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Some recent research trends in fixed point theory with applications

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Abstract: In this study, some fixed-point, coupled, tripled, and quadruple fixed-point results for generalized contractive type mappings under mild conditions in the setting of abstract spaces are obtained. Also, we established some solutions to integral equations, functional equations, functional Volterra-Fredholm integral equations, 2D Volterra integral equations, Riemann-Liouville integrals, Atangana-Baleanu integral operators, and the boundary value problem for singular coupled fractional differential equations are established. Moreover, a unique stationary distribution for the Markov process is discussed. Furthermore, a new faster iterative algorithm is created than the previous sober algorithms. The proposed algorithm is used to analyze convergence, stability, and data-dependence results. Finally, two numerical examples are also provided to highlight the behavior and effectiveness of our approach.

Keywords: Fixed point technique, abstract space, integral equation, fractional differential equation, stationary distribution, Markov process, iterative scheme, stability result, strong convergence.

1 Introduction

Fixed-point technologies offer a focal concept with many diverse uses. It has been and still is an important theoretical tool in many fields and various disciplines, such as topology, game theory, optimal control, artificial intelligence, logic programming, dynamical systems (and chaos), functional analysis, differential equations, and economics. More clearly, for example, the technique of the fixed point is used to find the solution to the equilibrium troubles in economics and game theory. It is used to find analytical and numerical solutions to Fredholm integral equations [1,2,3,4,5,6,7,8,9,10,11,12].

The fundamental result of Banach on ordinary metric spaces endowed with vector-valued metrics is defined by Perov [13]. Later, for a self-mapping on generalized metric spaces, the results of Perov were generalized by Filip and Petruşel [14]. They proved some fixed point results under this regard. In paper [15], the notions of mixed-monotone functions and coupled fixed points were initiated and studied. Under partially ordered metric spaces and abstract spaces, some main results in this direction have been driven; for broadening, see [16]. Thereafter, a triple fixed point was introduced by Berinde and Borcut [17] in 2011. They initiated it for self-mappings and established some exciting consequences in partially ordered metric spaces. For more details, see [18,19,20,21]. For a generalized case of a triple fixed point, Karapinar initiated the idea of a quadruple fixed point and showed some fixed point results on the topic [22]. Following this study, a quadruple fixed point is developed, and some related fixed point results are discussed in [23,24,25]. As an extension of a metric space, a $b-$metric space was presented by Bakhtin [26], a metric-like space was discussed by Amini-Harandi [27], and a cone metric space was introduced by Huang and Zhang [28]. They discussed some fixed point theorems with applications in such spaces.

In the last years, boundary value problems of nonlinear fractional differential equations with a variety of boundary conditions have been studied by various researchers [29,30,31,32,33,34,35,36,37,38]. Fractional differential equations appear naturally in diverse fields of science and engineering. They constitute an important field of research. It should be noted that most papers dealing with the existence of solutions of nonlinear initial value problems to fractional differential equations mainly

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use techniques of nonlinear analysis such as fixed point techniques, stability, the Leray–Schauder result, etc. Relatively, fractional calculus and fractional differential/integral equations are very fresh topics for the researchers. For instance, in [39], the authors resolved some fractional differential equations with multiple delays in relation to chaos neuron models by using the results of Lou [40], E. de Pascale and L. de Pascale [41].

One of the important trends in fixed-point technique is the study of the behavior and performance of algorithms that contribute greatly to real-world applications. One of the well-established principles of the fixed point theory is Banach's contraction principle. This principle is significant as a source of existence and uniqueness in various parts of science. The Banach contraction principle depends on the Picard one-step iteration, which is given by:

$$v_{i+1} = \Xi v_i, \ \forall i \geq 1,$$

where Ξ is a contraction mapping defined in a complete metric space. When the existence of the fixed point theorem is guaranteed in a complete metric space, Banach contraction principle is not well applied to nonexpansive mapping because Picard's iteration gives poor results for the convergence of the fixed point. So, many authors tended to create many iterative methods for approximating fixed points in terms of improving the performance and convergence behavior of algorithms for nonexpansive mappings. Moreover, data-dependent results and the stability results with respect to Ξ via these methods have been introduced. For more details, we refer to some iterative methods such as S algorithm [42], Picard-S algorithm [43], Thakur algorithm [44], K^*−algorithm [52] and Hammad et al. [45]. See also, [46, 47, 49, 50, 51].

In the present article, some fixed-point, coupled, tripled, and quadruple fixed-point results for generalized contractive type mappings under certain conditions in the context of abstract spaces are established. Furthermore, solutions to Riemann-Liouville integrals, Atangana-Baleanu integral operators, functional Volterra-Fredholm integral equations, functional Volterra equations, 2D Volterra equations, and the boundary value problem for singular coupled fractional differential equations are obtained by fixed-point techniques. Moreover, a unique stationary distribution for the Markov process is discussed. Finally, a novel iterative algorithm that is faster than the sober algorithms in the prior writing is developed. Also, convergence and stability results are examined using the suggested algorithm.

2 Preliminaries

This part is intended to give some basic facts, which will help the reader understand our review and be useful in the sequel.

Throughout this review, the study was done under the following spaces: a metric-like space [27], a generalized metric space [14], a partially ordered metric space [27], and a cone b−metric space [28]. It should be noted that the considered spaces are generalized from the standard metric space.

The notion of coupled fixed point introduced by Bhaskar and Lakshmikantham [15] as follows:

Definition 1.[15] *An element $(p,q) \in \Omega^2$ is called a coupled fixed point of the mapping $H : \Omega^2 \to \Omega$ if $H(p,q) = p$ and $H(q,p) = q$.*

A tripled fixed point concept was presented by Berinde and Borcut [17] as follows:

Definition 2.[17] *An element $(p,q,r) \in \Omega^3$ is called a tripled fixed point of the mapping $H : \Omega^3 \to \Omega$ if*

$$H(p,q,r) = p, \ H(q,r,p) = q \text{ and } H(r,p,q) = r.$$

Karapinar [23] introduced the idea of qudrable fixed point as the following:

Definition 3.*Let $\Omega \neq \emptyset$, and $H : \Omega^4 \to \Omega$ be a given mapping. An element $(p,q,r,t) \in \Omega^4$ is called a quadruple fixed point of H if*

$$H(p,q,r,t) = p, \ H(q,r,t,p) = q,$$
$$H(r,t,p,q) = r, \text{ and } H(t,p,q,r) = t.$$

Definition 4.[23] *The mapping $\bar{\omega} : \Omega^4 \to \mathbb{R}^m$ defined on a generalized metric space (Ω,ω) equipped with*

$$\begin{aligned}&\bar{\omega}((p_1,p_2,p_3,p_4),(q_1,q_2,q_3,q_4))\\ &= \omega(p_1,q_1)+\omega(p_2,q_2)+\omega(p_3,q_3)+\omega(p_4,q_4),\end{aligned}$$

defined a metric on Ω^4 for all $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4 \in \Omega$.

Definition 5.[53] *For a function $z(\tau) : (0,\infty) \to \mathbb{R}$, the Riemann–Liouville fractional integral of order ν is described as*

$$I^\nu z(\tau) = \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau-\theta)^{\nu-1} z(\theta) d\theta, \ \nu > 0,$$

provided that an integral exists.

Definition 6.[53] *The Caputo derivative of fractional order $\nu > 0$, $n-1 < \nu < n$, $n \in \mathbb{N}$, for the function $z(\tau) : (0,\infty) \to \mathbb{R}$ is described as*

$$^C D^\nu z(\tau) = \frac{1}{\Gamma(n-\nu)} \int_0^\tau \frac{z^n(\theta)}{(\tau-\theta)^{\nu-n+1}} d\theta, \ n = [\nu]+1,$$

where $[\nu]$ represents the integer part of the real number ν.

Now, assume that $\{\alpha_i\}$ and $\{\gamma_i\}$ are non-negative sequences in $[0,1]$. The following algorithm is known as K^*-algorithm [52]:

$$\begin{cases} \nu_\circ \in \Delta, \\ \varpi_i = (1-\alpha_i)\nu_i + \alpha_i \Xi \nu_i, \\ \wp_i = \Xi((1-\gamma_i)\varpi_i + \gamma_i \Xi \varpi_i), \\ \nu_{i+1} = \Xi \wp_i, \end{cases} \forall i \geq 1. \tag{1}$$

The following definitions are used in the last section:

Definition 7. *A mapping $\Xi : \Lambda \to \Lambda$ is called Suzuki generalized nonexpansive mapping (SGNM) if*

$$\frac{1}{2}\|\nu - \Xi\nu\| \leq \|\nu - \varpi\| \Rightarrow \|\Xi\nu - \Xi\varpi\| \leq \|\nu - \varpi\|, \ \forall \nu, \varpi \in \Lambda.$$

Definition 8. *A mapping $\Xi : \Lambda \to \Lambda$ is called almost contractive mapping (ACM), if there exist $\theta \in (0,1)$ and $\ell \geq 0$ so that*

$$\|\Xi\nu - \Xi\varpi\| \leq \theta\|\nu - \varpi\| + \ell\|\nu - \Xi\nu\|, \ \forall \nu, \varpi \in \Lambda. \tag{2}$$

3 Solution of nonlinear integral equation via fixed point of cyclic α_L^ψ−rational contraction mappings in metric-like spaces

In 2003, Kirk et al. [54] introduced cyclic contraction mappings in metric spaces and investigated the existence of fixed points for cyclic contraction mappings. Many authors proved the existence of fixed points for various types of cyclic contractions in a metric space [55,56]. In the present investigation, Hammad and De la Sen [5] generalized the concept of cyclic contraction mappings by introducing the notion of a cyclic α_L^ψ-rational contractive mapping and established the existence and uniqueness of fixed points for such mappings in complete metric-like spaces. Finally, an application to an integral equation is obtained to illustrate the usability of the theoretical results.

Let Ψ be the class of all functions $\psi : [0,\infty) \to [0,\infty)$, satisfying the following conditions:
(i) ψ is non-decreasing and continuous;
(ii) for all $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$.

The definition of a cyclic α_L^ψ-rational contractive mapping can be introduced as follows:

Definition 9. *Let (X,ω) be a metric-like space, $q \in \mathbb{N}$, $B_1, B_2, ...B_q$ be $\omega-$closed subsets of X, $Y = \cup_{i=1}^q B_i$ and $\alpha : Y \times Y \to [0,\infty)$ be a mapping. We say that S is a cyclic $\alpha_L^\psi-$rational contractive mapping if*
(a)

$$S(B_j) \subseteq B_{j+1}, \ j = 1,2,...,q, \ \text{where} \ B_{q+1} = B_1. \tag{3}$$

(b) for any $x \in B_i$ and $y \in B_{i+1}$, $i = 1,2,...,q$ where $B_{q+1} = B_1$ and $\alpha(x,Sx)\alpha(y,Sy) \geq 1$, we get

$$\psi(\omega(Sx,Sy)) \leq \psi(M_\omega(x,y)) - LM_\omega(x,y), \tag{4}$$

where $\psi \in \Psi$, $0 < L < 1$ and

$$M_\omega(x,y) = \max\left\{ \begin{array}{c} \omega(x,y), \frac{\omega(x,Sx)\omega(y,Sy)}{\omega(x,y)}, \\ \frac{\omega(y,Sy)(\omega(x,Sx)+1)}{1+\omega(x,y)}, \\ \frac{\omega(x,Sy)+\omega(y,Sx)}{4} \end{array} \right\}.$$

The theorem concerned with the existence of fixed points under the mentioned contraction can be formulated below:

Theorem 1. *Let (X,ω) be a complete metric-like space, q be a positive integer, $B_1, B_2, ...B_q$ be nonempty $\omega-$closed subsets of X, $Y = \cup_{i=1}^q B_i$ and $\alpha : Y \times Y \to [0,\infty)$ be a mapping. Assume that $S : Y \to Y$ is a cyclic $\alpha_L^\psi-$rational contractive mapping satisfying the following conditions:*
(i) S is an $\alpha-$admissible mapping;
(ii) there exists $x_\circ \in Y$ such that $\alpha(x_\circ, Sx_\circ) \geq 1$;
(iii)(1) either S is $\alpha-$continuous, or;
(2) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$,
then $\alpha(x,Sx) \geq 1$.
Then, S has a fixed point $x \in \cap_{i=1}^q B_i$. Moreover, if
(iv) for all $x \in Fix(S)$ we have $\alpha(x,x) \geq 1$.
Then S has a unique fixed point $x \in \cap_{i=1}^q B_i$.

To support theoretical results, Theorem 1 is applied to prove the existence of solutions for the following integral equation:

$$x(t) = h(t) + \int_0^1 k(t,s)f(s,x(s))ds, \ t \in [0,1], \tag{5}$$

where $h : [0,1] \to \mathbb{R}$, $k : [0,1] \times [0,1] \to [0,\infty)$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ are three continuous functions.

