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Estimation of Diversification Effects/Benefits Using the Generalised Pareto Distribution - Extreme Value Gumbel Copula Model

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Abstract: The objective of this paper is to estimate the diversification effects/benefits of an investment in a portfolio consisting of the South African Industrial (J520) and the Financial (J580) Indices using the Generalised Pareto Distributions (GPDs) with an extreme value Gumbel copula. The GPD is used as the marginal distribution to both assets to better characterise the extreme risk of returns in both Indices tails. The extreme value Gumbel copula captures the dependence structure (co-movement) of the financial assets in the portfolio. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) goodness of fit tests and the scatterplots indicate that the upper tail of the gains (the larger gains) risk and the losses tail (the larger losses) are best captured using the extreme value Gumbel copula. Monte Carlo simulation of an equally weighted portfolio of the two Indices is used to estimate the portfolio risk. The univariate marginal risks and the portfolio risks are used to calculate the diversification effects/benefits. The results show that there are benefits in diversification since the riskiness of the portfolio is less than the sum of the risk of the two financial assets. This implies that VaR, although not additive theoretically, is sub-additive in this practical situation. This property of sub-additivity represents the benefits of diversification for a portfolio. The implication is that investors investing in individual risky assets can benefit from constructing such a portfolio to reduce extreme risk. Due to high dependence and contagion between developed markets/Global markets, this is useful information for local and international investors seeking a portfolio which includes developing countries' market Indices, such as South African assets, which are less correlated with other Global markets, thereby reducing the risk of contagion.

Keywords: Diversification effects/benefits, Expected Shortfall, extreme value Gumbel copula, Generalised Pareto Distribution, Monte Carlo simulation, Value-at-Risk.

1. Introduction

The estimation of portfolio risk is important in reserving capital to finance the investment risk taken. Portfolio risk allows one to account for diversification effects. Accounting for diversification provides information that allows effective institution-wide risk management and other key business decisions and business processes [1]. According to [2, 3, 4] there is higher dependence and correlation between developed markets/Global markets (which mainly excluded developing countries' markets/emerging markets), indicating that the diversification effects/benefits of international investing are decreasing. This implies that diversification effects /benefits of international investing are minimal if developing countries' markets are not included in the portfolios. Due to the high dependence and contagion between developed markets/Global markets, the findings of this study are useful information for local and international investors seeking a portfolio which includes developing countries' stock market Indices containing South African financial assets.

This will benefit investors and practitioners in improving investment diversification, as developing countries' markets are less correlated with developed markets/Global markets. This study extends studies by [5, 6, 7, 8], who used bivariate portfolios to estimate bivariate portfolio risk. However, this study goes further in estimating diversification effects/benefits for the portfolio.

This paper provides investors and practitioners with a model framework that will allow for portfolio risk and diversification effects/benefits to be estimated more accurately whilst providing information on investment in South African assets. Portfolio investment risk is one measure used in controlling/minimising the risk in a specific country. At the industry/sector level, efforts have been made to investigate the interdependence of stock returns [9]. The South African Industrial and Financial sectors are under investigation in this study. This study is confined to a bivariate case, although, in principle, the ideas discussed can be applied to higher-dimension models. [10] confirmed that the extreme value Gumbel copula dependence structure can be extended to higher dimensions. Investors ought to mitigate the impact of rising dependence and inflation among and between developed markets/Global markets by investing in portfolios which include developing countries' markets, e.g., containing the South African stock market assets, to reduce the risk of contagion.

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The portfolio Value at Risk (VaR), Expected Shortfall (ES) are estimated using the GPD-extreme value Gumbel copula model. The Monte-Carlo simulation of an equally weighted portfolio is used to estimate portfolio VaR/ES metrics which are used to calculate diversification effects/benefits. [11] showed that VaR is incoherent and fails to correctly estimate the risk measures when the return distributions are heavy-tailed. This strongly challenges the reliability of the conventional VaR risk metric based on the Normal distribution [12]. To address the weaknesses of the Normal distribution related VaR, [13] proposed to use a combination of the extreme value distributions and copula functions to address the drawbacks of the conventional VaR Normal distribution-based methodologies. According to [10} VaR is sub-additive when it is estimated using extreme value copula with heavy-tailed marginal distributions. [14] stated that copula models are appropriate for describing the dependence structure, especially when extreme values are present in the data set.

1.1 Why diversification matters?

Diversification is a strategy of spreading investments around so that exposure to any one type of asset/industry/country is minimised. This strategy is designed to minimise the volatility of the overall portfolio. Diversification does not guarantee a profit or guarantee against loss. Diversification aims to minimise the effect of volatility on the portfolio. The addition of South African assets to the global portfolio for any international investor will reduce risk in the global portfolio since developing countries' assets are less correlated with global assets. The benefits apply to local investment in local assets, as shown in this paper.

1.2 Justification and statement of the problem

The challenge is to estimate extreme portfolio risk and diversification effects/benefits resulting from the uncertainties in the dependence structure between two risky assets, specifically the South African Industrial and Financial Indices. Investors and practitioners use diversification to minimise the portfolio risk of investing in risky assets. Portfolio diversification has always been a challenge for investors and practitioners who have to deal with decision-making processes requiring them to estimate portfolio risk and diversification benefits/ effects. The GPD-extreme value Gumbel copula model is used to estimate the portfolio risk and to account for diversification effects using a portfolio of the two risky assets. This is preferable to the traditional, correlation-based approach, as the model can capture non-linear dependencies [8]. The GPD is used as the marginal distribution, instead of the Normal distribution, used to better characterise the extreme returns of the two risky assets. The extreme Gumbel copula model is used to construct the dependence structure (co-movement). A positive dependence can be very harmful as any extreme adverse outcome of one risk may cause simultaneous extreme adverse outcomes of the other. Diversification effects must be estimated accurately to mitigate such extreme risk.

Given the limited studies on the dependence structure (co-movement) of the South African sector Indices, extreme correlations/dependencies and diversification effects at a sector level, the bivariate portfolio consisting of the South Africa Industrial and Financial Indices is considered in this paper. This paper provides valuable information for investors and practitioners about portfolio risk and diversification effects/benefits in South Africa and around the two Indices in particular.

1.3 Objectives of the Study

The objective of the paper is to estimate the risk of an equally weighted portfolio consisting of the South African Industrial Index (J520) and the South African Financial Index (J580) whilst accounting for the diversification benefits/effects.

The specific objectives are:

- To fit the GPD marginals to the two Indices returns.
- Estimating univariate VaR and ES using the GPD model.
- To determine which bivariate copula is to be fitted to the bivariate distribution (in this case, it was found to be the extreme value Archimedean Gumbel copula, which was fitted to the GPD marginal distribution).
- Estimating the portfolio VaR and ES using the resultant joint distribution and interpreting the risk measures associated with the portfolio.
- estimating the diversification benefits/effects thereof.

The GPD-extreme value Gumbel copula model is important as it can be used as a framework to help investors and practitioners estimate the portfolio risk and diversification benefits/effects more accurately, especially in extremely risky scenarios. The study is organised as follows: Section 2: Literature review, Section 3: Methodology: Section 4: Results and Section 5: Conclusions.

