0,b] \to \mathbb{R}$ is a $(\bar{h}$-$m)$-convex function, if f is non-negative and for all $x,y \in [0,b]$, $m \in [0,1]$ and $t \in (0,1)$ one has*

$$f(tx+m(1-t)y) \le \bar{h}(t)f(x) + m\bar{h}(1-t)f(y).$$

If we choose $m=1$, then we have $\bar{h}$-convex function. If $\bar{h}(t)=t$, we obtain non-negative m-convex function. If $m=1$ and $\bar{h}(t)=t$, we get convex function.

Definition 5.*[17] For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $f := [0,\infty) \to \mathbb{R}$ is said to be $(s-m)$- convex on I if*

$$f(tx+m(1-t)y) \le t^s f(x) + m(1-t)^s f(y) \tag{7}$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 6.*[18] Let $\alpha, m \in [0,1]$. The function $f := I \to \mathbb{R}$ is said to be (α,m)-convex if*

$$f(tx+m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{8}$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 7.*[19] Let $\bar{h}: (0,1) \subseteq J \to \mathbb{R}$ be a non-negative and non-zero function and I be interval in R. We say that $f := I \to \mathbb{R}$ is a (p,h)-convex function or f belongs to the class $ghx(h,p,I)$, if f is non-negative and*

$$f([\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}}) \le \bar{h}(\alpha)f(x) + \bar{h}(1-\alpha)f(y), \tag{9}$$

for all $x,y \in I$ and $\alpha \in (0,1)$.

In the forthcoming section we prove q-h-integral inequalities for monotone convex functions. The q-h-Hermite-Hadamard type inequalities for generalized convex functions are established. Several variants of q-Hermite-Hadamard inequalities are deducible in from main results.

2 Generalized q-h-Hermite Hadamard inequalities

Theorem 4.*Let $g: I \to \mathbb{R}$ be a real valued q-h-integrable function, and $h \ge 0$.*
(i) If g is decreasing and convex, then the q-h-integral satisfies the following inequality:

$$\frac{\int_a^x g(x)_{h_1} d_qx}{(1-q)(x-a)+qh_1} + \frac{\int_x^b g(x)_{h_2} d_qx}{(1-q)(b-x)+qh_2}$$
$$\le \frac{g(a)+g(x)+q(g(x)+g(b))}{1-q^2}, \tag{10}$$

where $h_1 = (x-a)h$ and $h_2 = (b-x)h$, $x \in [a,b]$, $a,b \in I$ and $a < b$.
(ii) If g is increasing and concave function, then the reverse of above inequality holds:

Proof.(i) For $h \geq 0$, $k \in \mathbb{N}$, $q \in (0,1)$ and $x \in [a,b]$, we have

$$q^k a + (1-q^k)x + kq^k h \geq q^k a + (1-q^k)x. \tag{11}$$

By using monotonicity and convexity of g, one can have

$$q^k g(q^k a + (1-q^k)x + kq^k h) \leq q^k(q^k g(a) + (1-q^k)g(x)).$$

From which one can get the following expression for infinite sums:

$$\sum_{k=0}^{\infty} q^k g(q^k a + (1-q^k)x + kq^k h) \leq \sum_{k=0}^{\infty} q^k(q^k g(a) + (1-q^k)g(x)).$$

Keeping in view Definition 4, for left hand side, while calculating the sum of right hand side, the following inequality is yielded:

$$\frac{1}{(1-q)(x-a)+qh_1}\int_a^x g(x)_{h_1} d_q x \leq \frac{g(a)+qg(x)}{1-q^2}. \tag{12}$$

Also, we have that

$$q^k x + (1-q^k)b + kq^k h \geq q^k x + (1-q^k)b. \tag{13}$$

By using monotonicity and convexity of g, one can have

$$q^k g(q^k x + (1-q^k)b + kq^k h) \leq q^k(q^k g(x) + (1-q^k)g(b)).$$

From which one can get the following expression for infinite sums:

$$\sum_{k=0}^{\infty} q^k g(q^k x + (1-q^k)b + kq^k h) \leq \sum_{k=0}^{\infty} q^k(q^k g(x) + (1-q^k)g(b)).$$

Keeping in view Definition 5 for right hand side, while calculating the sum of right hand side, the following inequality yielded:

$$\frac{1}{(1-q)(b-x)+qh_2}\int_x^b g(x)_{h_2} d_q x \leq \frac{g(x)+qg(b)}{1-q^2}. \tag{14}$$

From (12) and (14), one can constitute the required inequality.
(ii) The proof is similar to the proof of (10).