Let $X = C([0,1],\mathbb{R})$ be the set of real continuous functions on $[0,1]$. Take the metric-like space

$$\omega_\infty(x,y) = \sup_{t\in[0,1]} \{\| x(t)| - |y(t)\|\} \text{ for all } x,y \in X.$$

Then (X,ω_∞) is a complete metric-like space.

Suppose that $\pi : X \times X \to \mathbb{R}$ is a function with the following properties:
• $\pi(x,y) \geq 0 \Longrightarrow \pi(Sx,Sy) \geq 0$,
• there exists $x_\circ \in X$ such that $\pi(x_\circ Sx_\circ) \geq 0$,
• if $\{x_n\}$ is a sequence in X such that $\pi(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\pi(x,Sx) \geq 0$, where

$$Sx(t) = h(t) + \int_0^1 k(t,s)f(s,x(s))ds, \text{ for all } t \in [0,1].$$

Let $(a,b) \in X \times X$, $(a_\circ, b_\circ) \in \mathbb{R}^2$ such that

$$a_\circ \leq a(t) \leq b(t) \leq b_\circ, \text{ for all } t \in [0,1]. \tag{6}$$

Assume that for all $t \in [0,1]$, we get

$$a(t) \leq h(t) + \int_0^1 k(t,s) f(s,b(s)) ds \tag{7}$$

and

$$b(t) \geq h(t) + \int_0^1 k(t,s) f(s,a(s)) ds. \tag{8}$$

Let for all $s \in [0,1]$, $f(s,.)$ be a decreasing function, that is,

$$x, y \in \mathbb{R},\ x \geq y \Rightarrow f(s,x) \leq f(s,y). \tag{9}$$

$$\sup_{t \in [0,1]} \int_0^1 k(t,s) ds \leq 1. \tag{10}$$

Also, suppose that for all $s \in [0,1]$, $x, y \in \mathbb{R}$ with ($x \leq b_\circ$ and $y \geq a_\circ$) or ($x \geq a_\circ$, $y \leq b_\circ$) and $\pi(y, Sy) \geq 1$ and $\pi(x, Sx) \geq 1$, we have

$$||f(s,x)| - |f(s,y)|| \leq r \max \left\{ \begin{array}{c} ||x|-|y||, \\ \frac{||x|-|Sx|| ||y|-|Sy||}{||x|-|y||}, \\ \frac{||y|-|Sy||(||x|-|Sx||+1)}{1+||x|-|y||}, \\ \frac{||x|-|Sy||+||y|-|Sx||}{4} \end{array} \right\}, \tag{11}$$

where $0 < r < 1$. Based on Conditions (6)-(18), the integral equation (5) has a solution $\{x \in X : a \leq x(t) \leq b$ for all $t \in [0,1]\}$.

4 A technique of tripled coincidence points for solving a system of nonlinear integral equations in POCML spaces

In 2014, the idea of $C-$type functions, which cover a large class of contractive conditions was presented by Ansari [57] as follows:

Definition 10.*([57]) A function $\Lambda : [0,\infty)^2 \to \mathbb{R}$ is called $C-$type function if it is continuous and fulfills the following hypotheses:*
(1) $\Lambda(\lambda,\mu) \leq \lambda$,
(2) $\Lambda(\lambda,\mu) = \lambda$ implies that either $\lambda = 0$ or $\mu = 0$ for all $\lambda,\mu \in [0,\infty)$.
We symbolize the $C-$type functions as $\complement$.

Here, via this notion, some recent theoretical consequences related to tripled coincidence points for non-self mappings are initiated in the context of a partially ordered complete metric-like space (POCML space, for short). These contributions unify and extend some previous studies in the literature. As an application, some theoretical results are applied to discuss the existence of the solution to a system of nonlinear integral equations.

Assume that $\Pi = \{\pi : [0,\infty) \to [0,\infty)$ is non-decreasing function and lower semi-continuous such that $\pi(\nu) = 0 \Leftrightarrow \nu = 0\}$.

Firstly, the main result of this part is as follows:

Theorem 2.*Assume that $\Re : \chi^3 \to \chi$ and $\Im : \chi \to \chi$ are two-mappings on a POCML space $(\chi, \precsim, \xi)$ such that*
(i) $\Re(\chi^3) \subseteq \Im(\chi)$;
(ii) $\Re$ is continuous;
(iii) $\Im$ is continuous and commutes with $\Re$;
(iv) $\Re$ has a mixed $\Im-$monotone property;
(v) there is $\pi \in \Pi$, $\zeta \geq 0$ and $\Lambda \in \complement$ such that

$$\xi(\Re(\wp,\hbar,\eth), \Re(x,y,z)) \leq \Lambda \left(\begin{array}{c} \pi\left(\max \left\{ \begin{array}{c} \xi(\Im(\wp),\Im(x)), \\ \xi(\Im(\hbar),\Im(y)), \\ \xi(\Im(\eth),\Im(z)) \end{array} \right\} \right), \\ \zeta \max \left\{ \begin{array}{c} \xi(\Im(\wp),\Im(x)), \\ \xi(\Im(\hbar),\Im(y)), \\ \xi(\Im(\eth),\Im(z)) \end{array} \right\} \end{array} \right), \tag{12}$$

for any $\wp,\hbar,\eth,x,y,z \in \chi$, for which $\Im(\wp) \precsim \Im(x)$, $\Im(y) \precsim \Im(\hbar)$ and $\Im(\eth) \precsim \Im(z)$. If there exists $\wp_\circ,\hbar_\circ,\eth_\circ \in \chi$ such that $\Im(\wp_\circ) \precsim \Re(\wp_\circ,\hbar_\circ,\eth_\circ)$, $\Im(\hbar_\circ) \succsim \Re(\hbar_\circ,\wp_\circ,\eth_\circ)$ and $\Im(\eth_\circ) \precsim \Re(\eth_\circ,\hbar_\circ,\wp_\circ)$. Then $\Re$ and $\Im$ have a tripled coincidence point.

Next, choosing $\Im = I_\chi$, (where I_χ is the identity mapping) and replace a mixed-monotone with a monotone-increasing property on Theorem 2, the following result holds:

Corollary 1.*Let $(\Re, \precsim, \xi)$ be a POCML space. Suppose that $\Re : \chi^3 \to \chi$ is mapping such that:*
(i) $\Re$ is continuous;
(ii) $\Re$ is non-decreasing with respect to $\precsim$;
(iii) there exists three elements $\wp_\circ,\hbar_\circ,c_\circ \in \chi$ such that $\wp_\circ \precsim \Re(\wp_\circ,\hbar_\circ,\eth_\circ)$, $\hbar_\circ \succsim \Re(\hbar_\circ,\wp_\circ,\hbar_\circ)$ and $c_\circ \precsim \Re(c_\circ,\hbar_\circ,\wp_\circ)$;
(v) here is $\pi \in \Pi$, $\zeta \geq 0$ and $\Lambda \in \complement$ such that

$$\xi(\Re(\wp,\hbar,\eth), \Re(x,y,z)) \leq \Lambda \left(\begin{array}{c} \pi\left(\max \left\{ \begin{array}{c} \xi(\wp,x), \\ \xi(\hbar,y), \\ \xi(\eth,z) \end{array} \right\} \right), \\ \zeta \max \left\{ \begin{array}{c} \xi(\wp,x), \\ \xi(\hbar,y), \\ \xi(\eth,z) \end{array} \right\} \end{array} \right), \tag{13}$$

for any $\wp,\hbar,\eth,x,y,z \in \chi$, and for which $\wp \precsim x$, $y \precsim \hbar$ and $\eth \precsim z$. Then, there is a tripled coincidence point of $\Re$.

For an application, let Ω be a class functions $\omega : [0,\infty) \to [0,\infty)$ such that ω is increasing and there exists $\pi \in \Pi$, $\zeta \geq 0$ and $\Lambda \in \complement$ such that $\omega(\mu) = \frac{1}{3}\Lambda(\pi(\mu), \zeta\mu)$ for all $\mu \in [0,\infty)$.

Next, Corollary 1 is applied to show the existence of solution to the following problem:

$$\wp(\nu) = \varphi(\nu) + \int_p^q \left(\begin{array}{c} r_1(\nu,\rho) \\ +r_2(\nu,\rho) \\ +r_3(\nu,\rho) \end{array} \right) \left[\begin{array}{c} p_1(\rho,\wp(\rho)) \\ +p_2(\rho,\hbar(\rho)) \\ +p_3(\rho,\eth(\rho)) \end{array} \right] d\rho, \tag{14}$$

for all $\nu \in [p,q]$ under the following assumptions:

(i) $\varphi : [p,q] \to \mathbb{R}$ is continuous;

(ii) $p_i, r_i (i = 1,2,3) : [p,q] \times \mathbb{R} \to \mathbb{R}$ are continuous;

(iii) for all $\wp, \eth \in \mathbb{R}$, there is $\varkappa, \tau, \sigma$ such that

$$0 \leq p_1(\rho,\wp) - p_1(\rho,\eth) \leq \varkappa\omega(\wp - \eth),$$
$$0 \leq p_2(\rho,\wp) - p_2(\rho,\eth) \leq \tau\omega(\wp - \eth),$$

and

$$0 \leq p_3(\rho,\wp) - p_3(\rho,\eth) \leq \sigma\omega(\wp - \eth);$$

(iv) we assume that

$$\max\{\varkappa,\tau,\sigma\}\left(\sup_{t\in[p,q]}\int_p^q \begin{bmatrix} r_1(\nu,\rho) \\ +r_2(\nu,\rho) \\ +r_3(\nu,\rho)\end{bmatrix} d\rho\right) \leq 1;$$

(v) there is continuous functions $\alpha, \beta, \gamma : [p,q] \to \mathbb{R}$ such that

$$\begin{aligned}\alpha(\nu) \leq & \int_p^q r_1(\nu,\rho)\begin{bmatrix} p_1(\rho,\alpha(\rho)) \\ +p_2(\rho,\beta(\rho)) \\ +p_3(\rho,\gamma(\rho))\end{bmatrix} d\rho \\ &+ \int_p^q r_2(\nu,\rho)\begin{bmatrix} p_1(\rho,\delta(\rho)) \\ +p_2(\rho,\eta(\rho)) \\ +p_3(\rho,\delta(\rho))\end{bmatrix} d\rho \\ &+ \int_p^q r_3(\nu,\rho)\begin{bmatrix} p_1(\rho,\gamma(\rho)) \\ +p_2(\rho,\beta(\rho)) \\ +p_3(\rho,\alpha(\rho))\end{bmatrix} d\rho,\end{aligned}$$

$$\begin{aligned}\beta(\nu) \geq & \int_p^q r_1(\nu,\rho)\begin{bmatrix} p_1(\rho,\beta(\rho)) \\ +p_2(\rho,\eta(\rho)) \\ +p_3(\rho,\beta(\rho))\end{bmatrix} d\rho \\ &+ \int_p^q r_2(\nu,\rho)\begin{bmatrix} p_1(\rho,\gamma(\rho)) \\ +p_2(\rho,\beta(\rho)) \\ +p_3(\rho,\alpha(\rho))\end{bmatrix} d\rho \\ &+ \int_p^q r_3(\nu,\rho)\begin{bmatrix} p_1(\rho,\alpha(\rho)) \\ +p_2(\rho,\beta(\rho)) \\ +p_3(\rho,\gamma(\rho))\end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}\gamma(\nu) \leq & \int_p^q r_1(\nu,\rho)\begin{bmatrix} p_1(\rho,\gamma(\rho)) \\ +p_2(\rho,\beta(\rho)) \\ +p_3(\rho,\alpha(\rho))\end{bmatrix} d\rho \\ &+ \int_p^q r_2(\nu,\rho)\begin{bmatrix} p_1(\rho,\alpha(\rho)) \\ +p_2(\rho,\beta(\rho)) \\ +p_3(\rho,\gamma(\rho))\end{bmatrix} d\rho \\ &+ \int_p^q r_3(\nu,\rho)\begin{bmatrix} p_1(\rho,\beta(\rho)) \\ +p_2(\rho,\alpha(\rho)) \\ +p_3(\rho,\beta(\rho))\end{bmatrix} d\rho.\end{aligned}$$

Let $\chi = C([p,q],\mathbb{R})$ be the set of real continuous functions on $[p,q]$ endowed with

$$\begin{aligned}\xi(\wp,\hbar) &= \|\wp - \hbar\|_\infty \\ &= \sup_{\nu\in[p,q]}\{|\wp(\nu) - \hbar(\nu)|\},\end{aligned}$$

for all $\wp, \hbar \in \chi$. Then the pair (χ,ξ) is a complete metric-like space. We endow χ with the partial ordered $\precsim$ as

$$\wp \precsim \hbar \Leftrightarrow \wp(\nu) \leq \hbar(\nu), \ \forall \nu \in [p,q].$$

Subsequently, $(\chi,\precsim,\xi)$ is a POCML space if $\wp \precsim x$, $y \precsim \hbar$ and $\eth \precsim z$ whenever, $\wp(\nu) \leq x(\nu)$, $y(\nu) \leq \hbar(\nu)$ and $\eth(\nu) \leq z(\nu)$, for all $\wp, \hbar, \eth, x, y, z \in \chi$ and $\nu \in [p,q]$.