2. Literature Review

According to studies in literature, the GPD-extreme value Gumbel copula model offers investors and practitioners a powerful tool to model the portfolio's extreme risk using a joint distribution. This is preferable to the traditional, correlation-based approach. [6] also states that the estimates of traditional univariate and bivariate Normal distribution-based models are less accurate in calculating risk for most cases with extreme observations.

[7] estimated portfolio risk using the EVT-copula approach and the GARCH model. The study investigated the dependence structure (co-movement) of a bivariate portfolio consisting of the TEPIX Index (Iran) and NASDAQ Index (USA). The GARCH and EVT models were used as marginal distributions to characterise the return distributions. The copula functions were used to estimate the bivariate distribution's dependence structure (co-movement), and portfolio VaR was estimated using Monte-Carlo simulation. The study revealed that copula models capture portfolio risk more accurately than traditional Normal distributions-based methods.

[5] used a GPD-copula model to estimate the VaR of a portfolio. They used a portfolio consisting of two financial assets: IBOVESPA and MERVAL Indices from Brazil. The financial returns were modelled using the ARMA-GARCH models, and a copula put together the joint distribution. The GPD marginals were used to characterise the distribution's left tail (large losses). The method was used to estimate the portfolio VaR of the two financial assets compared to other, more traditional approaches. The results revealed that the GPD-copula model with subsequent Monte Carlo simulation provides more accurate portfolio risk estimates than the other traditional simulation methods.

[15] used the Monte-Carlo simulation method to estimate and optimise the credit risk of a portfolio with 10 Italian obligors. The study used Student-t-copula to estimate the dependence structure/joint probability distribution with the GPD marginals characterising the distribution's tails. The GPD-copula model-based VaR was estimated. The findings were that the GPD-copula model provides better VaR forecasts than the traditional VaR forecasts.

[16] used copulas to estimate the dependence structure of bivariate financial data, viz: the daily prices of the IBOVESPA (Brazilian) and S&P500 (United States) Indices. The study revealed that diversification benefits under the Gaussian copula were minimal compared to the Student-t copula under heavier tails.

[17] used the GPD-Archimedean copula models approach to analyse equity portfolios. The study used eight data sets consisting of four stock indices, viz:-JSE/FTSE All Share Index, Bovespa Index, Indice de Precios Cotizaciones Index and Shanghai Composite Index from South Africa, Brazil, Mexico and China, respectively, and four developed market indices – the S&P500, FTSE100, DAX and CAC40 from the USA, Britain, Germany and France respectively The results showed that the lower tail was best modelled using the Clayton copula, and the Gumbel copula was the best for modelling the upper tail.

Many other studies have applied the bivariate Archimedean copula functions with different statistical distributions as marginals to estimate bivariate portfolio risk. These studies include [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. However, this study extends these studies by estimating extreme portfolio VaR and ES and in estimating diversification effects/benefits. This study differs from other studies in that it adopts the GPD-extreme value Gumbel copula model in estimating the VaR, ES and the diversification effects of the portfolio consisting of two South African financial assets, viz: - the South African Industrial Index (J520) and the South African Financial Index (J580). Studies on risk in developing countries like South Africa are very few, if any, especially around the two main sector Indices chosen herein. This study also differs from the other studies discussed as it uses an unconditional approach involving the direct application of GPD as the marginal distribution to model extreme risk in South African assets.

3. Research Methodology

This paper uses the GPD model to characterise the tails of distributions and the extreme value Gumbel copula for estimating the dependence structure (co-movement) of the two financial assets. Firstly, the GPD is used to estimate the univariate extreme risks of the J520 and J580 returns. Secondly, the extreme value Archimedean Gumbel copula is employed to estimate the dependence structure (co-movement) between the two assets and then estimate the portfolio risk using the Monte Carlo simulation of an equally weighted portfolio. Thirdly, the diversification effects formula is used to calculate the portfolio's diversification effects/benefits value using the estimated univariate VaR/ES and the portfolio VaR/ES. The models are discussed in this section.

3.1 Generalised Pareto Distributions (GPD)

The GPD is fitted to financial returns data for both upper and lower tails. Applying the GPD to characterise the extreme tail distributions will improve the accuracy of the portfolio risk measurement, especially at the extremities in losses/gains. The GPD model characterises the behaviour of returns that exceed certain high thresholds. It is a distribution for modelling extreme/excess losses/gains.

Excess Distribution

The GPD model is an application of two theorems: [30, 31] , to the distribution of interest:

$$F_u(x) = P(X - u \leq x | X > u) = \frac{F(x+u) - F(u)}{1 - F(u)} = 1 - \frac{1 - F(x+u)}{1 - F(u)}, \quad (1)$$

Which is a distribution of the values that exceed a threshold u.

Theorem: Pickands (1975), Balkema and de Haan (1974)

The two theorems by [30, 31] estimate the returns distribution exceeding a certain threshold u with two parameters. The two-parameter GPD is estimated with a scale parameter β and a shape parameter ξ.

The CDF for the GPD of returns which exceed a certain threshold is:

$$F(x) = \begin{cases} 1 - \left(1 + \frac{x}{\xi\beta}\right)^{-\xi}, \xi \neq 0 \\ 1 - \exp\left(-\frac{x}{\beta}\right), \xi = 0 \end{cases} \text{ where } x = y - u \quad (2)$$

Where, $x > 0$ when $\xi \geq 0, 0 \leq x \leq -\beta/\xi$ when $\xi < 0$ and $\beta > 0$

The differentiation of (1) gives the PDF of the distribution of interest:

$$f(x) = \begin{cases} \frac{1}{\beta}\left(1 + \frac{x}{\xi\beta}\right)^{-(\xi+1)}, \xi \neq 0 \\ \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right), \xi = 0 \end{cases}, \quad (3)$$

Where. $x = y - u$ is the exceedance, ξ represents the shape parameter, and $\sigma > 0$ represents the scale parameter of the GPD. $F(x)$is is called the GPD with a shape parameter ξ and a scale parameter β. The shape parameter ξ indicates the level of heavy-tailedness of the distribution. A large positive value for ξ indicates a very heavy tail, a negative value indicates a short tail (bounded), and a zero value indicates a light tail.

Peaks-over-Threshold (POT)

The Peaks-over-Threshold (POT) method is used to select the excesses/extremes above a certain high threshold, u.

The distribution function of the tail estimator is given as follows:

$$\widehat{F(x)} = 1 - \frac{N_u}{n}\left(1 + \frac{\hat{\xi}}{\hat{\beta}}(y - u)\right)^{-1/\hat{\xi}}, \quad (4)$$

Where, n – total number of return variables, N_u - number of excesses above a certain threshold.

3.2 Threshold Determination

The threshold in this study is determined by the pareto Q-Q plot using a linear regression equation as a tangent to the observations on the plot [32].

3.3 Parameter Estimation using MLE

$Y = (Y_1, Y_2, Y_3, \ldots, Y_n)$ are random variables that are independent and identically distributed (iid) random variables from a distribution F, and $Y_i > u$. The shape parameter and scale parameter are estimated from a certain threshold u.