*Remark.*Let $h \leq 0$ in Theorem 4. (i) If g is increasing and convex, then (10) also holds. (ii) Moreover, if g is decreasing and concave, then (10) holds in reverse order.

Theorem 5.*Let $g : I \to \mathbb{R}$ be $(\bar{h}\text{-}m)$-convex function and g be q-h-integrable, and $h \geq 0$.*
(i) If g is decreasing and convex, then the q-h-integral satisfies the following inequality:

$$\frac{\int_a^x g(x)_{h_1} d_q x}{(1-q)(x-a)+qh_1} + \frac{\int_x^b g(x)_{h_2} d_q x}{(1-q)(b-x)+qh_2} \leq (g(a)+g(x))\sum_{k=0}^{\infty} q^k \bar{h}\left(q^k\right) + m\sum_{k=0}^{\infty} q^k \times \bar{h}\left(1-q^k\right)\left(g\left(\frac{x}{m}\right)+g\left(\frac{b}{m}\right)\right), \tag{15}$$

where h_1, h_2 are same as in Theorem 4, $x \in [a,b]$, $a,b \in I$, $a > 0$ and $m \neq 0$.
(ii) If g is increasing and concave function then the reverse of above inequality holds.

Proof.(i) By using monotonicity and $(\bar{h}\text{-}m)$-convexity of g, one can have the following inequality from (11):

$$q^k g\left(q^k a + m(1-q^k)\frac{x}{m} + kq^k h\right) \leq q^k\left(\bar{h}(q^k)g(a) + m\bar{h}(1-q^k)g\left(\frac{x}{m}\right)\right).$$

From which one can get the following expression for infinite sums: $\sum_{k=0}^{\infty} q^k\left(g\left(q^k a + m\left(1-q^k\right)\frac{x}{m} + kq^k h\right)\right) \leq \sum_{k=0}^{\infty} q^k\left(\bar{h}\left(q^k\right)g(a) + m\bar{h}\left(1-q^k\right)g\left(\frac{x}{m}\right)\right)$.
Keeping in view Definition 4 for left hand side, while calculating the sum of right hand side, the following inequality is yielded:

$$\frac{1}{(1-q)(x-a)+qh_1}\int_a^x g(x)_{h_1} d_q x \leq \sum_{k=0}^{\infty} q^k \bar{h}(q^k)g(a) + m\sum_{k=0}^{\infty} q^k \bar{h}(1-q^k)g\left(\frac{x}{m}\right). \tag{16}$$

On the other hand by using monotonicity and $(\bar{h}\text{-}m)$-convexity of g, one can have the following inequality from (13):

$$q^k g\left(q^k x + m(1-q^k)\frac{b}{m} + kq^k h\right) \leq q^k\left(\bar{h}(q^k)g(x) + m\bar{h}(1-q^k)g\left(\frac{b}{m}\right)\right)$$

From which one can get the following expression for infinite sums:

$$\sum_{k=0}^{\infty} q^k g\left(q^k x + m(1-q^k)\frac{b}{m} + kq^k h\right)\Bigg) \leq \sum_{k=0}^{\infty} q^k\left(\bar{h}(q^k)g(x) + m\bar{h}(1-q^k)g\left(\frac{b}{m}\right)\right).$$

Keeping in view Definition5 for right hand side, while calculating the sum of right hand side, the following inequality is yielded:

$$\frac{1}{(1-q)(b-x)+qh_2}\int_x^b g(x)_{h_2}d_qx$$
$$\leq \sum_{k=0}^{\infty} q^k\bar{h}\left(q^k\right)g(x)+m\sum_{k=0}^{\infty} q^k\bar{h}\left(1-q^k\right)g\left(\frac{b}{m}\right) \quad (17)$$

From(16) and(17) , one can constitute the required inequality.