According to Hypotheses (i)-(v), problem (14) has a solution in χ^3, where $\chi = C([p,q],\mathbb{R})$.

5 Generalized dynamic process for an extended multi-valued $F-$contraction in metric-like spaces with applications

The system of dynamic programming consists of two main sections. One is the decision space, which is the spectrum of decisions taken to find a solution to the problem. The other is the state space, which is a family of parameters acting in different states, containing action states, transitional states, and initial states. This direction formulates the problems of mathematical optimization and computer programming by using the fixed point technique; see [58,59,6].

This process is formulated in the sense of the fixed point technique by Arshad et al. [60] as follows:

Definition 11.[60] *A generalized dynamic process of mappings $Q : Y \to Y$ and $Z : Y \to CB(Y)$ is defined as:*

$$\Theta(Z,Q,\Omega_\circ) = \left\{ \begin{array}{c} \{\Omega_j\}_{j\in N\cup\{0\}} : \\ \Omega_{j+1} = Q(\Omega_j) \in Z(\Omega_{j-1}) \end{array}\right\},$$

for all $j \in N$, where $\Omega_\circ \in Y$ is a starting point. For short, we denote $\Theta(Z,Q,\Omega_\circ)$ with $Q(\Omega_j)$.

In 2012, Wardowski [61] introduced a type of contraction called $F-$contraction and proved some fixed point theorems concerning $F-$contraction. In this way, he generalized the Banach contraction principle in a different manner from the well-known results from the literature.

Definition 12.[61] *Let (Y,d) be a metric space. A mapping $Q : Y \longrightarrow Y$ is called an $F-$contraction if there exist $F \in \Pi$ and $\tau > 0$ such that*

$$d(Q\Omega_1, Q\Omega_2) > 0$$

implies $\tau + F(d(Q\Omega_1, Q\Omega_2)) \leq F(d(\Omega_1,\Omega_2))$,

for all $\Omega_1, \Omega_2 \in Y$, where Π is the class of functions $F : (0,+\infty) \to \mathbb{R}$ verifying the following:

(i) For all $\iota, \kappa > 0$ such that $\iota < \kappa$, $F(\iota) < F(\kappa)$, (i.e., F is strictly increasing);

(ii) For each sequence $\{\iota_n\}_{n\in N}$ in $(0,+\infty)$, $\lim_{n\to\infty}\iota_n = 0$ if and only if $\lim_{n\to\infty} F(\iota_n) = -\infty$;

(iii) There exists $\lambda \in (0,1)$ such that $\lim_{i\to 0^+} \iota^\lambda F(\iota) = 0$.

According to the above two definitions, Hammad et al. [6] developed some fixed-point results satisfying an extended multi-valued $F-$contraction mapping under a generalized dynamic process in the context of metric-like spaces. Also, the solution of a system of functional equations is obtained as an application.

The notion of an extended multi-valued $F-$contraction mapping can be formulated as follows:

Definition 13. *Let Q be a self-mapping on a metric-like space (Y,θ). A mapping $Z : Y \to CB(Y)$ is called a generalized multi-valued $F-$contraction with respect to generalized dynamic process $\Theta(Z,Q,\Omega_\circ)$, $\Omega_\circ \in Y$, if for some $F \in \Pi$ and $\tau > 0$, we have $\theta(Q(\Omega_j)), Q(\Omega_{j+1})) > 0$, implies*

$$\tau H(\Omega_{j-1},\Omega_j) + F\left(\theta(Q(\Omega_j), Q(\Omega_{j+1}))\right) \le F\left(H(\Omega_{j-1},\Omega_j)\right),$$

where

$$H(\Omega_{j-1},\Omega_j) = \max\left\{\begin{array}{c} \theta(Q(\Omega_{j-1}), Q(\Omega_j)), \\ \theta(Q(\Omega_{j-1}), Z(\Omega_{j-1})), \\ \theta(Q(\Omega_j), Z(\Omega_j)), \\ \frac{\theta(Q(\Omega_{j-1}), Z(\Omega_j)) + \theta(Q(\Omega_j), Z(\Omega_{j-1}))}{4} \end{array}\right\},$$

for all $\Omega_1,\Omega_2 \in \Theta(Z,Q,\Omega_\circ)$, $j \in N$.

Based on Definition 13, the existence of fixed points can be obtained from the following theorem:

Theorem 3. *Suppose that (Y,θ) is a complete metric-like space and $Q : Y \to Y$ is a self-mapping. Let $Z : Y \to CB(Y)$ be a generalized multi-valued $F-$contraction with respect to generalized dynamic process $\Theta(Z,Q,\Omega_\circ)$. Then Z and Q possess a common fixed point.*

For applications, Theorem 3 is applied to show the existence of a solution for a system of functional equations in the form of

$$\left\{\begin{array}{l} \sigma(\Omega_1) = \sup_{\Omega_2\in\Xi}\left\{\begin{array}{c}\psi(\Omega_1,\Omega_2)+\phi(\Omega_1,\Omega_2, \\ \sigma(\varpi(\Omega_1,\Omega_2)))\end{array}\right\} \\ \sigma^*(\Omega_1) = \sup_{\Omega_2\in\Xi}\left\{\begin{array}{c}\psi^*(\Omega_1,\Omega_2)+\phi^*(\Omega_1,\Omega_2, \\ \sigma^*(\varpi(\Omega_1,\Omega_2)))\end{array}\right\}\end{array}\right., \quad (15)$$

for $\Omega_1 \in \Pi$, where the mappings

$\phi,\phi^* : \Pi\times\Xi\times\mathbb{R} \to \mathbb{R}$,
$\psi,\psi^* : \Pi\times\Xi \to \mathbb{R}$,
$\varpi : \Pi\times\Xi \to \Pi$.

Suppose that Y_1 and Y_2 are quasi-Banach spaces. Consider a state space $\Pi \subset Y_1$ (is the initial space, set of actions, and transposition model of the process) and a decision space $\Xi \subset Y_2$ (is the set of a possible action that can be considered to facilitate the problem). For more details, see [60].

Assume that $B(\Pi)$ is the set of all bounded real valued functions on Π. For an arbitrary point $k \in B(\Pi)$, take $\|k\| = \sup_{\Omega\in\Pi}|k(\Omega)|$. The pair $(B(\Pi),\|.\|)$ is a quasi-Banach space equipped with the metric-like defined by

$$\theta(m_1,m_2) = \sup_{\Omega\in\Pi}\left||m_1(\Omega)|-|m_2(\Omega)|\right|,$$

for all $m_1,m_2 \in B(\Pi)$.

Define $\delta : B(\Pi) \to B(\Pi)$ by

$$\left\{\begin{array}{l} \delta_1(m)(\Omega) = \sup_{\omega\in\Xi}\left\{\begin{array}{c}\psi_1(\Omega,\omega) \\ +\phi_1(\Omega,\omega,m(\varpi(\Omega,\omega)))\end{array}\right\} \\ \delta_2(m)(\Omega) = \sup_{\omega\in\Xi}\left\{\begin{array}{c}\psi_2(\Omega,\omega) \\ +\phi_2(\Omega,\omega,m(\varpi(\Omega,\omega)))\end{array}\right\}\end{array}\right., \quad (16)$$

for all $m \in B(\Pi)$ and $\Omega \in \Pi$, where ϕ_1,ϕ_2,ψ_1 and ψ_2 are bounded and continuous functions. Suppose that for all $\tau > 0$,

$$|\phi_1(\Omega,\omega,m_1(\Omega)) - \phi_2(\Omega,\omega,m_2(\Omega))| \le e^{-\tau}H(m_1(\Omega),m_2(\Omega)),$$

for all $m_1,m_2 \in B(\Pi)$, where $\Omega \in \Pi$ and $\omega \in \Xi$.

Based on what has been assumed, the solution to the system can be obtained by the following theorem:

Theorem 4. *Let $\delta_1,\delta_2 : B(\Pi) \to B(\Pi)$ be a semi-continuous mappings defined as (16). Assume that the following assumptions are satisfied:*

(i) ϕ_1,ϕ_2,ψ_1 and ψ_2 are bounded and continuous,

(ii) for all $m \in B(\Pi)$, there exists $\ell \in B(\Pi)$ such that

$$\delta_1(m)(\Omega) = \delta_2(\ell)(\Omega),$$

(iii) δ_1 and δ_2 are weakly compatible, i.e., there exists $m \in B(\Pi)$ such that

$$\delta_1(m)(\Omega) = \delta_2(m)(\Omega)$$

implies $\delta_1\delta_2(m)(\Omega) = \delta_2\delta_1(m)(\Omega)$.

Then the system (15) possesses a bounded solution.

6 A tripled fixed point technique for solving a tripled-system of integral equations and Markov process in CCbMS

In 2007, the concept of a cone-metric space was introduced by Huang and Zhang [62]. They talked about some fixed-point theorems that extended certain results of this kind to the complete metric space. Great papers combine $b-$metric spaces with cone-metric spaces clarified by Hussain and Shah in [63] to form a cone-metric space (CbMS, for short), where some topological properties in such spaces are investigated and recent results for contractive mappings in a CbMS are established.

Alignment for this work, Hammad and De la Sen [21] investigated a new generalized nonlinear contraction mapping and discussed the existence of tripled fixed points in complete CbMS. To support their results, a solution for the tripled system of integral equations and a unique stationary distribution for the Markov process are presented.

The form of the contractive condition is as follows:

Definition 14. *Let (χ,ξ) be a CbMS with the coefficient $s \geq 1$, $Q \cup (-Q) = B$ (i.e. Q is a total ordering cone) and $\Im : \chi \to \chi$ be a mapping. We will say that a mapping $\Re : \chi^3 \to \chi$ is a $\Im_\theta-$contraction if there is $\theta \in [0,1)$ such that*

$$\xi(\Im\Re(p,q,r),\Im\Re(p^*,q^*,r^*)) \precsim \theta \max\left\{\begin{array}{c}\xi(\Im p,\Im p^*),\\ \xi(\Im q,\Im q^*),\\ \xi(\Im r,\Im r^*)\end{array}\right\}, \tag{17}$$

for all $p,q,r,p^*,q^*,r^* \in \chi$.

Under this contractive condition, the existence of fixed points is demonstrated in the following theorem:

Theorem 5. *Assume that the mapping $\Re$ is $\Im_\theta-$contraction (31) defined on CCbMS (χ,ξ) with the coefficient $s \geq 1$, such that B is a solid cone with $Q \cup (-Q) = B$ and $\Im : \chi \to \chi$ is a continuous and one-to-one, then for all $p,q,r,p^*,q^*,r^* \in \chi$,*

(a) there exist $\upsilon_{p_\circ},\upsilon_{q_\circ},\upsilon_{r_\circ} \in \chi$ and iterative sequences $\Re^n(p_\circ,q_\circ,r_\circ) = p_n$, $\Re^n(q_\circ,p_\circ,q_\circ) = q_n$ and $\Re^n(r_\circ,q_\circ,p_\circ) = r_n$ such that

$$\lim_{n\to\infty} \Im\Re^n(p_\circ,q_\circ,r_\circ) = \upsilon_{p_\circ},$$

$$\lim_{n\to\infty} \Im\Re^n(q_\circ,p_\circ,q_\circ) = \upsilon_{q_\circ},$$

and $\lim_{n\to\infty} \Im\Re^n(r_\circ,q_\circ,p_\circ) = \upsilon_{r_\circ}$;

(b) $\Re^n(p_\circ,q_\circ,r_\circ)$, $\Re^n(q_\circ,p_\circ,q_\circ)$ and $\Re^n(r_\circ,q_\circ,p_\circ)$ have a convergent subsequence, whenever $\Im$ is subsequentially convergent;

(c) there exist a unique $\sigma_{p_\circ},\sigma_{q_\circ},\sigma_{r_\circ} \in \chi$ such that $\Re(\sigma_{p_\circ},\sigma_{q_\circ},\sigma_{r_\circ}) = \sigma_{p_\circ}$, $\Re(\sigma_{q_\circ},\sigma_{p_\circ},\sigma_{q_\circ}) = \sigma_{q_\circ}$ and $\Re(\sigma_{r_\circ},\sigma_{q_\circ},\sigma_{p_\circ}) = \sigma_{r_\circ}$;

(d) for every $p_\circ,q_\circ,r_\circ \in \chi$, the sequences $\Re^n(p_\circ,q_\circ,r_\circ)$, $\Re^n(q_\circ,p_\circ,q_\circ)$ and $\Re^n(r_\circ,q_\circ,p_\circ)$ converges to $\sigma_{p_\circ},\sigma_{q_\circ},\sigma_{r_\circ} \in \chi$ respectively, provided that $\Im$ is subsequentially convergent.