$x = (Y_1 - u, Y_2 - u, Y_3 - u, \ldots, Y_{n-u})$ for $Y_i - u > 0$ only

The parameters are estimated using the MLE method:

For the GPD when $\xi \neq 0$, the density function is given as,

$$f(x_i) = \frac{1}{\beta}\left(1 + \frac{x_i}{\xi\beta}\right)^{-(\xi+1)}, \xi \neq 0, \text{ where } x_i = y_i - u \quad (5)$$

The likelihood function is given as follows:

$$L(\mathbf{x}; \xi, \beta) = \prod_{i=1}^{n} \frac{1}{\beta}\left(1 + \frac{x_i}{\xi\beta}\right)^{-(\xi+1)} = \frac{1}{\beta^n}\prod_{i=1}^{n}\left(1 + \frac{x_i}{\xi\beta}\right)^{-(\xi+1)} \quad (6)$$

The following log-likelihood function is derived:

$$\log\big(L(\mathbf{x};\xi,\beta)\big) = \log\left(\frac{1}{\beta^n}\prod_{i=1}^{n}\left(1+\frac{x_i}{\xi\beta}\right)^{-(\xi+1)}\right)$$

$$= -n\log\beta - (\xi+1)\log\left(\prod_{i=1}^{n}\left(1+\frac{x_i}{\xi\beta}\right)\right)$$

$$= -n\log\beta - (\xi+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\xi\beta}\right) \quad (7)$$

When $\xi \neq 0$, the log-likelihood function above is differentiated and it gives the following result for the scale and shape parameters:

$$\frac{\partial\log\big(L(\mathbf{x};\xi,\beta)\big)}{\partial\beta} = -\frac{n}{\beta} - \frac{(\xi+1)}{\beta}\sum_{i=1}^{n}\frac{x_i}{x_i+\xi\beta} = 0$$

$$\therefore 0 = n + (\xi+1)\sum_{i=1}^{n}\frac{x_i}{\mathrm{x}_i+\xi\beta} \quad (8)$$

$$\frac{\partial\log\big(L(\mathbf{x};\xi,\beta)\big)}{\partial\xi} = -\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\xi\beta}\right) + \frac{\xi+1}{\xi}\sum_{i=1}^{n}\frac{x_i}{\mathrm{x}_i+\xi\beta}$$

$$\therefore 0 = (\xi+1)\sum_{i=1}^{n}\frac{x_i}{\mathrm{x}_i+\xi\beta} - \xi\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\xi\beta}\right) \quad (9)$$

The solutions to the above equations are complex. Therefore, numerical methods or relevant computer software can be used to estimate $\hat{\sigma}$ and $\hat{\xi}$.

3.4 Copula functions

Copula functions combine marginal statistical distributions into a joint distribution function to form a copula [33]. Sklar's Theorem distinguishes between the dependent structure and marginal distributions.

Sklar's Theorem is stated below:

i) Let F be a joint distribution function with marginal functions $F_1, F_2, \ldots, F_d$ then there is a copula C such that for all $x_1,\ldots,x_d \in$ [-∞, ∞],

$$\mathrm{F}(x_1,\ldots,x_d) = \mathrm{C}(F_1(x_1),\ldots,F_d(x_d)). \quad (10)$$

ii) If F_1 , F_2,..., F_d are continuous, the C is unique. Conversely, if Cis a copula and F_1, F_2,..., Fd is distribution functions, F, as defined above, is a joint distribution function with marginal distributions $F_1, F_2, \ldots, F_d$.

Let X and Y be random variables of $\mathrm{F_x} = \mathrm{u_1}$ and $\mathrm{F_y} = \mathrm{u_2}$. [34] showed the joint distribution function $F(x, y)$ can be written as a function of a copula $\mathrm{C(u_1, u_2)}$:

$$\mathrm{F(x, y) = C(u_1, u_2)}. \quad (11)$$

Where $\mathrm{F(x, y)}$ is a coupling of $\mathrm{F_x(x)}$ and $\mathrm{F_y(y)}$.

3.4.1 Modelling tail dependence

Extreme tail dependence deals with multivariate extreme events which occur in the upper or lower tails. Alternatively, tail dependence of, say, a bivariate return series is a measure of the co-movements of the variables in the tails of the distribution. This study uses the extreme value Gumbel copula to quantify tail dependence in a bivariate portfolio. The two stock market Indices may exhibit no correlation but show tail dependence in the extremities.

3.4.2 The Archimedean copula

In many financial applications, there is strong lower tail dependence for extreme losses when compared to extreme gains [35]. Since they have closed-form expressions, such asymmetries can be easily modelled using Archimedean copulas. They contain a parameter that regulates the tail dependence between two risks. The Archimedean copulas' unique property is their capacity to model heavy tail dependence. They enable the modelling of dependence structures where upper or lower tail dependence, or tail dependence that only exists on one side of the distribution is present.

The Archimedean copula has many forms for representing different structures of dependence. The three most common forms of Archimedean copula functions are the Frank and Clayton and Gumbel (or extreme value Gumbel copula) copulas [36]. The Frank copula is suitable when there no upper nor lower dependence The Clayton copula captures lower tail dependence. In this paper, the Archimedean Gumbel copula (also known as the extreme value Gumbel copula) is deemed appropriate as the suspicion and interest are in upper tail dependence in the returns of the two Indices, which might have a disastrous consequence when losses are under consideration.

3.4.3 Archimedean Frank Copula

The asymmetric Frank copula given by:

$$C(u_1 , u_2) = -\frac{1}{\theta}\ln \left(1 + \frac{(e^{-\theta u_1}-1)((e^{-\theta u_2}-1)}{e^{\theta}-1}\right), \tag{12}$$

Where, u_1 and u_2 are the cumulative marginal distributions of the two Indices.

3.4.4 Archimedean Clayton Copula

The asymmetric Clayton copula displays dependence in the lower tail and no dependence in the upper tail.

The formula for this copula is shown below:

$$C(u_1 , u_2 ; \theta) = (u_1{}^{-\theta} + u_2{}^{-\theta} - 1)^{\frac{-1}{\theta}}) \tag{13}$$

Where, θ is the dependence parameter, u_1 and u_2 are marginal cumulative distributions of the two Indices and are uniformly distributed.

3.4.5 extreme value Gumble copula (Archimedean Gumbel copula)

The extreme value Gumbel copula is an asymmetric and exhibits upper tail dependence in the one corner. The bivariate Archimedean Gumbel copula is applied in this study and is also known as the Gumbel-Hougard copula. [37] defined the extreme value Gumbel copula as:

$$C^{Gu} (u_1 , u_2 , \theta) = \exp\left(-[(-\log u_1)^{\theta} + (-\log u_2)^{\theta}]^{\frac{1}{\theta}}\right) , \tag{14}$$

Where, $\theta \in [1, \infty)$, u_1 and u_2 are marginal distributions of the two Indices

The parameter θ estimates the degree of dependency. When $\theta = 1$, independence is obtained and when $\theta \to \infty$, the Gumbel copula converges to perfect dependence (minimum copula).

According to [38, 35, 39], the extreme value Gumbel copula is the most widely used Archimedean copula in engineering, economics, finance, applied statistics and mathematics. It is suitable in situations exhibiting greater dependence in the upper tail than in the lower tail. The extreme value Gumbel copula is tractable, which facilitates simulations and maximum likelihood estimation.