(ii) The proof is similar to the proof of(15).

*Remark.*Let $h \leq 0$ in Theorem 4. (i) If g is increasing and convex, then(15) also holds. (ii) Moreover, if g is decreasing and concave, then (15) holds in reverse order.

Theorem 6.*Let $g: I \to \mathbb{R}$ be $(\alpha$-$m)$-convex function and g be q-h-integerable and $h \geq 0$.*
(i) If g is decreasing and convex, then the q-h-integral satisfies the following inequality :

$$\frac{\int_a^x g(x)_{h_1}d_qx}{(1-q)(x-a)+qh_1}+\frac{\int_x^b g(x)_{h_2}d_qx}{(1-q)(b-x)+qh_2}$$
$$\leq \frac{(1-q)(g(a)+g(x))+mq(1-q^{\alpha})\left(g\left(\frac{x}{m}\right)+g\left(\frac{b}{m}\right)\right)}{(1-q)(1-q^{1+\alpha})}, \quad (18)$$

where h_1, h_2 are same as in Theorem 4, $x \in [a,b]$, $a,b \in I$, $a > 0$ and $m \neq 0$.
(ii) If g is increasing and concave function then the reverse of above inequality holds.

*Proof.*We leave the proof for reader.

*Remark.*Let $h \leq 0$ in Theorem 4. (i) If g is increasing and convex, then (18) also holds. (ii) Moreover, if g is decreasing and concave, then (18) holds in reverse order.

Theorem 7.*Let I be an interval in $\mathbb{R}$, $g: I \to \mathbb{R}$ be $qa < b$.*
(i) If g is symmetirc about $\frac{a+z}{2}$, $z \in (a,b)$, then left q-h-integrals satisfy the following inequality:

$$\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{a+z}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(z-a)+qh_1}\int_a^z g(t)_{h_1}d_qt$$
$$\leq g(z)\int_0^1 \hbar(t)_h d_qt + m\times g\left(\frac{a}{m}\right)\int_0^1 \hbar(1-t)_h d_qt. \quad (19)$$

(ii) If g is symmetirc about $\frac{z+b}{2}$, $z \in (a,b)$, then right q-h-integrals satisfy the following inequality:

$$\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{z+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-z)+qh_2}\int_z^b g(t)_{h_2}d_qt$$
$$\leq g(b)\int_0^1 \hbar(t)_h d_qt + m\times g\left(\frac{z}{m}\right)\int_0^1 \hbar(1-t)_h d_qt, \quad (20)$$

where h_1 and h_2 is same as in Theorem 4.

*Proof.*Here we proof the inequality (19) for left q-h-integral and leave the proof of (20) for the reader.
(i) It is given that g is $(\bar{h}$-$m)$-convex, hence the following inequality holds:

$$\frac{1}{\bar{h}\left(\frac{1}{2}\right)}g\left(\frac{a+z}{2}\right) \leq g(ta+(1-t)z)$$
$$+mg\left(\frac{tz+(1-t)a}{m}\right), t \in [0,1].$$

Applying q-h-integral on the above inequality one can get

$$\frac{1}{\bar{h}\left(\frac{1}{2}\right)}g\left(\frac{a+z}{2}\right) \leq \frac{(1-q)}{(1-q)+qh}\left(\int_0^1 g(ta+(1-t)z)_h d_qt\right.$$
$$\left.+mg\left(\frac{tz+(1-t)a}{m}\right)_h d_qt\right), t \in [0,1]. \quad (21)$$

By using the condition $g\left(\frac{a+z-u}{m}\right) = f(u)$ for all $u \in (a,z)$, one can have:

$$\frac{1}{\bar{h}\left(\frac{1}{2}\right)}g\left(\frac{a+z}{2}\right) \leq \frac{(1+m)(1-q)}{(1-q)+qh}\int_0^1 g(a+(z-a)t)_h d_qt. \quad (22)$$