Theorem 5 has been supported by using it to study the existence and uniqueness of the solution for the following integral system:

$$\begin{cases} p(\daleth) = \aleph(\daleth) + \int_0^{\wp} \pounds(\daleth,\mho)\begin{bmatrix}\eth_1(\mho,p(\mho))\\ +\eth_2(\mho,q(\mho))\\ +\eth_3(\mho,r(\mho))\end{bmatrix} d\mho\\ q(\daleth) = \aleph(\daleth) + \int_0^{\wp} \pounds(\daleth,\mho)\begin{bmatrix}\eth_1(\mho,q(\mho))\\ +\eth_2(\mho,p(\mho))\\ +\eth_3(\mho,q(\mho))\end{bmatrix} d\mho,\\ r(\daleth) = \aleph(\daleth) + \int_0^{\wp} \pounds(\daleth,\mho)\begin{bmatrix}\eth_1(\mho,r(\mho))\\ +\eth_2(\mho,q(\mho))\\ +\eth_3(\mho,p(\mho))\end{bmatrix} d\mho.\end{cases} \tag{18}$$

for all $\daleth \in [0,\wp]$, under the following assumptions:

(i) $\aleph : [0,\wp] \to \mathbb{R}$ and $\pounds : [0,\wp]\times\mathbb{R} \to \mathbb{R}$ are continuous;

(ii) $\eth_l(l = 1,2,3) : [0,\wp]\times\mathbb{R} \to \mathbb{R}$ are continuous;

(iii) There is a constant $\alpha > 0$ such that for all $p,q \in \mathbb{R}$:

$$0 \precsim \Gamma_1(\mho,p) - \Gamma_1(\mho,q) \precsim \alpha(p-q),$$
$$0 \precsim \Gamma_2(\mho,p) - \Gamma_2(\mho,q) \precsim \alpha(p-q),$$
$$0 \precsim \Gamma_3(\mho,p) - \Gamma_3(\mho,q) \precsim \alpha(p-q).$$

(iv) we consider

$$\max_{\daleth\in[0,\wp]}\left(\int_0^{\wp} \pounds(\daleth,\mho)\,d\mho\right)^2 \precsim \frac{1}{21}.$$

We assume that $\chi = C([0,\wp],\mathbb{R})$ is a set of all continuous real-valued functions on $[0,\wp]$ and taking values in $\mathbb{R}$ and $Q = \{\pi \in B : \pi \geq 0\}$. Set $\xi : \chi\times\chi \to B$ as $\xi(o,O) = e^{\daleth}\max_{\daleth\in[0,\wp]} |o(\daleth) - O(\daleth)|^2$. It is obvious that (χ,ξ) is a CCbMS.

Now, the theorem concerned with the existence of a solution to the above system is formulated as follows:

Theorem 6. *[21] Under hypotheses $(i)-(iv)$, problem (18) has a solution in χ^3, where $\chi = C([0,\wp],\mathbb{R})$.*

Another application involved in Theorem 5 is the study of a unique stationary distribution for the Markov process.

Suppose that $\mathbb{R}^n_+ = \{(\delta_1,\delta_2,...,\delta_n) = \delta : \delta_\ell \geq 0,\ \ell \geq 1\}$ and $\Omega^3_{n-1} = \left\{\begin{array}{c}\sigma = (p,q,r) \in \mathbb{R}^n_+\times\mathbb{R}^n_+\times\mathbb{R}^n_+\\ : \sum_{\ell=1}^n \sigma_\ell = \sum_{\ell=1}^n (p_\ell+q_\ell+r_\ell) = 1\end{array}\right\}$ refer to $3(n-1)$ dimensional unit simplex and $\sigma \in \Omega^3_{n-1}$ could be considered a possibility over $3n$ possible statuses. Here, Markov process is a stochastic process which verifies that $3n$ statuses is realized in each period $\daleth = 1,2,...$with the probability contingent on the current achieved status. Assume that $a_{\ell\kappa}$ refer to the probability contingent that status ℓ is achieved in the subsequent period beginning in status κ. Hence, in period $\daleth$ and $\daleth+1$, the prior probability vector $\sigma^{\daleth}$ and the posterior probability $\sigma^{\daleth+1}$ given by $\sigma^{\daleth+1}_\ell = \sum_\ell a_{\ell\kappa}\sigma^{\daleth}_\kappa$ for each $\ell \geq 1$. To write this in matrix style, we consider $\sigma^{\daleth}$ is a column vector, then $\sigma^{\daleth+1} = \Im\sigma^{\daleth}$. Note that $a_{\ell\kappa} \geq 0$ and $\sum_{\ell=1}^n a_{\ell\kappa} = 1$ which is required for conditional probability. $\sigma^{\daleth}$ is called a stationary distribution of the Markov process at any period, if $\sigma^{\daleth} = \sigma^{\daleth+1}$. It's mean that, the problem of finding a stationary distribution is equivalent to the fixed point problem $\Im\sigma^{\daleth} = \sigma^{\daleth}$.

Let $\delta_\ell = \min_\kappa a_{\ell\kappa}$ and define $\delta = \sum_{\ell=1}^n \delta_\ell$.

Then, there exists a unique stationary distribution for the Markov process, provided that $a_{\ell\kappa} \geq 0$.

7 Quadruple fixed point techniques for solving integral equations involved with matrices and Markov process in generalized metric spaces

The fundamental result of Banach contraction mappings on metric spaces with vector-valued metrics is presented by Perov [13]. Later, Perov's results are refined for a self-mapping on generalized metric spaces by Filip and PetruÅŸel [14]. Also, they proved some fixed-point sequences in this space.

Motivated by the results in [13,14], Hammad and Abdeljwad [25] established some quadruple fixed point and coincidence point results under the same space. In addition, these results were exploited to obtain some applications.

Here, the symbols $M_{m,m}(\mathbb{R}^+)$, ϑ, I and $\Phi \in M_{m\times m}(\mathbb{R}^+)$, represent the set of all $m\times m$ matrices with components in $\mathbb{R}^+$, zero and identity matrices, and a matrix, respectively. Further, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and ZM refers to the set of all matrices $\Phi \in M_{m,m}(\mathbb{R}^+)$, where $\Phi^n \to \vartheta$.

The results can be started with the following definitions:

Definition 15. *For given mappings $\Im_i : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$. We say $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h satisfy the condition (O) if*

$$\begin{aligned}&\omega\left(\Im_i\left(e^1,e^2,e^3,e^4\right),\Im_j\left(r^1,r^2,r^3,r^4\right)\right)\\ &\leq \Phi\begin{bmatrix}\omega\left(h(e^1),\Im_i\left(e^1,e^2,e^3,e^4\right)\right)\\ +\omega\left(h(r^1),\Im_j\left(r^1,r^2,r^3,r^4\right)\right)\end{bmatrix}\\ &+\Psi\left(\omega\left(h(e^1),h(r^1)\right)\right), \qquad (19)\end{aligned}$$

for $e^1,e^2,e^3,e^4,r^1,r^2,r^3,r^4 \in \Omega$ with $h(e^1)\leq h(r^1)$, $h(r^2)\leq h(e^2)$, $h(e^3)\leq h(r^3)$, $h(r^4)\leq h(e^4)$ or $h(e^1)\geq h(r^1)$, $h(r^2)\geq h(e^2)$, $h(e^3)\geq h(r^3)$, $h(r^4)\geq h(e^4)$, $I\neq\Phi=\left(\phi_{ij}\right)$, $I\neq\Psi=\left(\psi_{ij}\right)\in M_{m,m}(\mathbb{R}^+)$, $(\Phi+\Psi)(\Phi-I)^{-1}\in ZM$.

Definition 16. *We say $\Im_0$ and h have a mixed quadruple transcendence point (MQTP), if there exists $e^1_\circ,e^2_\circ,e^3_\circ,e^4_\circ \in \Omega$ so that*

$$\begin{aligned}&\Im_\circ\left(e^1_\circ,e^2_\circ,e^3_\circ,e^4_\circ\right)\geq h(e^1_\circ),\\ &\Im_\circ\left(e^2_\circ,e^3_\circ,e^4_\circ,e^1_\circ\right)\leq h(e^2_\circ),\\ &\Im_\circ\left(e^3_\circ,e^4_\circ,e^1_\circ,e^2_\circ,\right)\geq h(e^3_\circ),\\ \text{and } &\Im_\circ\left(e^4_\circ,e^1_\circ,e^2_\circ,e^3_\circ\right)\leq h(e^4_\circ), \qquad (20)\end{aligned}$$

provided that $\Im_\circ$ and h have a non-decreasing transcendence point in $e^1_\circ,e^3_\circ$ and a non-increasing transcendence point in $e^2_\circ,e^4_\circ$.

The following lemma is very important in the sequel:

Lemma 1. *Suppose that $\Im_i : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ are two mappings on a partially ordered complete generalized metric space (POCGMS, for short) $(\Omega,\omega,\preceq)$. Assume also $\{\Im_i\}_{n\in\mathbb{N}_0}$ have mixed $h-$monotone property (MhMP) with $\Im_i(\Omega^4)\subseteq h(\Omega)$. If $\Im_\circ$ and h have a (MQTP), then*

(i) there are sequences $\{e^1_n\},\{e^2_n\},\{e^3_n\}$ and $\{e^4_n\}$ in Ω so that

$$\begin{aligned}he^1_n &= \Im_{n-1}\left(e^1_{n-1},e^2_{n-1},e^3_{n-1},e^4_{n-1}\right),\\ he^2_n &= \Im_{n-1}\left(e^2_{n-1},e^3_{n-1},e^4_{n-1},e^1_{n-1}\right),\\ he^3_n &= \Im_{n-1}\left(e^3_{n-1},e^4_{n-1},e^1_{n-1},e^2_{n-1}\right),\\ \text{and } he^4_n &= \Im_{n-1}\left(e^4_{n-1},e^1_{n-1},e^2_{n-1},e^3_{n-1}\right).\end{aligned}$$

(ii) $\{he^1_n\},\{he^3_n\}$ are non-decreasing sequences and $\{he^2_n\},\{he^4_n\}$ are non-increasing sequences.

Using the conditions of Lemma 1, the theorem below can be obtained.

Theorem 7. *Let the conditions of Lemma 1 hold, suppose that $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h are compatible, weakly reciprocally continuous, where h is continuous, satisfies the condition (O), monotonic nondecreasing and $h(\Omega)\subseteq\Omega$ is complete. Then $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h have a quadruple coincidence point provided that $h(\Omega)$ is regular and Φ,Ψ are nonzero matrices in ZM.*

By taking $h = Id$ (where Id is the identity mapping), the result below holds.

Corollary 2. *Let $(\Omega,\omega,\preceq)$ be a (POCGMS) and $\{\Im_i\}_{i\in\mathbb{N}\cup\{0\}} : \Omega^4\to\Omega$ be a mixed monotone sequence of mappings, where $\{\Im_i\}_{i\in\mathbb{N}\cup\{0\}}$ and $Id : \Omega\to\Omega$ satisfy the condition (O). Also $\Im_0$ and Id have a (MQTP) and $Id(\Omega)$ is regular. Then there is $(e^1,e^2,e^3,e^4)\in\Omega^4$ so that $\Im_i\left(e^1,e^2,e^3,e^4\right)=e^1$, $\Im_i\left(e^2,e^3,e^4,e^1\right)=e^2$, $\Im_i\left(e^3,e^4,e^1,e^2\right)=e^3$ and $\Im_i\left(e^4,e^1,e^2,e^3\right)=e^4$ for $i\in\mathbb{N}_0$.*

If we omit some conditions of Corollary 2 and take Φ as a zero matrix and expand the distance $\omega\left(\left(e^1\right),\left(r^1\right)\right)$, we get the following result:

Corollary 3. *Let $(\Omega,\omega,\preceq)$ be a (POCGMS) and $\urcorner : \Omega^4\to\Omega$ be a mixed monotone mapping, such that*

$$\begin{aligned}&\omega\left(\Im_i\left(e^1,e^2,e^3,e^4\right),\Im_j\left(r^1,r^2,r^3,r^4\right)\right)\\ &\leq\Psi\left(\omega\left(\left(e^1,e^2,e^3,e^4\right),\left(r^1,r^2,r^3,r^4\right)\right)\right).\end{aligned}$$

If $\urcorner$ has a (MQTP). Then $\urcorner$ has a (QFP) in Ω.