3.4.6 Kendall's tau and the tail dependence

According to [36], the Archimedean copula functions are attractive because the copula parameter (θ) is related to the tail dependence coefficient and Kendall's tau. Kendall's tau highlights the strength of the association of the two variables being compared [40].

Table 1: The Archimedean copula functions parameter (θ) and the relationship with tail dependence coefficients: Kendall's tau (τ) and tail dependence.

Archimedean Copula function	**Range of θ**	**Kendall's tau** $\boldsymbol{\tau}$	**Upper tail dependence measure** $\boldsymbol{\lambda_U}$	**Lower tail dependence measure** $\boldsymbol{\lambda_L}$
Gumbel	[1, ∞]	$\frac{\theta-1}{\theta}$	λ_U= 2-$2^{-\theta}$	λ_L= 0
Clayton	[-1, ∞]	$\frac{\theta}{\theta+2}$	λ_U= 0	λ_L= $2^{\frac{-1}{\theta}}$
Frank	[-∞, ∞]	$\frac{1+4(D_1(\theta)-1)}{\theta}$	0	0

In Table 1, the Archimedean copula functions are characterised by one dependence parameter (θ) that needs to be estimated. This study uses the inference for margins (IFM) approach to evaluate the parameter.

3.5 Model selection: AIC and BIC for Archimedean Copula functions and scatterplot method

The AIC and BIC are used to determine the Archimedean copula function that provides the best fit. The criterion with the lowest values AIC or BIC indicates that the copula function which provides the best fit to the data set.

3.5.1 Akaike information criterion (AIC) and Bayesian information criterion (BIC) goodness of fit test methods

The AIC and BIC values are estimated using equation 15 and 16:

AIC = *2∗loglikelihood + 2k [41]* (15)

BIC = *2∗loglikelihood + ln(n)∗k [42]* (16)

Where, k is the number of parameters and n is the number of observations.

3.5.2 Scatterplot Method

The scatterplots of the bivariate returns are used to determine the type of copula to fit the bivariate data of the two Indices.

3.6.0 Risk Measures

The risk measures adopted are associated with gains and losses for the GPD-extreme value Gumbel copula model. The portfolio VaR and ES estimation is estimated using the Monte Carlo simulation method of an equally weighted portfolio. The GPD marginals are used to characterise extreme risk in the tails of the distributions. The GPD-extreme value Gumbel copula model estimates and improves the accuracy of the univariate VaR/ES and portfolio VaR/ES forecasts, especially at the extremities. The diversification effects are then calculated.

A good risk measure has certain attributes, including the coherence property. According to [43], X and Y are, say, two securities, and a risk measure ρ(·) is said to be coherent if the four given axioms are satisfied:

Axiom 1: Monotonicity

$\rho(Y) \geq \rho(X)$ if $X \leq Y$

A higher predicted loss necessitates holding more capital.

Axiom 2: Sub-additivity

$\rho(X+Y) \leq \rho(X) + \rho(Y)$

Adding two or more risky assets will not increase the risk level. The portfolio risk should be lower than or equal to the sum of the risk of the individual financial assets. This Axiom 2 is the mathematical description of diversification applied in this study.

Axiom 3: Homogeneity

For any number $k > 0, \rho\,(kX) = k\rho(X)$ where k is a constant positive amount.

If we say k=2, then doubling the size of the loss situation will double the risk

Axiom 4: Translation Invariance

$\rho\,(X + rk) \leq \rho(X) - k$ where k is a constant

If we add a certain amount to the observed loss, the capital required to mitigate the loss grows by the same amount.

The VaR and ES are the main metrics of risk measures often cited when quantifying risk. According to [5] , VaR is the potential loss/gain (at a certain confidence level) from extreme market movements over a certain period. According to [41], ES is a risk measure which estimates the amount of loss or gain which exceeds VaR. VaR is criticised for lacking the sub-additivity property, implying it is not coherent. [44] states that the VaR metric is sub-additive in most practical situations, as confirmed by results of this study. This property of sub-additivity (Axiom 2) represents a portfolio's diversification effects/benefits [45] and is an important argument applied in this study. The ES is a coherent risk measure and hence does not have the problem associated with VaR. However, it is more complex to estimate. The portfolio VaR and ES are simulated using the Monte Carlo simulation of an equally weighted portfolio of the two financial assets.

3.6.1 Portfolio simulation

In the estimation procedure, [5, 6] follow a five-step process when fitting the GPD-extreme value Gumbel copula model.

Step 1: Log returns are calculated, and the GPD distribution is fitted to the two data sets, viz: the two Indices and losses and gains are treated separately as well. Estimate the univariate VaR and ES from the parameters.

Step 2: The log-return series of the two Indices are transformed into standard uniform (0, 1) variates and assumed to be i.i.d observations. A plot of the uniform marginals will help determine the appropriate Archimedean copula for the pair of transformed data series.

Step 3: Determining which bivariate copula (in this case, it was found to be the extreme value Gumbel copula) is to be fitted to the bivariate return series for each of the gains and losses separately and estimating the parameter τ, Kendall's tau in each case. θ is estimated using the Inference Function for Margins (IFM) estimation method.

Step 4: Use the estimated extreme value Gumbel copula model parameter to facilitate the simulation of N (N = 5 000 in this study) uniform random numbers from the joint distribution and transform the simulations to the original scales of the log returns using the inverse quantile function of the joint distribution. Use the average of the input parameters as the new GPD parameters.

Step 5: Estimate portfolio VaR/ES using the Monte Carlo simulation method of an equally weighted portfolio. In this study, the portfolio weights of the two Indices were arbitrarily considered equal and can be varied freely [15]. Once portfolio VaR and the ES are estimated, diversification effects are calculated.

3.6.2 Diversification effects

Diversification is more effective when assets are uncorrelated or negatively correlated with one another, allowing certain sections of the portfolio to fall while others rise [46]. Investors and practitioners find it very important to estimate portfolio risk because the risk reduction benefits of portfolio diversification rely on this statistic. The GPD-extreme value Gumbel copula model is applied to calculate the diversification effects/benefits and to capture the portfolio VaR more accurately. According to [47, 48, 49], the difference between the portfolio risk (aggregate or diversified value) and the simple sum of individual component risks (undiversified values) expressed as a percentage of the simple sum of individual component risks (undiversified value) is a measure of the diversification effects.

The diversification effects formula is given as follows:

$$\text{Diversification effects for VaR} = \frac{\textit{Simple Sum VaR} - \textit{Aggregate VaR}}{\textit{Simple Sum VaR}} \times 100\% \tag{17}$$

$$\text{Diversification effects for ES} = \frac{\textit{Simple Sum ES} - \textit{Aggregate ES}}{\textit{Simple Sum ES}} \times 100\% \tag{18}$$

Where,

Simple sum VaR –is the total sum from the addition of the VaR values of the two risk factors used in this study and is greater than the portfolio VaR (i.e., $Simple\ Sum\ VaR = VaR_1 + VaR_2 > VaR_{portifolio}$)

Simple sum ES –is the total sum from the addition of the ES values of the two risk factors used in this study and is greater than the portfolio ES (i.e., $Simple\ Sum\ ES = ES_1 + ES_2 > ES_{portifolio}$)

Aggregate VaR – the portfolio VaR of the two risky factors= $VaR_{portifolio}$

Aggregate ES – the portfolio ES of the two risky factors= $ES_{portifolio}$

The diversification effects formula illustrates the advantages of using a copula when analysing extreme risk in a portfolio compared to a univariate treatment of risk (silo) approach. In this study, formulas 17 and 18 will be used to estimate the diversification effects of a portfolio.