This further leads to the following inequality, for left q-h-integrals:

$$\frac{((1-q)+qh)}{(1-q)(z-a)+qh_1}\int_a^z g(t)_{h_1}d_qt$$
$$=((1-q)+qh)\sum_{k=0}^{\infty} q^k g(q^k a+(1-q^k)z+kq^k(z-a)h)$$
$$=\int_0^1 g(a+(z-a)t)_h d_qt. \quad (23)$$

Again using the $(\bar{h}$-$m)$-convexity on the last term of above inequality the following inequality is yielded:

$$\int_0^1 g(a+(z-a)t)_h d_qt \leq g(z)\int_0^1 \hbar(t)_h d_qt$$
$$+mg\left(\frac{a}{m}\right)\int_0^1 \hbar(1-t)_h d_qt.$$

The inequality (23) leads to the upcoming inequality:

$$\frac{((1-q)+qh)}{(1-q)(z-a)+qh_1}\int_a^z g(t)_{h_1}d_qt \leq g(z)\int_0^1 \bar{h}(t)_h d_qt$$
$$+mg\left(\frac{a}{m}\right)\int_0^1 \bar{h}(1-t)_h d_qt. \quad (24)$$

Inequalities (22), (23) and (24) constitute the required inequality (19).

Corollary 1. *The upcoming inequalities hold for left and right q-integrals, by setting $h = 0$, in (19) and (20) respectively:*

$$\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{a+z}{2}\right) \leq \frac{1+m}{z-a}\int_a^z g(t)d_qt \leq g(z)\int_0^1 \bar{h}(t)d_qt + mg\left(\frac{a}{m}\right) \times \int_0^1 \hbar(1-t)d_qt,$$

$$\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{z+b}{2}\right) \leq \frac{1+m}{b-z}\int_z^b g(t)_{h_2}d_qt \leq g(b)\int_0^1 \hbar(t)_hd_qt + mg\left(\frac{z}{m}\right) \times \int_0^1 \hbar(1-t)_hd_qt.$$

Theorem 8. *From Theorem 4, the following inequality can also be obtained:*

$$\frac{1}{\bar{h}\left(\frac{1}{2}\right)}g\left(\frac{a+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-a)+qh_3}\int_a^b g(t)_{h_3}d_qt \leq g(b)\int_0^1 \bar{h}(t)_hd_qt + mg\left(\frac{a}{m}\right) \times \int_0^1 \bar{h}(1-t)_hd_qt, \quad (25)$$

where $(b-a)h = h_3$.

Proof. If we put $z = b$ in (19), we get the following inequality:

$$\frac{1}{\bar{h}(\frac{1}{2})}g\left(\frac{a+b}{2}\right) \leq \frac{((1-q)+qh)}{(1-q)(b-a)+qh_3}\int_a^b g(t)_{h_3}d_qt \leq g(b)\int_0^1 \bar{h}(t)_hd_qt + mg\left(\frac{a}{m}\right)\int_0^1 \bar{h}(1-t)_hd_qt. \quad (26)$$

If we put $z = a$ in (20), we get the following inequality:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}g\left(\frac{a+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-a)+qh_3}\int_a^b g(t)_{h_3}d_qt \leq g(b)\int_0^1 \hbar(t)_hd_qt + mg\left(\frac{a}{m}\right) \times \int_0^1 \hbar(1-t)_hd_qt. \quad (27)$$

From (31) and (32), one can get the required inequality (25).

Corollary 2. *The upcoming inequality hold for q-integrals, by setting $h = 0$, in (25):*

$$\frac{1}{\bar{h}(\frac{1}{2})}g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(t)d_qt \leq g(b)\int_0^1 \bar{h}(t)d_qt + mg\left(\frac{a}{m}\right)\int_0^1 \bar{h}(1-t)d_qt. \quad (28)$$