Now, Corollary 2 is used to discuss the existence and uniqueness of the solution of the following integral equations system:

$$\begin{cases} e^1(\delta)=\int_0^\varrho\begin{pmatrix}w\left(\delta,\sigma,e^1(\sigma)\right)+x\left(\delta,\sigma,e^2(\sigma)\right)\\ +y\left(\delta,\sigma,e^3(\sigma)\right)+z\left(\delta,\sigma,e^4(\sigma)\right)\end{pmatrix}d\sigma\\ \qquad +b(\delta),\\ e^2(\delta)=\int_0^\varrho\begin{pmatrix}w\left(\delta,\sigma,e^2(\sigma)\right)+x\left(\delta,\sigma,e^3(\sigma)\right)\\ +y\left(\delta,\sigma,e^4(\sigma)\right)+z\left(\delta,\sigma,e^1(\sigma)\right)\end{pmatrix}d\sigma\\ \qquad +b(\delta),\\ e^3(\delta)=\int_0^\varrho\begin{pmatrix}w\left(\delta,\sigma,e^3(\sigma)\right)+x\left(\delta,\sigma,e^4(\sigma)\right)\\ +y\left(\delta,\sigma,e^1(\sigma)\right)+z\left(\delta,\sigma,e^2(\sigma)\right)\end{pmatrix}d\sigma\\ \qquad +b(\delta),\\ e^4(\delta)=\int_0^\varrho\begin{pmatrix}w\left(\delta,\sigma,e^4(\sigma)\right)+x\left(\delta,\sigma,e^1(\sigma)\right)\\ +y\left(\delta,\sigma,e^2(\sigma)\right)+z\left(\delta,\sigma,e^3(\sigma)\right)\end{pmatrix}d\sigma\\ \qquad +b(\delta),\end{cases} \qquad (21)$$

for all $\delta,\sigma\in[0,\varrho]$, for some $\varrho>0$.

As usual, we consider $\Omega = C([0,\varrho],\mathbb{R})$ is continuous real functions created on $[0,\varrho]$ and equipped with a metric

$$\omega(e^1,e^2)=\begin{pmatrix}\max_{0\leq\delta\leq\varrho}\left|e^1(\delta)-e^2(\delta)\right|\\ \max_{0\leq\delta\leq\varrho}\left|e^1(\delta)-e^2(\delta)\right|\end{pmatrix}.$$

Define a partial order " $\leq$ " on Ω as follows:

$$\text{for } e^1, e^2 \in \Omega, \text{ for any } \delta \in [0,\varrho], \ e^1 \leq e^2 \text{ iff } e^1(\delta) \leq e^2(\delta).$$

Thus, $(\Omega, \omega, \leq)$ is a (POCGMS).

System (21) will be taken under the hypotheses below:

$(\dagger_i)$ the functions $w,x,y,z : [0,\varrho]\times[0,\varrho]\times\mathbb{R} \to \mathbb{R}^2$ and $b : [0,\varrho] \to \mathbb{R}$ are continuous;

$(\dagger_{ii})$ for all $e^1, e^2 \in \Omega$, there is $\mu : [0,\varrho] \to M_{2\times 2}([0,\varrho])$, so that

$$\begin{cases} 0 \leq \left| w\left(\delta,\sigma,e^1(\sigma)\right) - w\left(\delta,\sigma,e^2(\sigma)\right)\right| \leq \mu_1(\delta)\omega(e^1,e^2), \\ 0 \leq \left| x\left(\delta,\sigma,e^2(\sigma)\right) - x\left(\delta,\sigma,e^1(\sigma)\right)\right| \leq \mu_2(\delta)\omega(e^1,e^2), \\ 0 \leq \left| y\left(\delta,\sigma,e^1(\sigma)\right) - y\left(\delta,\sigma,e^2(\sigma)\right)\right| \leq \mu_3(\delta)\omega(e^1,e^2), \\ 0 \leq \left| z\left(\delta,\sigma,e^2(\sigma)\right) - z\left(\delta,\sigma,e^1(\sigma)\right)\right| \leq \mu_4(\delta)\omega(e^1,e^2), \end{cases} \tag{22}$$

for all $\delta,\sigma \in [0,\varrho]$ with $\mu(\delta) \leq \Phi = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and $\mu(\delta) \leq \Psi = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$. This holds because of $\Phi, \Psi \in ZM$;

$(\dagger_{iii})$ consider $\mu_1(\delta)+\mu_2(\delta)+\mu_3(\delta)+\mu_4(\delta) < 1$ and

$$\mu(\delta) = \max\{\mu_1(\delta),\mu_2(\delta),\mu_3(\delta),\mu_4(\delta)\};$$

$(\dagger_{iv})$ there exist continuous functions $\lambda,\rho,\upsilon,\varkappa : [0,\varrho] \to \mathbb{R}$ so that

$$\begin{cases} \lambda \leq \int_0^{\varrho} \begin{pmatrix} w(\delta,\sigma,\lambda(\sigma)) + x(\delta,\sigma,\rho(\sigma)) \\ +y(\delta,\sigma,\upsilon(\sigma)) + z(\delta,\sigma,\varkappa(\sigma)) \end{pmatrix} d\sigma + b(\delta), \\ \rho \geq \int_0^{\varrho} \begin{pmatrix} w(\delta,\sigma,\rho(\sigma)) + x(\delta,\sigma,\upsilon(\sigma)) \\ +y(\delta,\sigma,\varkappa(\sigma)) + z(\delta,\sigma,\lambda(\sigma)) \end{pmatrix} d\sigma + b(\delta), \\ \upsilon \leq \int_0^{\varrho} \begin{pmatrix} w(\delta,\sigma,\upsilon(\sigma)) + x(\delta,\sigma,\varkappa(\sigma)) \\ +y(\delta,\sigma,\lambda(\sigma)) + z(\delta,\sigma,\rho(\sigma)) \end{pmatrix} d\sigma + b(\delta), \\ \varkappa \geq \int_0^{\varrho} \begin{pmatrix} w(\delta,\sigma,\varkappa(\sigma)) + x(\delta,\sigma,\lambda(\sigma)) \\ +y(\delta,\sigma,\rho(\sigma)) + z(\delta,\sigma,\upsilon(\sigma)) \end{pmatrix} d\sigma + b(\delta). \end{cases}$$

Based on the assumptions $(\dagger_i)-(\dagger_{vi})$, the problem (21) has a unique solution in Ω.

Also, Corollary 3 is used to discuss the unique stationary distribution for the Markov process. Suppose that $\mathbb{R}^n_+ = \{(e^1,e^2,...,e^n) = e : e_i \geq 0, i \geq 1\}$ and

$$\Lambda^4_{n-1} = \{\wp = (e^1,e^2,e^3,e^4) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^n \wp_i = \sum_{i=1}^n \left(e^1_i + e^2_i + e^3_i + e^4_i\right) = 1\},$$

refer to $4(n-1)$ dimensional unit simplex and $\wp \in \Lambda^4_{n-1}$ could be considered a possibility over $4n$ possible statuses. For more details in this direction, see Section 6.

Hence, if $a_{ij} \geq 0$, then there exists a unique stationary distribution for the Markov process.

8 Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann-Liouville fractional integrals and Atangana-Baleanu integral operators

Fractional differential equations appear naturally in diverse fields of science and engineering. They constitute an important field of research. It should be noted that most papers dealing with the existence of solutions to nonlinear initial value problems of fractional differential equations mainly use techniques of nonlinear analysis such as fixed point methods, stability, the Leray–Schauder result, etc. Numerous academics have investigated the existence and uniqueness of solutions to differential and integral equations involving fractional operators using fixed point theorems; for instance, see [64,65,34].

In this section, a new space is introduced called double-controlled metric spaces (in short, $\eta^{\nu}_{\beth}$-metric spaces), and some fixed-point results for generalized contractive type mappings under mild conditions are obtained. These results are applied to discuss the existence and uniqueness of solutions of 2D Volterra integral equations, Riemann-Liouville integrals, and Atangana-Baleanu integral operators.

The novel space is formulated as follows:

Definition 17. *Suppose that $\wp$ is a non-empty set and $\beth,\nu : \wp\times\wp \to [1,\infty)$ are given functions. Let $\eta^{\nu}_{\beth} : \wp\times\wp \to [0,\infty)$ be a distance function on $\wp$. We list the following hypotheses, for all $\varsigma,\upsilon,\tau \in \wp$,*

(J_1) $\eta^{\nu}_{\beth}(\varsigma,\upsilon) = 0$ iff $\varsigma = \upsilon$;

(J_2) $\eta^{\nu}_{\beth}(\varsigma,\upsilon) = \eta^{\nu}_{\beth}(\upsilon,\varsigma)$;

(J_3) $\eta^{\nu}_{\beth}(\varsigma,\upsilon) \leq \beth(\varsigma,\upsilon)\left[\eta^{\nu}_{\beth}(\varsigma,\tau)+\eta^{\nu}_{\beth}(\tau,\upsilon)\right]$ (an extended triangle inequality);

(J_4) $\eta^{\nu}_{\beth}(\varsigma,\upsilon) \leq \beth(\varsigma,\tau)\eta^{\nu}_{\beth}(\varsigma,\tau)+\beth(\tau,\upsilon)\eta^{\nu}_{\beth}(\tau,\upsilon)$ (controlled triangle inequality);

(J_5) $\eta^{\nu}_{\beth}(\varsigma,\upsilon) \leq \beth(\varsigma,\tau)\eta^{\nu}_{\beth}(\varsigma,\tau)+\nu(\tau,\upsilon)\eta^{\nu}_{\beth}(\tau,\upsilon)$ (double controlled triangle inequality).

$\eta^{\nu}_{\beth}$ is called:

- *an extended $b-$metric [66] if $\eta^{\nu}_{\beth}$ satisfies $(J_1)-(J_3)$,*
- *a controlled metric type [67] if $\eta^{\nu}_{\beth}$ satisfies (J_1), (J_2) and (J_4),*
- *a double controlled metric type [68] if $\eta^{\nu}_{\beth}$ satisfies (J_1), (J_2) and (J_5).*

The pair $(\wp,\eta^{\nu}_{\beth})$ is called an extended $b-$metric/ a controlled metric type/ a double controlled metric type space if $\eta^{\nu}_{\beth}$ is an extended $b-$metric/ a controlled metric type/ a double controlled metric type on $\wp$.

The topological properties of this space are described as

Definition 18. *Let $(\wp,\eta^{\nu}_{\beth})$ be an $\eta^{\nu}_{\beth}-$metric space. A sequence $\{\varsigma_n\} \in \wp$ is called:*

- *convergent if there is $\varsigma \in \wp$ such that $\lim_{n\to\infty}\eta^{\nu}_{\beth}(\varsigma_n,\varsigma) = 0$, and this notation leads to $\lim_{n\to\infty}\varsigma_n = \varsigma$;*

• *Cauchy iff* $\eta_{\daleth}^{\nu}(\varsigma_n,\varsigma_m)\to 0$ *as* $n,m\to\infty$.

An $\eta_{\daleth}^{\nu}-$*metric space is complete if every Cauchy sequence in* $\wp$ *is convergent.*

The existence and uniqueness of fixed points are investigated as follows:

Theorem 8. *Let* $(\wp,\eta_{\daleth}^{\nu})$ *be a complete* $\eta_{\daleth}^{\nu}-$*metric space and the mappings* $\eth,\mho:\wp\to\wp$ *satisfy*

$$a^b\eta_{\daleth}^{\nu}(\eth\varsigma,\mho\upsilon)\le\sigma\eta_{\daleth}^{\nu}(\varsigma,\upsilon)+\rho\frac{\eta_{\daleth}^{\nu}(\upsilon,\mho\upsilon)\left[1+\eta_{\daleth}^{\nu}(\upsilon,\mho\upsilon)\right]}{1+\eta_{\daleth}^{\nu}(\eth\varsigma,\mho\upsilon)}+\mu\frac{\eta_{\daleth}^{\nu}(\upsilon,\mho\upsilon)\left[1+\eta_{\daleth}^{\nu}(\varsigma,\eth\varsigma)\right]}{1+\eta_{\daleth}^{\nu}(\varsigma,\upsilon)},\quad(23)$$

for all $\varsigma,\upsilon\in\wp$, *where* σ,ρ *and* μ *are non-negative real numbers with* $\sigma+\rho+\mu<1$ *and* $a,b\ge 1$. *Consider* $\varsigma_n=\mho^n\varsigma_\circ$, *for* $\varsigma_\circ\in\wp$, *then* $\eth$ *and* $\mho$ *have a unique common fixed point, provided that the following hypotheses hold:*

(*i*)

$$\sup_{l\ge 1}\lim_{j\to\infty}\frac{\daleth(\varsigma_{j+1},\varsigma_{j+2})}{\daleth(\varsigma_j,\varsigma_{j+1})}\nu(\varsigma_{j+1},\varsigma_l)<\frac{1}{\Theta},\quad(24)$$

where $\Theta=\frac{\sigma}{a^b-\rho-\mu}$;

(*ii*) $\lim_{n\to\infty}\daleth(\varsigma,\varsigma_n)$ *and* $\lim_{n\to\infty}\nu(\varsigma,\varsigma_n)$ *exist and are finite for all* $\varsigma\in\wp$.