3.7 Test for stationarity, heteroscedasticity and autocorrelation

Table 2: Summary of Tests for stationarity, heteroscedasticity and autocorrelation.

Test	Method
Stationarity	The ADF test (also known as unit root or non-stationary test) is used to test for stationarity [50] in the J520 and J580 return series.
Heteroscedasticity	To test for the presence of Arch effects, the Lagrange Multiplier (LM) Test is used to test for the presence of heteroscedasticity [51] in residuals of the J520 and J580 return series.
Auto-correlation	The Ljung-Box test is used to test for autocorrelation [52] of the J520 and J580 returns series.

These three tests in Table 2 are used for stationarity, heteroscedasticity and autocorrelation, respectively of the two financial asset returns series.

4. Results and discussion

This section applies the GDP-extreme value Gumbel copula model to two financial assets traded on the Johannesburg Stock Exchange (JSE): the J520 and J580 returns are used in analysing the bivariate portfolio and the diversification effects thereof.

4.1 Software Used and Research Data

This study analysed the data using R statistical packages: actuar, Copula, fCopulae, QRM, Mass, evir, eva, fitdistrplus, fExtremes and extRemes. This study uses the South African stock market secondary data extracted from the website iress expert: https://expert.inetbfa.com (with permission). The analysis involved the J520 and J580 return distributions (spanning the years: 1995–2018). These Indices are calculated from the values of stocks of Industrial sector companies and financial services sector companies listed on the JSE and represent the performance of specific industries in the stock market. The Industrial and Financial sectors are currently the most represented sectors on the JSE by market capitalisation [53].

[54] state that, although the South Africa's All Share Index (ALSI) is weakly efficient, its sub-Indices may not be informationally efficient. This may allow investors to make excess profits/losses when invested in the sub-Indices. This implies that the sub-Indices returns, viz: the J520 and J580 returns, may be modelled using extreme value distributions such as the GPD. The bivariate portfolio is constructed from two stock Indices returns. The resultant portfolio is used to estimate the VaR and ES using the GPD-extreme value Gumbel copula model, and diversification benefits are then calculated.

The monthly logarithmic returns are estimated as follows:

$$r_t = ln\ M_t/M_{t-1} \tag{19}$$

Where, r_t are the log returns of the month t, M_{t-1} are the returns in month $t-1$, and ln is the natural logarithm.

In this study, the results of the return series' upper tails for gains and for losses were analysed separately. To model the upper tail gains, the data is used as is (x_t), and for the losses, the sign of the Index returns data were changed so that x_t = - r_t for $r_t < 0$. x_t is the loss function in this instance.

4.2 Exploratory Data Analysis

This section presents the descriptive data analysis and initial intuition on distributed datasets.

4.2.1 Descriptive Statistics

Table 3 below gives the descriptive statistics of the two data sets that inform the portfolio.

Table 3: Summary of statistics.

	Industrial Index (J520) returns	**Financial Index(J580) returns**
No of Observations	271	271
Minimum	-0.140273	-0.216516
Maximum	0.328471	0.511949
Mean	-0.009366	-0.008353
Median	-0.010478	-0.010155
Variance	0.003302	0.003651
Skewness	1.016932	2.194276
Kurtosis	4.420852	19.439520

The descriptive statistics for the J520 and J580 returns are given in Table 3. The maxima and minima values for the two indices returns are quite far apart from the mean, confirming presence of extreme returns. The mean returns for the two Indices are small, which indicates that there are no significant trends in the data sets. The skewness is positive for both indices return, which implies that the extreme values are present in the returns series. The results exhibit excess kurtosis, implying the returns distributions are heavy-tailed. The various attributes of returns distributions, such as skewness and high kurtosis, are present, which allows one to conclude that both Indices returns are heavy-tailed. The two financial asset returns are heavy-tailed; therefore, the GPD will be used as marginal distribution in this paper. In literature it is confirmed that stock market Indices do not always follow the normal distribution as they tend to be skewed, peaked and have extreme values.

4.2.2 Tests results for Stationarity, Heteroscedasticity and Auto-correlation

Table 4: Test results for stationarity, heteroscedasticity and autocorrelation.

Test	**Test Results**
Stationarity	The Augmented Dickey-Fuller (ADF) tests for stationarity in the returns. A p-value of less than 0.01 for both Indices returns leads to the rejection of Ho (presence of a unit root); hence, the returns series data are stationary.
Heteroscedasticity	The Arch (LM) test is applied to test for heteroscedasticity in the J520 and J580 returns series. The Arch (LM) test checks for the presence of ARCH effects. There were no ARCH effects in the J520) returns: χ^2 = 8.3670, df = 12, p-value = 0.7558) and J580 returns: χ^2 = 6.2386, df = 12, p-value = 0.9036). The p-values are greater than 0.05 for both returns series data, which led to the acceptance of Ho (there are no Arch effects).
Auto-correlation	The Ljung-Box test is used for auto-correlation in the J520 and J580 returns series. The results reveal a p-value > 0.05 for each return series data, indicating that the null hypothesis of no auto-correlation is accepted. This means that the return distribution is independently distributed. The Indices returns series are each independent and identically distributed. Therefore, applying the GPD to the return series is appropriate, as each is independently and identically distributed. The two series, however, may depend on each other, especially at the extremities. This is discussed in the next section.

The stationarity, heteroscedasticity and autocorrelation results are discussed in Table 4, and it is concluded that the datasets are stationary. It has no Arch effects and no autocorrelation. Studies in literature confirms that conditional volatility is absent from monthly return distributions, but the heavy-tails remain present Therefore, the monthly financial returns of J520 and J580 Indices are ideal for fitting the GPD-extreme value Gumbel copula using the unconditional method.

4.3 Analysing the gains and losses

The threshold value u needs to be estimated to fit a GPD to both the returns distributions' upper tail (gains) and lower tail (losses). The GPD is then fitted to the two log returns series. Since the interest is in both the upper and lower tails, the gains and losses are treated separately and modelled using the GPD model. The thresholds were estimated using the pareto Q-Q plot.

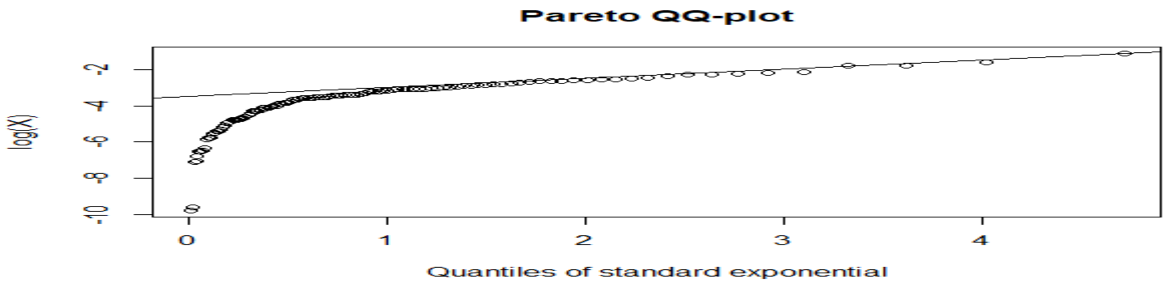

Fig.1: The pareto Q-Q plot estimates the threshold for the J520 losses.