Theorem 9. *Under the assumptions of Theorem 2 and let $\bar{h}(t) = t^s$ in (19) and (20) and using $(a+b)^s \leq a^s + b^s$ where $0 < s < 1$, we get the inequalities for $(s-m)$-convex function:*

$$\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{a+z}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(z-a)+qh_1}\int_a^z g(t)_{h_1}d_qt \leq ((1-q)+qh)\left(g(z)\left(\sum_{k=0}^{\infty} q^k \times (1-q^k)^s + \frac{kh}{1-q^{2s}}\right) + mg\left(\frac{a}{m}\right)\left(\frac{1+kh}{1-q^{2s}}\right)\right). \quad (29)$$

$$\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{z+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-z)+qh_2}\int_z^b g(t)_{h_2}d_qt \leq ((1-q)+qh)\left(g(b)\left(\sum_{k=0}^{\infty} q^k \times (1-q^k)^s + \frac{kh}{1-q^{2s}}\right) + mg\left(\frac{z}{m}\right)\left(\frac{1+kh}{1-q^{2s}}\right)\right). \quad (30)$$

Theorem 10. *Let $g : I \to \mathbb{R}$ be $(\alpha\text{-}m)$-convex function differentiable on (a,b) and $q \in (a,b)$.*
(i) If g is symmetric about $\frac{a+z}{2}$, $z \in (a,b)$, then left q-h-integrals satisfy the following inequality:

$$2g\left(\frac{a+z}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(z-a)+qh_1}\int_a^z g(t)_{h_1}d_qt \leq \left(g(z) - mg\left(\frac{a}{m}\right)\right)\left(\sum_{K=0}^{\infty} q^k \times ((1-q^k)^\alpha + \frac{kh}{1-q^{2\alpha}}\right) + \frac{m}{1-q}g\left(\frac{a}{m}\right). \quad (31)$$

(ii) If g is symmetric about $\frac{z+b}{2}$, $z \in (a,b)$, then right q-h-integrals satisfy the following inequality:

$$2g\left(\frac{z+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-z)+qh_2}\int_z^b g(t)_{h_2}d_qt \leq \left(g(b) - mg\left(\frac{z}{m}\right)\right)\left(\sum_{K=0}^{\infty} q^k \times ((1-q^k)^\alpha + \frac{kh}{1-q^{2\alpha}}\right) + \frac{m}{1-q}g\left(\frac{z}{m}\right), \quad (32)$$

where h_1 and h_2 is same as in Theorem 2.

Proof. Here we proof the inequality (31) for left q-h-integral and leave the proof of (32) for the reader.
(i) It is given that g is $(\alpha\text{-}m)$-convex, hence the following inequality holds:

$$2g\left(\frac{a+z}{2}\right) \leq g(ta+(1-t)z) + mg\left(\frac{tz+(1-t)a}{m}\right), t \in [0,1].$$

Applying q-h-integral on the above inequality one can get

$$2g\left(\frac{a+z}{2}\right) \leq \frac{(1-q)}{(1-q)+qh}\left(\int_0^1 g(ta+(1-t)z)_h d_qt + m \times g\left(\frac{tz+(1-t)a}{m}\right)_h d_qt\right), t\in[0,1] \quad (33)$$

By using the condition $f\left(\frac{a+z-u}{m}\right) = f(u)$ for all $u\in(a,z)$, one can have

$$2g\left(\frac{a+z}{2}\right) \leq \frac{(1+m)(1-q)}{(1-q)+qh}\int_0^1 g(a+(z-a)t)_h d_qt. \quad (34)$$

Again by using the (α-m)-convexity on the last term of (23) the following inequality yielded:

$$\frac{((1-q)+qh)}{(1-q)(z-a)+qh_1}\int_a^z g(t)_{h_1}d_qt \leq f(z)\int_0^1 t_h^\alpha d_qt + mf\left(\frac{a}{m}\right)\int_0^1 (1-t^\alpha)_h d_qt \quad (35)$$

Hence from (23), (34) and (35), we get the following inequality:

$$2g\left(\frac{a+x}{2}\right) \leq \frac{(1+m)(1-q)}{(1-q)(x-a)+qh_1}\int_a^x g(t)_{h_1}d_qt \leq g(x)\int_0^1 t_h^\alpha d_qt + mg\left(\frac{a}{m}\right)\int_0^1 (1-t^\alpha)_h d_qt. \quad (36)$$