If we take $a=b=1$, $\mu=\rho=0$ and $\eth=\mho$ in Theorem 8, we get the following corollary:

Corollary 4. *Let* $(\wp,\eta_{\daleth}^{\nu})$ *be a complete* $\eta_{\daleth}^{\nu}-$*metric space and let the mapping* $\eth:\wp\to\wp$ *satisfy*

$$\eta_{\daleth}^{\nu}(\eth\varsigma,\eth\upsilon)\le\sigma\eta_{\daleth}^{\nu}(\varsigma,\upsilon),$$

for all $\varsigma,\upsilon\in\wp$, *where* σ *is a non-negative real number with* $\sigma<1$. *Choose* $\varsigma_n=\mho^n\varsigma_\circ$, *for* $\varsigma_\circ\in\wp$, *then* $\eth$ *has a unique fixed point, provided that the following assumptions are satisfied:*

• $\sup_{l\ge 1}\lim_{j\to\infty}\frac{\daleth(\varsigma_{j+1},\varsigma_{j+2})}{\daleth(\varsigma_j,\varsigma_{j+1})}\nu(\varsigma_{j+1},\varsigma_l)<\frac{1}{\sigma}$;

• $\lim_{n\to\infty}\daleth(\varsigma,\varsigma_n)$ *and* $\lim_{n\to\infty}\nu(\varsigma,\varsigma_n)$ *exist and are finite for all* $\varsigma\in\wp$.

To address the applications, Corollary 4 was applied to find a solution to the two-dimensional (2D) Volterra integral equations, which take the form:

$$\Lambda(\lambda,\hbar)=\xi(\lambda,\hbar)+\int_0^{\lambda}\int_0^{\hbar}\Xi_1(\varsigma,\upsilon,\Lambda(\varsigma,\upsilon))d\varsigma d\upsilon+\daleth\int_0^{\lambda}\Xi_2(\hbar,\upsilon,\Lambda(\lambda,\upsilon))d\upsilon+\nu\int_0^{\hbar}\Xi_3(\lambda,\varsigma,\Lambda(\hbar,\varsigma))d\varsigma,\quad(25)$$

for all $\lambda,\hbar,\varsigma,\upsilon\in[0,1]$, where $\Lambda\in\wp=C([0,1]\times[0,1])$, and $\xi:[0,1]\times[0,1]\to\mathbb{R}^2$; $\Xi_i(i=1,2,3):[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$. Define the distance $\eta_{\daleth}^{\nu}:\wp\times\wp\to[0,\infty)$ on the set of all continuous functions $\wp=C\left([0,1]^2,\mathbb{R}^+\right)$ from $[0,1]^2$ onto $\mathbb{R}^+$ as follows:

$$\eta_{\daleth}^{\nu}(\Lambda(\lambda,\hbar),\varpi(\lambda,\hbar))=|\kappa(\lambda,\hbar)-\varpi(\lambda,\hbar)|^2,$$

for all $\kappa,\varpi\in\wp$. Let the functions $\daleth,\nu:\wp\times\wp\to[1,\infty)$ be defined by

$$\daleth(\Lambda(\lambda,\hbar),\varpi(\lambda,\hbar))=\frac{|\Lambda(\lambda,\hbar)|+|\varpi(\lambda,\hbar)|}{2}+2,$$
$$\nu(\Lambda(\lambda,\hbar),\varpi(\lambda,\hbar))=\frac{|\Lambda(\lambda,\hbar)|+|\varpi(\lambda,\hbar)|}{1+|\Lambda(\lambda,\hbar)|+|\varpi(\lambda,\hbar)|}+2.$$

Then $\left(\wp,\eta_{\daleth}^{\nu}\right)$ is clearly an $\eta_{\daleth}^{\nu}-$metric space.

Problem (25) was considered via the following assumptions:

(i) $\Xi_i(i=1,2,3):[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$ are continuous functions satisfying

$$\begin{aligned}&|\Xi_1(\varsigma,\upsilon,\hbar_1(\varsigma,\upsilon)-\Xi_1(\varsigma,\upsilon,\hbar_2(\varsigma,\upsilon)|\\ \le\ &¥_1\sqrt{|\hbar_1(\varsigma,\upsilon)-\hbar_2(\varsigma,\upsilon)|},\\ &|\Xi_2(\varsigma,\upsilon,\hbar_1(\varsigma,\upsilon)-\Xi_2(\varsigma,\upsilon,\hbar_2(\varsigma,\upsilon)|\\ \le\ &¥_2\sqrt{|\hbar_1(\varsigma,\upsilon)-\hbar_2(\varsigma,\upsilon)|},\\ &|\Xi_3(\varsigma,\upsilon,\hbar_1(\varsigma,\upsilon)-\Xi_3(\varsigma,\upsilon,\hbar_2(\varsigma,\upsilon)|\\ \le\ &¥_3\sqrt{|\hbar_1(\varsigma,\upsilon)-\hbar_2(\varsigma,\upsilon)|},\end{aligned}$$

for constants $¥_1,¥_2,¥_3\ge 0$ and $\hbar_1,\hbar_2\in\mathbb{R}^2$;

(ii) $¥_1+|\daleth|¥_2+|\nu|¥_3\le\sigma$; where $0<\sigma<1$.

According to the above conditions, Problem (25) has a unique solution.

Again, Corollary 4 is applied to find a unique solution to Reimann-Lioville fractional integrals, which takes the form

$${}^{RL}_{\ \hbar}I^{\tau}_{\kappa}\varsigma(\kappa)=\frac{1}{\Gamma(\tau)}\int_{\hbar}^{\kappa}(\kappa-\ell)^{\tau-1}\varsigma(\ell)d\ell;\ \Gamma(\tau)>0,\quad(26)$$

where $\tau\in\mathbb{R}$, $\varsigma(\kappa)\in\wp=C([0,1],\mathbb{R})(C([0,1],\mathbb{R})$ is the set of all continuous functions from $[0,1]$ onto $\mathbb{R}$) and $\kappa,\ell\in[0,1]$ which is the fractional integral. Define the distance $\eta_{\daleth}^{\nu}:\wp\times\wp\to[0,\infty)$ by

$$\eta_{\daleth}^{\nu}(\varsigma,\upsilon)=|\varsigma(\kappa)-\upsilon(\kappa)|^2,$$

for all $\varsigma(\kappa),\upsilon(\kappa)\in\wp$ and $\kappa\in[0,1]$. Also, define $\daleth,\nu:\wp\times\wp\to[1,\infty)$ by

$$\daleth(\varsigma(\kappa),\upsilon(\kappa))=\frac{|\varsigma(\kappa)|+|\upsilon(\kappa)|}{2}+2,$$
$$\nu(\varsigma(\kappa),\upsilon(\kappa))=\frac{|\varsigma(\kappa)|+|\upsilon(\kappa)|}{1+|\varsigma(\kappa)|+|\upsilon(\kappa)|}+2.$$

Then $\left(\wp,\eta_{\daleth}^{\nu}\right)$ is an $\eta_{\daleth}^{\nu}-$metric space.

By helping Corollary 4, Problem (26) has a unique solution provided that the following condition holds:

$$\frac{1}{\Gamma^2(\tau+1)}\frac{(\kappa-\ell)^{\tau-1}(\kappa-\hbar)^{2\tau}}{\left|(\kappa-\ell)^{\tau-1}\right|}<\sigma;$$

where $\sigma\in(0,1)$ and $\kappa\neq\ell$.

In 2016, Atangana and Baleanu [69] developed more general definitions of fractional derivative and integral operator targeting non-local and non-singular kernel, this operator takes the form (27).

Let $\wp=C([0,1],\mathbb{R})$ be a set of all continuous functions from $[0,1]$ onto $\mathbb{R}$. Define the distance $\eta_{\daleth}^{\nu}:\wp\times\wp\to[0,\infty)$ by

$$\eta_{\daleth}^{\nu}(\varsigma,\upsilon)=|\varsigma(\kappa)-\upsilon(\kappa)|^2,$$

for all $\varsigma(\kappa),\upsilon(\kappa)\in\wp$ and $\kappa\in[0,1]$. Also define $\daleth,\nu:\wp\times\wp\to[1,\infty)$ by

$$\begin{aligned}\daleth(\varsigma(\kappa),\upsilon(\kappa)) &= \frac{|\varsigma(\kappa)|+|\upsilon(\kappa)|}{2}+2,\\ \nu(\varsigma(\kappa),\upsilon(\kappa)) &= \frac{|\varsigma(\kappa)|+|\upsilon(\kappa)|}{1+|\varsigma(\kappa)|+|\upsilon(\kappa)|}+2.\end{aligned}$$

Then $\left(\wp,\eta_{\daleth}^{\nu}\right)$ is an $\eta_{\daleth}^{\nu}-$metric space.

Atangana-Baleanu fractional integral type of order $\Re$ of a function $\varsigma(\kappa)$ is exemplified as:

$$ {}^{AB}_{\hbar}I^{\Re}_{\kappa}\varsigma(\kappa)=\frac{1-\Re}{\beta(\Re)}\varsigma(\kappa)+\frac{\Re}{\beta(\Re)\Gamma(\Re)}\int_{\hbar}^{\kappa}\varsigma(\ell)(\kappa-\ell)^{\Re-1}d\ell. \tag{27}$$

where $\Re\in(0,1]$, $\varsigma(\kappa)\in\wp$ and $\kappa,\ell\in[0,1]$ which is the fractional integral.

Using Corollary 4, the fractional integral (27) has a unique solution if the following assumption is true:

$$\frac{1-\Re}{\beta(\Re)}+\frac{\hbar^{\Re}}{\beta(\Re)\Gamma(\Re)}<\sigma;\text{ where }\sigma\in(0,1).$$

9 Solving singular coupled fractional differential equations with integral boundary constraints by coupled fixed point methodology

Differential equations of fractional order are frequently used to provide multiple perspectives on control systems, fluid dynamics, and other topics. The fractional order models are more accurate than the correct order models, and they also seem to have a greater degree of freedom; for more details, see [70,71]. The integral boundary conditions play a prominent role in many applications, such as thermoelasticity, population dynamics, problems with blood flow, and underground water supply. For a full and comprehensive explanation of the terms with integral boundaries, see [72,73,33].

Continuing the same approach, some coupled fixed point results are obtained for two rational contractive mappings under mild conditions in $b-$metric spaces. Furthermore, the findings are used to debate the existence and uniqueness of the solution to the following singular coupled fractional differential equation (CFDE, for short) of order ν:

$$\begin{cases} {}^{c}D^{\nu}z(\tau)+\Xi(\tau,z(\tau),w(\tau))=0,\ \tau\in(0,1),\\ {}^{c}D^{\nu}w(\tau)+\Xi(\tau,w(\tau),z(\tau))=0,\ \tau\in(0,1),\\ \Lambda'''(0)=\Lambda''(0)=0,\\ \Lambda'=\Lambda(1)=\alpha\int_0^1\Lambda(\theta)d\theta, \end{cases}\tag{28}$$

where $\Lambda\in C[0,1]\times C[0,1]$ is given by $\Lambda(\tau)=(z(\tau),w(\tau))$, $\nu\in(3,4)$, $\alpha\in(0,2)$, $c\mathcal{D}^{\nu}$ is the Caputo fractional derivative, and Ξ may be singular at $z=0$ and $w=0$.

The theorem concerned with the existence of fixed points is formulated as follows:

Theorem 9.*Assume that (Ω,μ_r) is a complete $b-$metric space with a coefficient $r(=b)\geq1$ and let the mappings $\Upsilon,\Xi:\Omega\times\Omega\to\Omega$ satisfy*

$$\begin{aligned}&\mu_r(\Upsilon(\rho,a),\Xi(\sigma,b))\\ \leq\ &\lambda\frac{\mu_r(\rho,\sigma)+\mu_r(a,b)}{2}\\ &+\tau\frac{[1+\mu_r(\rho,\Upsilon(\rho,a))]\mu_r(\sigma,\Xi(\sigma,b))}{(1+\mu_r(\rho,\sigma)+\mu_r(a,b))}\\ &+\zeta\frac{\mu_r(\sigma,\Upsilon(\rho,a))\mu_r(\rho,\Xi(\sigma,b))}{(1+\mu_r(\rho,\sigma)+\mu_r(a,b))}.\end{aligned}$$

for all $\rho,a,\sigma,b\in\Omega$ and $\lambda,\tau,\zeta\geq0$ with $r\lambda+\tau<1$ and $\lambda+\zeta<1$. Then there is a unique common coupled fixed point of Υ and Ξ.