In Figure 1, the tangent line crosses the y-axis at -3.5, the exponent is 0.03. The threshold is therefore 0.03 for the negative returns (losses).

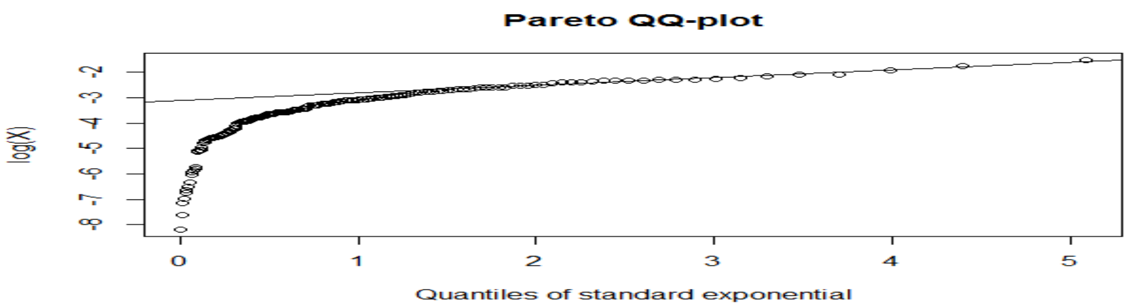

Fig.2: The pareto Q-Q plot estimates the threshold for the J580 losses.

In Figure 2, the tangent line crosses the y-axis at -3.1, which implies that the threshold is 0.045 (the exponent of -3.1 is 0.045). Figure 1 and Figure 2 show how the thresholds for losses are estimated for the two indices. The data points obtained above the threshold are fitted to the GPD, and the parameters are estimated. These parameters (see Table 7) are used to estimate the univariate VaR and ES (see Table 8). The two other diagrams for the gains were done but not presented in this study. The thresholds for the losses are 0.03 for J520 Index and 0.045 for J580 Index.

4.4 Threshold stability plots

Threshold stability plots were used to compare the parameters of the model. The parameters were found to be in the same range. This confirms the accuracy of models used in this thesis. Below is the threshold stability plot of the J580 Index losses using the GPD.

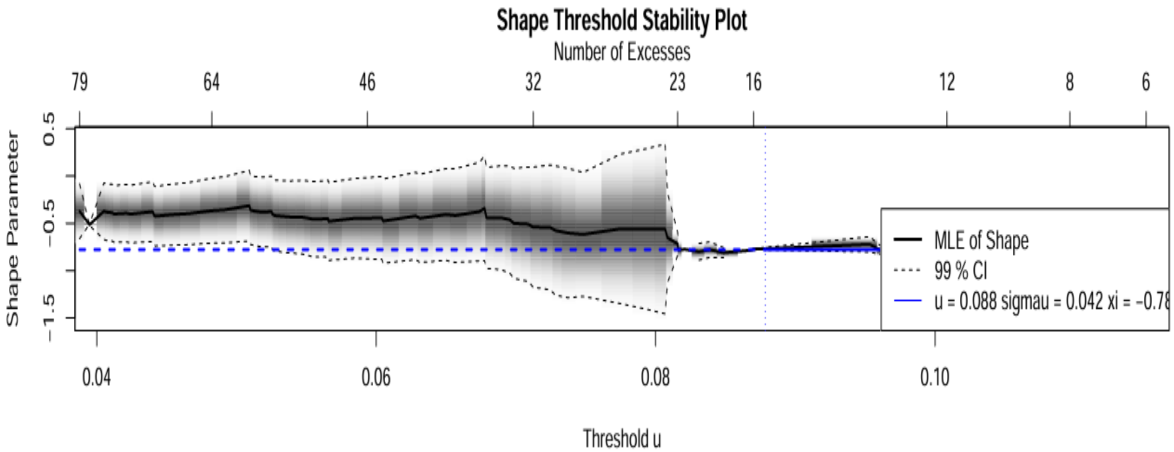

Fig.3: The GPD shape threshold stability plot for J580 losses

The shape threshold stability plot in Figure 3 shows that the shape parameter estimate is stable around 0.04 and above. The pareto Q-Q plot in Figure 2 has linear pattern starting from $log(x)$ = -3.1 and above. The corresponding threshold is 0.045. Therefore, the threshold selected for J580 Index is 0.045. The other plots for J580 gains and for J520 losses and gains were done but not presented.

4.5 Selection of appropriate bivariate Archimedean copula

The appropriate copula can be arrived at by visualisng scatter plots of the marginal distributions as shown in Figure 1 and Figure 2. The appropriate copula will be further confirmed using the AIC and BIC.

4.5.1 Selection of appropriate Archimedean copula using scatterplots

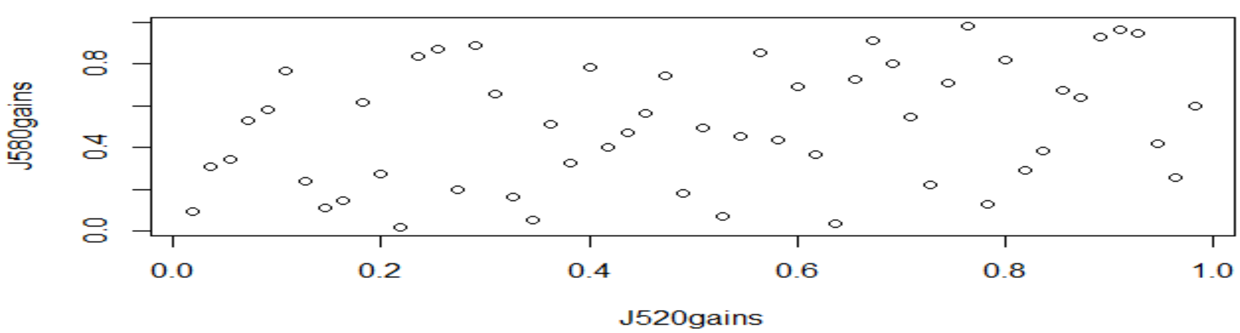

Fig. 4: The scatterplot of the marginal distributions for bivariate gains transformed into uniform variables u_1 and u_2, suggesting a Gumble copula.

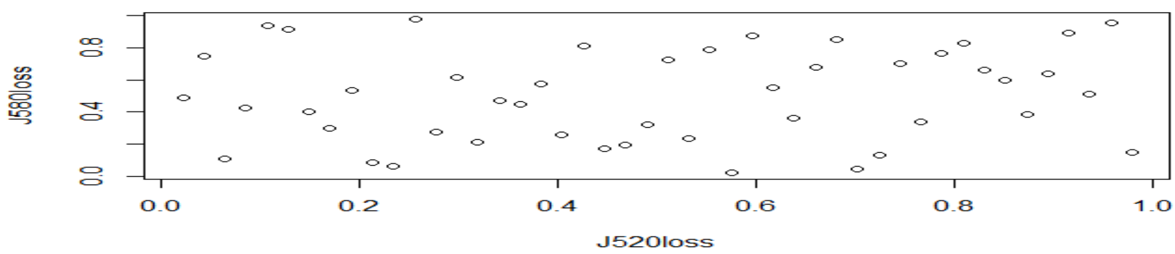

Fig. 5: The scatterplot of the marginal distributions for the bivariate losses transformed into uniform variables u_1 and u_2, suggesting, again somewhat, an extreme value Gumbel copula.