From definition, we have that

$$\int_0^1 t_h^\alpha d_qt = ((1-q)+qh)\sum_{k=0}^{\infty} q^k((1-q^k)+kq^kh)^\alpha,$$

by using $(a+b)^\alpha \leq a^\alpha + b^\alpha$ where $0<\alpha<1$ we have

$$\int_0^1 t_h^\alpha d_qt = ((1-q)+qh)\sum_{k=0}^{\infty} q^k((1-q^k)+kq^kh)^\alpha \leq ((1-q)+qh)\left(\sum_{k=0}^{\infty} q^k((1-q^k)^\alpha + \frac{kh}{1-q^{2\alpha}}\right). \quad (37)$$

Using (37) in (36), constitute the required inequalities.

Corollary 3. *The upcoming inequalities hold for left and right q-integrals, by setting $h=0$, in (31) and (32) respectively:*

$$2g\left(\frac{a+x}{2}\right) \leq \frac{1+m}{x-a}\int_a^x g(t)d_qt \leq \left(g(x)-mg\left(\frac{a}{m}\right)\right)\sum_{K=0}^{\infty} q^k(1-q^k)^\alpha + \frac{m}{1-q}g\left(\frac{a}{m}\right). \quad (38)$$

$$2f\left(\frac{x+b}{2}\right) \leq \frac{1+m}{b-x}\int_x^b f(t)d_qt \leq \left(f(b)-mf\left(\frac{x}{m}\right)\right)\sum_{K=0}^{\infty} q^k(1-q^k)^\alpha + \frac{m}{1-q}f\left(\frac{x}{m}\right). \quad (39)$$

Theorem 11. *From Theorem 10, the following inequality can also be obtained:*

$$2g\left(\frac{a+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-a)+qh_3}\int_a^b g(t)_{h_3}d_qt \leq \left(g(b)-mg\left(\frac{a}{m}\right)\right)\left(\sum_{K=0}^{\infty} q^k((1-q^k)^\alpha + \frac{kh}{1-q^{2\alpha}}\right) + \frac{m}{1-q}g\left(\frac{a}{m}\right). \quad (40)$$

Proof. If we put $x=b$ in (31), we get the following inequality:

$$2g\left(\frac{a+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-a)+qh_1}\int_a^b g(t)_{h_3}d_qt \leq \left(g(b)-mg\left(\frac{a}{m}\right)\right)\left(\sum_{K=0}^{\infty} q^k\times((1-q^k)^\alpha + \frac{kh}{1-q^{2\alpha}}\right) + \frac{m}{1-q}g\left(\frac{a}{m}\right). \quad (41)$$

If we put $x=a$ in (32), we get the following inequality:

$$2g\left(\frac{a+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-a)+qh_3}\int_a^b g(t)_{h_3}d_qt \leq \left(g(b)-mg\left(\frac{a}{m}\right)\right)\left(\sum_{K=0}^{\infty} q^k\times((1-q^k)^\alpha + \frac{kh}{1-q^{2\alpha}}\right) + \frac{m}{1-q}g\left(\frac{a}{m}\right). \quad (42)$$

From (41) and (42), one can get the required inequality (40).

Corollary 4. *If $h=0$ in (40) we have*

$$2g\left(\frac{a+b}{2}\right) \leq \frac{1+m}{b-a}\int_a^b g(t)d_qt \leq \left(g(b)-mg\left(\frac{a}{m}\right)\right)\sum_{K=0}^{\infty} q^k((1-q^k)^\alpha + \frac{m}{1-q}g\left(\frac{a}{m}\right).$$

Theorem 12. *Let $\bar{h}:(0,1)\subseteq J\to\mathbb{R}$ be a non-negative and non-zero function and let $g:=I\to\mathbb{R}$ is a $(p,\bar{h})$-convex function then,*
(i) If g is symmetric about $\frac{a^p+x^p}{2}$, $x\in(a,b)$, then for left q-h-integrals the following inequality holds:

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+x^p}{2}\right)^{\frac{1}{p}} \leq \frac{2(1-q)}{(1-q)(x^p-a^p)+qh_4}\int_{a^p}^{x^p} g(t)_{h_4}^{\frac{1}{p}}d_qt \leq g(x^p)\int_0^1 \bar{h}(t)_h d_qt + g(a^p)\int_0^1 \bar{h}(1-t)_h d_qt, \quad (43)$$

where $(x^p - a^p)h = h_4$.
(ii) *If* g *is symmetric about* $\frac{x^p+b^p}{2}$, $x \in (a,b)$, *then we have the following inequality for right q-h-integrals.*

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{x^p+b^p}{2}\right)^{\frac{1}{p}} \le \frac{2(1-q)}{(1-q)(b_p-x^p)+qh_5}\int_{x^p}^{b^p} g(t)_{h_5}^{\frac{1}{p}} d_qt$$
$$\le g(b^p)\int_0^1 \bar{h}(t)_h d_qt + g(x^p)\int_0^1 \bar{h}(1-t)_h d_qt, \tag{44}$$

where $(b^p - x^p) = h_5$.

*Proof.*Here we proof the inequality (43) for left q-h-integral and leave the proof of (44) for the reader.
(i) It is given that g is (p,h)-convex, hence the following inequality holds:

$$g\left(\frac{a^p+x^p}{2}\right)^{\frac{1}{p}}$$
$$\le \hbar\left(\frac{1}{2}\right)\left(g(ta^p+(1-t)x^p)^{\frac{1}{p}} + g(tx^p+(1-t)a^p)^{\frac{1}{p}}\right),$$
$$t \in [0,1].$$

Applying q-h-integral on the above inequality, one can get the

$$\frac{1}{\bar{h}\left(\frac{1}{2}\right)} g\left(\frac{a^p+x^p}{2}\right)^{\frac{1}{p}}$$
$$\le \frac{(1-q)}{(1-q)+qh}\left(\int_0^1 g(ta^p+(1-t)x^p)_h^{\frac{1}{p}} d_qt\right.$$
$$\left. + g\left(tx^p+(1-t)a^p\right)^{\frac{1}{p}}{}_h d_qt\right), t \in [0,1]. \tag{45}$$

By using the condition $g(a^p+x^p-z)^{\frac{1}{p}} = g(z)^{\frac{1}{p}}$ for all $z \in (a^p, x^p)$, one can get

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+x^p}{2}\right)^{\frac{1}{p}}$$
$$\le \frac{2(1-q)}{(1-q)+qh}\int_0^1 g(a^p+(x^p-a^p)t)_h^{\frac{1}{p}} d_qt. \tag{46}$$

This further leads to the following inequality, for left q-h-integrals:

$$\frac{((1-q)+qh)}{(1-q)(x^p-a^p)+qh_4}\int_{a^p}^{x^p} g(t)_{h_4} d_qt$$
$$= ((1-q)+qh)\sum_{k=0}^{\infty} q^k g(q^k a^p + (1-q^k)x^p + kq^k(x^p-a^p)h)$$
$$= \int_0^1 g(a^p+(x^p-a^p)t)_h^{\frac{1}{p}} d_qt. \tag{47}$$

Again by using the $(p,\bar{h})$-convexity on the last term of above inequality the following inequality yielded:

$$\int_0^1 g(a^p+(x^p-a^p)t)_h^{\frac{1}{p}} d_qt$$
$$\le g(x^p)\int_0^1 \bar{h}(t)_h d_qt + g(a^p)\int_0^1 \hbar(1-t)_h d_q, \tag{48}$$

the inequality (47) leads to the upcoming inequality:

$$\frac{((1-q)+qh)}{(1-q)(x^p-a^p)+qh_4}\int_{a^p}^{x^p} g(t)_{h_4} d_qt$$
$$\le g(x^p)\int_0^1 \bar{h}(t)_h d_qt + g(a^p)\int_0^1 \bar{h}(1-t)_h d_qt. \tag{49}$$

Hence from (46), (47) and (49), one can get the required inequality (43).