The following corollary follows immediately from Theorem 9.

Corollary 5.*Let (Ω,μ_r) be a complete bMS with a coefficient $r\geq1$ and let the mapping $\Upsilon:\Omega\times\Omega\to\Omega$ verifies*

$$\mu_r(\Upsilon(\rho,a),\Upsilon(\sigma,b))\leq\lambda\frac{\mu_r(\rho,\sigma)+\mu_r(a,b)}{2},$$

for all $\rho,a,\sigma,b\in\Omega$ and $\lambda\geq0$ with $r\lambda<1$. Then Υ has a unique coupled fixed point.

The Green's function of the problem (28) is obtained as follows:

Lemma 2.*Given the pair $(a,q)\in(C(0,1)\cap L(0,1))\times(C(0,1)\cap L(0,1))$, $\nu\in(3,4)$, $\alpha\in(0,2)$, the unique solution of*

$$\begin{cases} {}^{c}\Theta^{\nu}(\Lambda(\tau))+(a(\tau),q(\tau))=0,\ \tau\in(0,1),\\ \Lambda'''(0)=\Lambda''(0)=0,\\ \Lambda'=\Lambda(1)=\alpha\int_0^1\Lambda(\theta)d\theta, \end{cases}\tag{29}$$

is

$$\Lambda(\tau)=\int_0^1 \Im(\tau,\theta)(a(\theta),q(\theta))d\theta,$$

where $\Lambda(\tau)=(z(\tau),w(\tau))$ *and*

$$\begin{aligned}&\Im(\tau,\theta)\\ =&\ \frac{1}{\nu(2-\alpha)\Gamma(\nu)}\\ &\times\begin{cases}\begin{pmatrix}\nu(2-\alpha)\\+2\alpha\tau(\nu-1+\theta)\end{pmatrix}(1-\theta)^{\nu-1} & \text{if } 0\le\theta\le\tau\le 1,\\ \quad -\nu(2-\alpha)(\tau-\theta)^{\nu-1}, & \\ \begin{pmatrix}\nu(2-\alpha)\\+2\alpha\tau(\nu-1+\theta)\end{pmatrix}(1-\theta)^{\nu-1}, & \text{if } 0\le\tau\le\theta\le 1,\end{cases}\end{aligned}$$

The next lemma estimates Green's function $\Im(\tau,\theta)$ of a fractional definitional equation with integral boundary stipulations described in (29) on $L_2(0,1)$.

Lemma 3.*Suppose that* $\nu\in(3,4)$, $\alpha\in(0,2)$, *such that* $\nu\neq\alpha$, *then for all* $\tau,\theta\in(0,1)$, *the Green's function* $\Im(\tau,.)\in L_2$ *verifies*

$$\int_0^1\left|\Im(\tau,\theta)\right|^2 d\theta<\frac{1}{\Gamma^2(\nu)}\left(\frac{4}{5}+\frac{8\alpha}{3|\alpha-2|}+\frac{4\alpha^2}{9(\alpha-2)^2}\right).$$

Assume that $\Xi(.,z(.),w(.))\in L_2$ for any $z,w\in C[0,1]$ and describe a mapping $\Upsilon: C[0,1]\times C[0,1]\to C[0,1]$ as follows:

$$\Upsilon(z(\tau),w(\tau))=\int_0^1\Im(\tau,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta, \tag{30}$$

where $\theta\to\Im(\tau,\theta)$ is continuous from $[0,1]$ to L_2.

The following lemma gives the solution of the boundary value problem (28):

Lemma 4.*Assume that* Υ *is a mapping described as (30) and* $z,w\in C[0,1]$. *The pair* $(z(\tau),w(\tau))$ *is a solution of the boundary value problem (28) if and only if it is a coupled fixed point of* Υ.

Finally, a uniqueness of the solution is presented by the following theorem:

Theorem 10.*Suppose that* $\nu\in(3,4)$ *and*

$$\lambda=\frac{1}{\Gamma^2(\nu)}\left(\frac{4}{5}+\frac{8\alpha}{3|\alpha-2|}+\frac{4\alpha^2}{9(\alpha-2)^2}\right)<1,$$

holds for any $\alpha\in(0,2)$. *Let* $\Xi(.,z(.),w(.))$ *be a function in* L_2 *for any* $z,w\in C[0,1]$ *and for any* $z^*,w^*\in C[0,1]$ *the inequality below holds*

$$\begin{aligned}&\left|\Xi(\theta,z(\theta),w(\theta))-\Xi(\theta,z^*(\theta),w^*(\theta))\right|^2\\ \le&\ \frac{|z(\theta)-z^*(\theta)|^2+|w(\theta)-w^*(\theta)|^2}{2},\ \theta\in[0,1],\end{aligned}$$

Then the mapping Υ *has a unique coupled fixed point, which is a unique solution to the boundary value problem (28).*

10 Applying faster algorithm for obtaining convergence, stability, and data dependence results with application to functional-integral equations

One of the important directions of fixed-point methods is the study of the behavior and performance of algorithms that contribute greatly to real-world applications. Many authors tended to create many iterative methods for approximating fixed points in terms of improving the performance and convergence behavior of algorithms for nonexpansive mappings [43,44,45].

In this section Hammad et al. [45] created a new faster iterative algorithm than the previous sober algorithms. In the setting of Banach spaces, this algorithm is used to analyze convergence and stability results. A basic numerical example is also provided to highlight the behavior and effectiveness of the approach. Ultimately, the proposed algorithm is used to solve the functional Volterra-Fredholm integral problem as an application.

The iterative scheme can be formulated as:

$$\begin{cases}\nu_\circ\in\Delta,\\ \varpi_i=(1-\alpha_i)\nu_i+\alpha_i\Xi\nu_i,\\ \wp_i=\Xi((1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i),\\ \Im_i=\Xi((1-\gamma_i)\wp_i+\gamma_i\Xi\wp_i),\\ \nu_{i+1}=\Xi\Im_i,\end{cases} \tag{31}$$

for each $i\ge 1$, where α_i, η_i and γ_i are sequences in $[0,1]$.

Note, the symbols $\longrightarrow$ and $\rightharpoonup$ refer to the strong and weak convergence, respectively.

Firstly, the rate of convergence of the iterative scheme for ACMs is presented in the following theorems:

Theorem 11.*Assume that* Λ *is a Banach space (BS, for short) and* Δ *is a closed convex subset (CCS) of* Λ. *Let* $\Xi:\Lambda\to\Lambda$ *be a mapping satisfying (2) with* $\Upsilon(\Xi)\neq\emptyset$. *Suppose that* $\{\nu_i\}$ *is the iterative sequence generated by (31) with* $\{\alpha_i\},\{\eta_i\},\{\gamma_i\}\in[0,1]$ *such that* $\sum_{i=0}^{\infty}\gamma_i=\infty$. *Then* $\{\nu_i\}\longrightarrow\zeta\in\Upsilon(\Xi)$.

Theorem 12.*Let* Δ *be a CCS of a BS* Λ *and* $\Xi:\Lambda\to\Lambda$ *be a mapping satisfying (2) with* $\Upsilon(\Xi)\neq\emptyset$. *Consider* $\{\nu_i\}$ *is the iterative sequence generated by the algorithm (31) with* $\{\alpha_i\},\{\eta_i\},\{\gamma_i\}\in[0,1]$ *such that* $0<\gamma\le\gamma_i\le 1$, *for all* $i\ge 1$. *Then the sequence* $\{\nu_i\}$ *converges faster to* ν *than the iterative scheme (1).*

Next, convergence results of the iteration procedure (31) for SGNMs in the setting of uniformly vonvex Banach spaces (UCBSs) are obtained as follows:

Lemma 5.*Let* Δ *be a CCS of a BS* Λ *and* $\Xi:\Lambda\to\Lambda$ *be SGNM with* $\Upsilon(\Xi)\neq\emptyset$. *If* $\{\nu_i\}$ *is the iterative sequence given by the algorithm (31), then* $\lim_{i\to\infty}\|\nu_i-\zeta\|$ *exists for each* $\zeta\in\Upsilon(\Xi)$.

Lemma 6.*Let $\varDelta$ be a non-empty CCS of a UCBS $\varLambda$ and $\varXi : \varLambda \to \varLambda$ be SGNM. If $\{\nu_i\}$ is the iterative sequence defined by the algorithm (31). Then $\varUpsilon(\varXi) \neq \emptyset$ if and only if $\{\nu_i\}$ is bounded and $\lim_{i\to\infty} \|\varXi\nu_i - \nu_i\| = 0$.*

Theorem 13.*Let $\varLambda$, $\varDelta$ and $\varXi$ be as in Lemma 6. If $\varLambda$ satisfies Opial's condition and $\varUpsilon(\varXi) \neq \emptyset$, then the sequence $\{\nu_i\}$ iterated by (31) converges weakly to a fixed point of $\varXi$, that is $\{\nu_i\} \rightharpoonup \zeta \in \varUpsilon(\varXi)$.*

Theorem 14.*Let $\varLambda$ be a UCBS and $\varDelta$ be a non-empty compact convex subset of $\varLambda$. Assume that $\varXi : \varDelta \to \varDelta$ is SGNM and $\{\nu_i\}$ is the iterative sequence given by (31). Then $\{\nu_i\} \longrightarrow \zeta \in \varUpsilon(\varXi)$.*

Theorem 15.*Let $\varLambda, \varDelta$ and $\varXi$ be described as Lemma 6 and $\{\nu_i\}$ be an iterative sequence defined by (31). Then*

$$\{\nu_i\} \longrightarrow \zeta \in \varUpsilon(\varXi) \Leftrightarrow \liminf_{i\to\infty} d(\nu_i, \varUpsilon(\varXi)) = 0,$$

where $d(\nu, \varUpsilon(\varXi)) = \inf\{\|\nu - \zeta\| : \zeta \in \varUpsilon(\varXi)\}$.

Additionally, the stability of the iteration process (31) is presented in the theorem below.

Theorem 16.*Let $\varLambda$ be a BS and $\varDelta$ be a CCS of $\varLambda$. Suppose that $\varXi : \varLambda \to \varLambda$ is a self-mapping satisfies (2) and $\{\nu_i\}$ is the iterative sequence iterated by (31) with $\{\alpha_i\}, \{\eta_i\}, \{\gamma_i\} \in [0,1]$ so that $\sum_{i=0}^{\infty} \gamma_i = \infty$. Then the algorithm (31) is $\varXi-$stable.*

Furthermore, the following example is used to investigate the behavior of the proposed algorithm.

*Example 1.*Let $\varLambda = \mathbb{R}$, $\varDelta = [0,50]$, and $\varXi : \varDelta \to \varDelta$ be a mapping defined by

$$\varXi(\nu) = \sqrt{\nu^2 - 9\nu + 54}.$$

Clearly, 6.0000 is a *FP* of the mapping $\varXi$. Take $\alpha_i = \eta_i = \gamma_i = \frac{1}{5(i+2)}$, with different initial values. Then we obtain the following Tables: 1-3 and graphs for comparison of the various iterative methods.

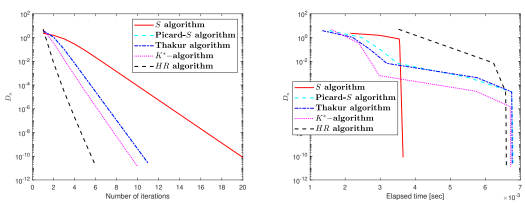

Fig. 1: Graphically comparison of the proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 1$.

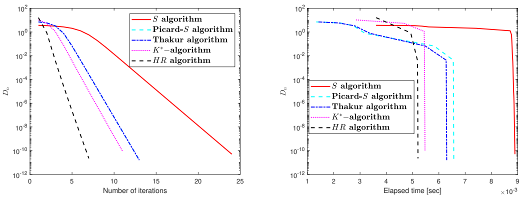

Fig. 2: Graphically comparison of the proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 23$.

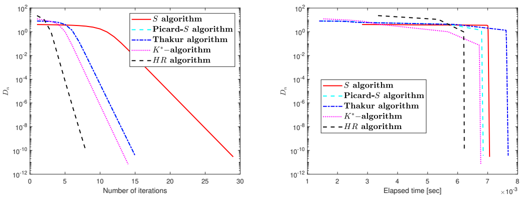

Fig. 3: Graphically comparison of proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 41$.