The bivariate series in Figure 4 and Figure 5 converge towards an upper tail. When losses are considered, a positive dependence can be extremely risky since it increases the likelihood that both risks will simultaneously experience extreme outcomes. Both Indices may be driven to extremes by a single, underlying economic factor. In risk modelling, the dependence of the random pair of maxima is crucial. The pattern suggests upper tail dependence; hence an extreme value Gumbel copula is used to model the data. The AIC, BIC and the scatter plots indicate that the extreme value Gumbel copula is the most appropriate model for the bivariate distribution. Therefore, the extreme value Gumbel copula method is applied in this study.

4.5.2 Selection of appropriate Archimedean copula using AIC and BIC goodness of fit tests

The AIC and BIC goodness of fit tests and scatterplots of the bivariate distributions of the losses and gains are used to determine an appropriate Archimedean copula function that best describes the dependence in the bivariate return series data.

Table 5: Estimation of AIC and BIC goodness of fit test statistics

	AIC	**BIC**	**AIC**	**BIC**
Clayton	-52.38	-49.87	-38.51	-35.99
Gumble	-92.63	-90.12	-48.12	-45.60
Frank	-80.95	-78.44	-47.13	-44.61

Table 5 shows that the Gumbel copula is the one that provides the best fit to the data since it has the lowest values for both criterions: the AIC and the BIC for both the losses and gains.

4.6 Estimation of Kendall's tau and tail dependence

This study estimates the extreme value Gumbel copula parameter and the dependence measures using Kendall's tau.

Table 6: The Kendall's tau, copula parameter, lower and upper tail dependence measures.

Family	**Kendal's tau ($\hat{\tau}$)**	**Copula Parameter ($\hat{\theta}$)**	**Upper tail $\hat{\lambda}_U$=(2-$2^{-\hat{\theta}}$)**	**Lower tail**
Gumble for the gains	0.0879	1.2553	0.2629	0
Gumble for the losses	0.2034	1.0964	0.1182	0

In Table 6, the estimated extreme value Gumbel copula parameters ($\hat{\theta}$) for the upper tails are 1.2553 and 1.0964, for the losses and gains respectively. The parameters imply the presence of tail dependence. This means that the two stock indices depend on the extremes for both the gains and losses. This means that large losses from the two stock indices have a greater probability to co-move together concurrently, and the same applies to the gains [35].

The upper tail dependency measure for the gains $\hat{\lambda}_U$ is 0.2629, and for the losses, $\hat{\lambda}_U$ is 0.1182. This suggests that the dependence structure among the two stock indices is asymmetric. The results show a significant upper-tail dependence which has important implications for financial risk management. Tail dependence measures indicate the degree of extreme co-movements of large gains/losses in the stock market. This gives an opportunity and allows investors and practitioners to quantify portfolio risk.

4.7 extreme value Gumbel copula density and contour plots

Figure 5 and Figure 6 show the extreme value of Gumble copula density and contour plots for the gains and losses.

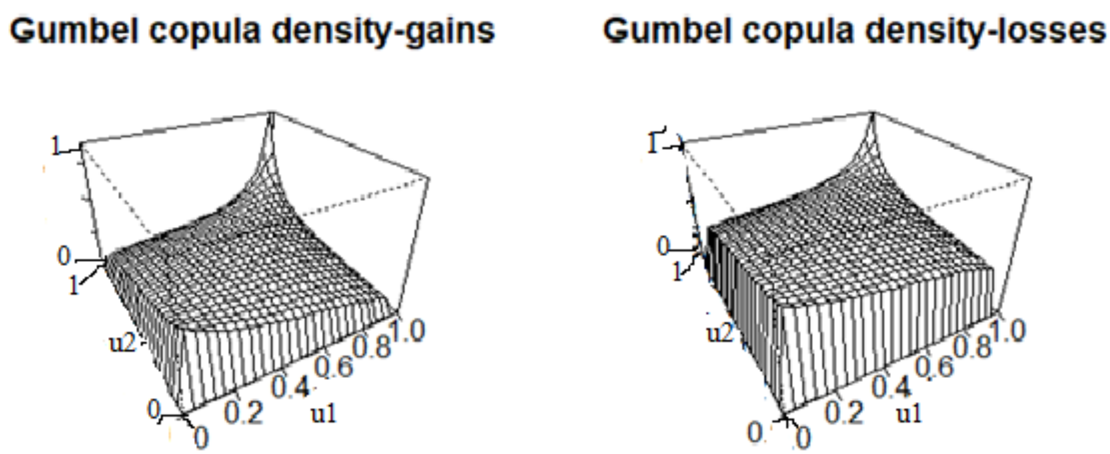

Fig. 6: The density plots of the bivariate distribution for the gains (θ=1.255263) and losses (θ=1.096398).

In Figure 6, the density plots confirm the presence of the upper tail dependence in the extreme value Gumble copula in the gains and losses, respectively, for the given parameters.

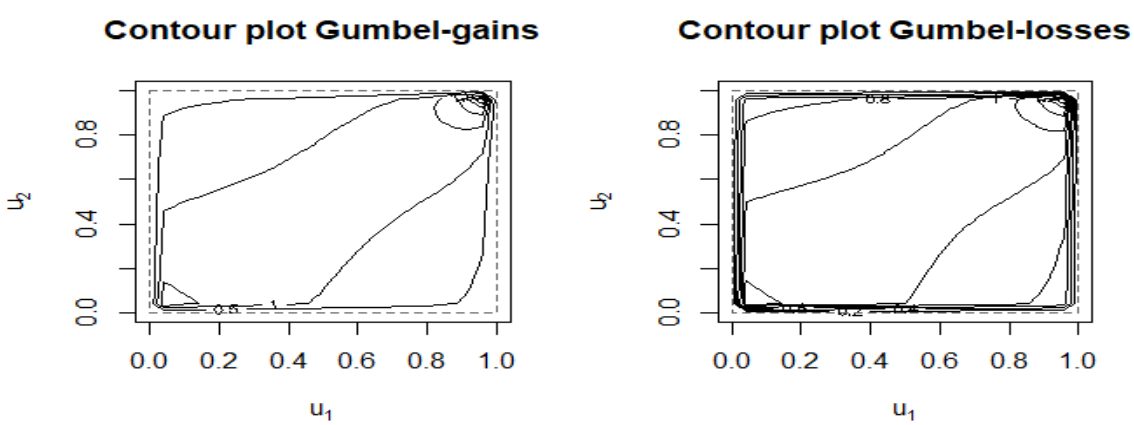

Fig. 7: The contour plots of the bivariate distribution for the gains (θ=1.255263) and the losses (θ=1.096398).

In Figure 7, the contour plots confirm the presence of the upper tail dependence in the extreme value Gumble copula in the gains and losses for the given parameters. The extreme value Gumbel copula is characterised by the upper tail dependence, as can be concluded from the density and contour plots.