Corollary 5.*The upcoming inequalities hold for left and right q-integrals, by setting* $h = 0$, *in* (43) *and* (44) *respectively:*

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+x^p}{2}\right)^{\frac{1}{p}} \le \frac{2}{x^p-a^p}\int_{a^p}^{x^p} g(t)^{\frac{1}{p}} d_qt$$
$$\le g(x^p)\int_0^1 \bar{h}(t)d_qt + g(a^p)\int_0^1 \bar{h}(1-t)d_qt,$$

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{x^p+b^p}{2}\right)^{\frac{1}{p}} \le \frac{2}{b_p-x^p}\int_{x^p}^{b^p} g(t)^{\frac{1}{p}} d_qt$$
$$\le g(b^p)\int_0^1 \hbar(t)d_qt + g(x^p)\int_0^1 \bar{h}(1-t)d_qt.$$

Theorem 13.*Under the assumptions of Theorem 8, we get the following inequality:*

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$$
$$\le \frac{2(1-q)}{(1-q)(b^p-a^p)+qh_6}\int_{a^p}^{b^p} g(t)_{h_6}^{\frac{1}{p}} d_qt$$
$$\le g(b^p)\int_0^1 \bar{h}(t)_h d_qt + g(a^p)\int_0^1 \hbar(1-t)_h d_qt, \tag{50}$$

where $(b^p - a^p) = h_6$.

Proof.(i) By setting $x = b$, in (43), we get the following inequality:

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$$
$$\le \frac{2(1-q)}{(1-q)(b^p-a^p)+qh_6}\int_{a^p}^{b^p} g(t)_{h_6}^{\frac{1}{p}} d_qt$$
$$\le g(b^p)\int_0^1 \bar{h}(t)_h d_qt + g(a^p)\int_0^1 \bar{h}(1-t)_h d_qt, \tag{51}$$

(ii)By setting $x = a$, in (44), we get the following inequality:

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \leq \frac{2(1-q)}{(1-q)(b_p-a^p)+qh_6}\int_{a^p}^{b^p} g(t)^{\frac{1}{p}}_{h_6}d_qt \leq g(b^p)\int_0^1 \hbar(t)_h d_qt + g(a^p)\int_0^1 \hbar(1-t)_h d_qt. \quad (52)$$

From(51) and (52), one can get the required inequality (50).

Corollary 6.*By setting $h = 0$ in (50), we get the following inequality:*

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \leq \frac{2}{b^p-a^p}\int_{a^p}^{b^p} g(t)^{\frac{1}{p}}d_qt \leq g(b^p)\int_0^1 \bar{h}(t)d_qt + g(a^p)\int_0^1 \hbar(1-t)d_qt, \quad (53)$$

Corollary 7.*By setting $\bar{h}(t) = t^s$ in (43) and (44) and using $(a+b)^s \leq a^s + b^s$ where $0 < s < 1$ we get the following inequality:*

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{a^p+x^p}{2}\right)^{\frac{1}{p}} \leq \frac{2(1-q)}{(1-q)(x^p-a^p)+qh_4}\int_{a^p}^{x^p} g(t)^{\frac{1}{p}}_{h_4}d_qt \leq ((1-q)+qh) \times \left((g(x^p)-g(a^p))\left(\sum_{k=0}^{\infty} qk(1-q^k)^s + \frac{kh}{1-q^{2s}}\right) + \frac{g(a^p)}{1-q}\right), \quad (54)$$

$$\bar{h}\left(\frac{1}{2}\right)g\left(\frac{x^p+b^p}{2}\right)^{\frac{1}{p}} \leq \frac{2(1-q)}{(1-q)(b_p-x^p)+qh_5}\int_{x^p}^{b^p} g(t)^{\frac{1}{p}}_{h_5}d_qt \leq ((1-q)+qh) \times \left((g(b^p)-g(x^p))\left(\sum_{k=0}^{\infty} qk(1-q^k)^s + \frac{kh}{1-q^{2s}}\right) + \frac{g(x^p)}{1-q}\right). \quad (55)$$

3 Conclusion

We established inequalities for q- and h-integrals in implicit forms. Inequalities for q-integrals were deduced from composite results. All the inequalities were analyzed for certain classes of functions closely related with convexity. Moreover, some symmetry and symmetry like conditions were required to impose for getting the required results.

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