*Remark.*Based on what has been shown in the tables and figures above, it is clear that the method is successful and the behavior of the algorithm is satisfactory compared to some sober iterations in this direction.

Finally, the considered algorithm is applied to solve the Volterra-Fredholm integral equation, which was suggested by Lungu and Rus [74].

Consider the following problem:

$$\begin{aligned}\xi(\nu,\varpi) &= \aleph(\nu,\varpi,z(\xi(\nu,\varpi)) \\ &\quad + \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,\xi(\kappa^*,\tau^*))\, d\kappa^* d\tau^*, \qquad (32)\end{aligned}$$

for all $\nu, \varpi \in \mathbb{R}_+$. Assume that $(\varGamma, |.|)$ is a *BS*, $s > 0$ and

$$\chi_s = \left\{ \begin{array}{l} \xi \in C\left(\mathbb{R}_+^2, \varGamma\right): \text{ there is } U(\xi) > 0 \\ \text{so that } |\xi(\nu,\varpi)| e^{-s(\nu+\varpi)} \leq U(\xi) \end{array} \right\}.$$

Define the norm on χ_s as follows:

$$\|\xi\|_s = \sup_{\nu,\varpi\in\mathbb{R}_+} \left(|\xi(\nu,\varpi)| e^{-s(\nu+\varpi)}\right).$$

It follows from the paper [75] that $(\chi_s, \|\xi\|_s)$ is a BS.

The following theorem helps us for proving our main result in this part.

Theorem 17.*[74] Assume that the postulates below are satisfied:*

$(P_i)$$\aleph \in C\left(\mathbb{R}_+^2 \times \varGamma, \varGamma\right)$ *and* $\mho \in C\left(\mathbb{R}_+^4 \times \varGamma, \varGamma\right)$;

Table 1: Example 1: Numerical effectiveness comparison of the proposed algorithm (HR **algorithm**) when $\nu_\circ = 1$.

Iter (n)	S algorithm	Picard-S algorithm	Thakur algorithm	K^*−algorithm	HR algorithm
1	8.70091704981746	7.16921920849454	7.16914772374443	6.36059443309194	6.00727524833131
2	7.16526232112099	6.10977177957558	6.10975923118057	6.01468466644452	6.00002780412529
3	6.39088232469395	6.00714403607375	6.00714316980988	6.00065450058063	6.00000018328059
4	6.10931564922512	6.00044721072023	6.00044715630748	6.00003168526056	6.00000000153606
5	6.02823416956883	6.00002793225376	6.00002792885454	6.00000161316022	6.00000000001474
6	6.00711636371271	6.00000174471605	6.00000174450373	6.00000008491151	6.00000000000015
7	6.00178220920879	6.00000010899371	6.00000010898045	6.00000000457683	
8	6.00044563525887	6.00000000680959	6.00000000680876	6.00000000025113	
9	6.00011139089900	6.00000000042547	6.00000000042542	6.00000000001397	
10	6.00002784178933	6.00000000002659	6.00000000002658	6.00000000000079	
11	6.00000695905778	6.00000000000166	6.00000000000166		
12	6.00000173945939				
13	6.00000043479852				
14	6.00000010868515				
15	6.00000002716811				
16	6.00000000679132				
17	6.00000000169767				
18	6.00000000042438				
19	6.00000000010609				
20	6.00000000002652				

Table 2: Example 1: Numerical effectiveness comparison of the proposed algorithm (HR **algorithm**) when $\nu_\circ = 23$.

Iter (n)	S algorithm	Picard-S algorithm	Thakur algorithm	K^*−algorithm	HR algorithm
1	19.3563152555029	15.9518056335603	15.9517798586745	12.5877267284391	6.85064626172682
2	15.9377280808459	10.1547429779848	10.1547063746371	7.21310921795745	6.00404110670722
3	12.8216267389821	6.83637415013381	6.83635232023217	6.07773436689889	6.00002666859578
4	10.1453665006161	6.07061076131832	6.07060799598213	6.00385998151440	6.00000022350839
5	8.09904894091384	6.00453182122915	6.00453163699128	6.00019677562843	6.00000000214472
6	6.83342864338475	6.00028356649349	6.00028355493982	6.00001035833149	6.00000000002245
7	6.26044908931579	6.00001771655983	6.00001771583789	6.00000055832830	6.00000000000025
8	6.07032543085564	6.00000110688254	6.00000110683744	6.00000003063582	
9	6.01796113338512	6.00000006915944	6.00000006915662	6.00000000170455	
10	6.00451434416413	6.00000000432139	6.00000000432122	6.00000000009591	
11	6.00112994220417	6.00000000027003	6.00000000027002	6.00000000000545	
12	6.00028253511929	6.00000000001687	6.00000000001687		
13	6.00007062920329	6.00000000000105	6.00000000000105		
14	6.00001765533615				
15	6.00000441334114				
16	6.00000110322227				
17	6.00000027578013				
18	6.00000006893931				
19	6.00000001723354				
20	6.00000000430809				
21	6.00000000107696				
22	6.00000000026923				
23	6.00000000006730				
24	6.00000000001683				

(P_{ii})*there exist* $z:\chi_s \to \chi_s$ *and* $\pi_z > 0$ *so that*

$$\left|z(\xi(\nu,\varpi)) - z(\xi^*(\nu,\varpi))\right| \le \pi_z \left\|\xi - \xi^*\right\| e^{s(\nu+\varpi)},$$

for all $\nu,\varpi \in \mathbb{R}_+$ *and* $\xi,\xi^* \in \chi_s$;

(P_{iii})*for all* $\nu,\varpi \in \mathbb{R}_+$ *and* $c,c^* \in \Gamma$, *there exists* $\pi_\aleph > 0$ *so that*

$$\left|\aleph(\nu,\varpi,c) - \aleph(\nu,\varpi,c^*)\right| \le \pi_\aleph \left|c - c^*\right|;$$

(P_{iv})*for all* $\nu,\varpi,\kappa^*,\tau^* \in \mathbb{R}_+$ *and* $c,c^* \in \Gamma$, *there exists* $\pi_\mho(\nu,\varpi,\kappa^*,\tau^*) > 0$ *so that*

$$\left|\mho(\nu,\varpi,\kappa^*,\tau^*,c) - \mho(\nu,\varpi,\kappa^*,\tau^*,c^*)\right| \le \pi_\mho(\nu,\varpi,\kappa^*,\tau^*)\left|c - c^*\right|;$$

$(P_v)\pi_\mho \in C\left(\mathbb{R}_+^4, \mathbb{R}_+\right)$ *and*

$$\int_0^\nu \int_0^\varpi \pi_\mho(\nu,\varpi,\kappa^*,\tau^*)\, e^{s(\kappa^*+\tau^*)} d\kappa^* d\tau^* \le \pi e^{s(\kappa^*+\tau^*)},$$

for all $\nu,\varpi \in \mathbb{R}_+$;

$(P_{vi})\pi_z \pi_\aleph + \pi < 1$.

Table 3: Example 1: Numerical effectiveness comparison of the proposed algorithm (HR **algorithm**) when $\nu_\circ = 41$.

Iter (n)	S algorithm	Picard-S algorithm	Thakur algorithm	K^*−algorithm	HR algorithm
1	36.9195393902310	32.9359459295575	32.9359410372494	28.6777050430159	18.0010100671820
2	32.9185209048623	25.1854713159820	25.1854637821918	19.1184050275537	6.99692674147933
3	28.9966652255498	17.9433658908193	17.9433563898048	11.5396125686070	6.00870649597241
4	25.1701649172411	11.6863804499820	11.6863697408416	7.09098245036886	6.00007316294814
5	21.4670877470757	7.49707008913586	7.49706191231517	6.07877285288080	6.00000070206652
6	17.9313757495734	6.15612940775542	6.15612775011077	6.00426075582864	6.00000000734890
7	14.6320375697776	6.01035659987680	6.01035647945727	6.00023000499555	6.00000000008173
8	11.6781275774842	6.00064966715110	6.00064965955411	6.00001262154899	6.00000000000095
9	9.23371552492475	6.00004060231534	6.00004060184040	6.00000070225384	
10	7.49350575600870	6.00000253705577	6.00000253702609	6.00000003951159	
11	6.53524954796329	6.00000015853311	6.00000015853125	6.00000000224357	
12	6.15563769964487	6.00000000990656	6.00000000990645	6.00000000012838	
13	6.04078294734902	6.00000000061907	6.00000000061906	6.00000000000739	
14	6.01032406840519	6.00000000003869	6.00000000003869	6.00000000000043	
15	6.00258903649963	6.00000000000242	6.00000000000242		
16	6.00064771546926				
17	6.00016194664418				
18	6.00004048534517				
19	6.00001012070523				
20	6.00000253001219				
21	6.00000063246434				
22	6.00000015810715				
23	6.00000003952473				
24	6.00000000988071				
25	6.00000000247007				
26	6.00000000061749				
27	6.00000000015437				
28	6.00000000003859				
29	6.00000000000965				

Then the problem (32) has a unique solution $\zeta \in \chi_S$ and the iterative sequence

$$\begin{aligned} &\xi_{i+1}(\nu,\varpi) \\ &= \aleph(\nu,\varpi,z(\xi(\nu,\varpi)) \\ &\quad + \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,\xi_i(\kappa^*,\tau^*))\,d\kappa^* d\tau^*, \end{aligned}$$

for all $i \geq 1$ converges uniformly to ζ.

Now, according to the above hypotheses, the main theorem of this part is as follows:

Theorem 18.*Let $\{\nu_i\}$ be an iterative sequence generated by (31) with sequences $\{\alpha_i\},\{\eta_i\},\{\gamma_i\} \in [0,1]$ so that $\sum_{i=0}^{\infty}\gamma_i = \infty$. If the postulates $(P_i)-(P_{vi})$ of Theorem 17 hold. Then the problem (32) has a unique solution $\zeta \in \chi_S$ and the intended algorithm (31) converges strongly to ζ.*

11 Conclusion

One of the important branches of mathematics is functional analysis. The development of this field progressed in parallel to the development of modern theoretical physics. The formal framework of functional analysis adheres closely to the laws of both quantum mechanics and quantum field theory. At the same time, these theoretical physics frameworks have a very relevant influence and provide links substantiating the body of problems and solution methodology of functional analysis. A branch was taken from this tree and was called fixed-point theory. Fixed point technologies offer a focal concept with many diverse applications. It has been and still is an important theoretical tool in many fields and various disciplines, such as topology, game theory, optimal control, artificial intelligence, logic programming, and dynamical systems. So, in this review, for generalized contractive type mappings in the context of abstract spaces, various fixed-point, coupled, tripled, and quadruple fixed-point results are established. Furthermore, several common examples are given to support our theoretical conclusions. Furthermore, solutions to Riemann-Liouville integrals, Atangana-Baleanu integral operators, functional equations, functional Volterra-Fredholm integral equations, 2D Volterra integral equations, and the boundary value issue for singularly coupled fractional differential equations are established. Also discussed is a unique stationary distribution for the Markov process. In addition, a new iterative algorithm that is faster than the previous article's sober algorithms is developed. Results for convergence and stability are examined using the suggested technique. Moreover, a simple numerical example is given to illustrate the behavior and efficacy of the proposed algorithm.

12 Future works

There are many open problems for the last section in this review. Some of these are summarized as follows:

(1)If the mapping Ξ is defined in a Hilbert space Δ endowed with inner product space, then we can use our iteration (31) to find a common solution to the variational inequality problem. This problem can be stated as follows: find $\wp^* \in \Delta$ such that

$$\langle \Xi\wp^*, \wp - \wp^*\rangle \geq 0 \text{ for all } \wp \in \Delta,$$

where $\Xi : \Delta \to \Delta$ is a nonlinear mapping. Variational inequalities are an important and essential modeling tool in many fields, such as engineering mechanics, transportation, economics, and mathematical programming, (see [76]).

(2)The proposed algorithm can be generalized to gradient and extra-gradient projection methods; these methods are very important for finding saddle points and solving many problems in optimization, (see [77]).

(3)The convergence of the considered algorithm can be accelerated by adding shrinking projection and CQ terms. These methods stimulate algorithms and improve their performance to obtain strong convergence; for more details; see [78].

(4)If the mapping Ξ is an $\alpha-$inverse strongly monotone function and the inertial term is included in the algorithm, the result is an inertial proximal point algorithm. This algorithm is used in many applications, such as monotone variational inequalities, image restoration problems, convex optimization problems, and split convex feasibility problems (see [79,80,81,82]). For more accuracy, these problems can be expressed as mathematical models such as machine learning and the linear inverse problem.

(5)A supposed algorithm can be applied to solve second-order differential equations and fractional differential equations, where these equations can be converted into integral equations by Green's functions.

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