4.8 Modelling the losses and gains using the GPD-extreme value Gumbel copula model

The gains and losses of the two distributions are fitted to the GPD before applying the extreme value Gumbel copula to estimate the joint distribution.

4.8.1 Estimation of univariate parameters

Table 7: Estimates of univariate GPD parameters.

	South African Industrial Index (J520)		South African Financial Index (J580)	
	Losses	Gains	Losses	Gains
	u=0.04	u = 0.03	u=0.07	u = 0.03
Shape Parameter, ξ	-0.0659	0.2825	-0.4821	0.1829
Scale Parameter, β	0.0361	0.0269	0.0389	0.0326

The parameters obtained for the gains/upper tail and losses/lower tail of the two financial asset return distributions are estimated and given in Table 7. The shape parameters for the losses are negative for the two Indices, implying that losses are short-tailed (bounded). For the gains, the shape parameters are positive, indicating that they are heavy-tailed. The risk measures estimated from positive shape parameters are riskier than those estimated from a negative shape parameter. The shape parameters suggest that extreme gains are more likely than extreme losses for one invested in the two South African Indices.

4.8.2 Estimation of univariate VaR and ES metrics

The univariate VaR and ES metrics for the two financial assets are estimated using the GPD model.

Table 8: Univariate risk measures for J520 and J580.

	South African Industrial Index (J520)		South African Financial Index (J580)	
Probability	VaR	ES	VaR	ES
Measures of Risk for gains				
0.950	0.1602	0.2294	0.1566	0.2439
0.990	0.2659	0.3587	0.2842	0.4217
0.995	0.3220	0.4273	0.3597	0.5269
Measures of Risk for losses				
0.950	0.1317	0.1379	0.1431	0.1710
0.990	0.1420	0.1448	0.1884	0.2134
0.995	0.1445	0.1465	0.2065	0.2304

Table 8 shows the univariate VaR ad ES of the gains and losses for the J520 and the J580 returns. For the J580 gains, the VaR and ES values are greater than those for J520 values at 99 % and 99.5% confidence levels, implying that the J580 gains are riskier. For the losses, the J580 VaR and ES values are greater at 95%, 99 % and 99.5% confidence levels, implying that the J580 losses are riskier. These risk measures are used to estimate the diversification benefits/effects of the portfolio. Generally, its riskier to invest in the J580 when compared to investing in the J520.

4.9 Estimation of portfolio risk using the GPD-extreme value Gumbel copula model.

The portfolio risk is estimated using the GPD-extreme value Gumbel copula model for the gains and losses. Firstly, the average GPD parameters for the losses and gains are calculated for the copula inverse distribution. Secondly the portfolio risk is then estimated.

Table 9: Average GPD parameters obtained from univariate J520 and J580 parameters.

Parameter	Shape	Scale	Threshold
Left Tail/Gains Average	0.2327	0.0298	0.030
Right Tail/Losses Average	-0.2740	0.0375	0.055

Table 9 summarises the average parameters for the copula inverse distribution for the gains and losses joint returns. The shape parameter for the losses is negative, implying that they are short-tailed (bounded). In the case of the gains, the shape parameter is positive, indicating that they are heavy-tailed. The heavy-tailed gains are riskier than the short-tailed losses, as shown in Table 9 below. However, the higher riskiness of the gains is favourable to the investors as it indicates a high probability of making extreme gains. The average of the GPD parameters are used as the new input parameters for the inverse quantile distribution utilised in the Monte-Carlo Simulation of an equally weighted portfolio to estimate risk.

Table 10: Estimation of portfolio risk.

Copula	**Marginals**	**portfolio VaR**			**portfolio ES**		
		95 %	99%	99.5 %	95 %	99%	99.5 %
Gains							
Gumbel	GPD	0.2217	0.4085	0.5357	0.3602	0.6569	0.8447
Losses							
Gumbel	GPD	0.1248	0.1646	0.1778	0.1499	0.1879	0.2048

Table 10 summarises the estimated VaR and ES for the portfolio. In the case of the gains, at a 95 % confidence level, the GPD-extreme value Gumble copula gives VaR and ES estimates of 0.2217 and 0.3602, respectively. VaR's results are interpreted as follows: the expected market portfolio gains will not go above 22.17 % (0.2217) at the 95% confidence level. If it goes beyond, it will average an ES value of 36.02 % (0.3602). The interpretation is the same for all the other estimates. The results also reveal that the portfolio gains are greater than the portfolio losses, which is favourable to the risk averse investor, as the probability of making an extreme profit is higher than that of making an extreme loss. Estimating the portfolio risk of financial assets is important because it has been widely accepted that investing in a portfolio of several assets has diversification benefits over investing in a single asset [55]. The estimated portfolio risk in a bivariate setting can be used to calculate and account for the diversification effects. For those wishing to invest in the South African Industrial and Financial markets, portfolio risk and diversification effects can provide quantitative indicators for use in their investment selection decisions to mitigate against exposure to a single risk.

4.10 Estimation of diversification effects

In this section, diversification effects are estimated.

Table 11: Estimation of diversification effects of the GPD- extreme value Gumbel copula model for the gains.

Alpha	**J520 tail risks**	**J580 tail risks**	**Simple Sum for tail risks**	**Portfolio risks**	**Diversification effects**
VaR for GPD-extreme value Gumbel copula-gains					
95	0.1602	0.1566	0.3169	0.1281	59.58 %
99	0.2659	0.2842	0.5501	0.2424	55.93 %
99.5	0.3220	0.3598	0.6818	0.31.23	54.19 %
ES for GPD-extreme value Gumbel copula-gains					
95	0.2294	0.2439	0.4733	0.2097	56.11%
99	0.3587	0.4217	0.7804	0.3755	51.85%
99.5	0.4273	0.5269	0.9542	0.4782	49.88%

In Table 11, the portfolio VaR and ES for the gains are less than the sum of simple component values of the two Indices. The estimated diversification effects imply a trade-off for less gains for the portfolio whilst there is greater protection against large losses. These results imply that there is a reduction in the portfolio risk. This also confirms that VaR, in this case, is sub-additive when modelling using extreme value Gumbel copula with heavy-tailed GPD model [10].

Table 12: Estimation of diversification effects of the GPD-extreme value Gumbel copula model for the losses.

Alpha	J520 tail risks	J580 tail risks	Simple Sum for tail risks	Portfolio Risks	Diversification effects
VaR for GPD- extreme value Gumbel copula-losses					
95	0.1317	0.1431	0.2749	0.0769	71.50%
99	0.1420	0.1884	0.3304	0.0991	70.00 %
99.5	0.1445	0.2065	0.3510	0.1054	69.67 %
ES for GPD- extreme value Gumbel copula-losses					
95	0.1379	0.1710	0.3089	0.0898	70.93%
99	0.1448	0.2134	0.3582	0.1072	70.07%
.99.5	0.1465	0.2304	0.3769	0.1122	70.23 %

In Table 12, the portfolio VaR and ES for the losses are less than the sum of simple component values of the two Indices and the diversified portfolio will come with lower returns/or gains. The diversification effects for the losses in Table 12, for both VaR and ES, are greater than for the gains in Table 11, implying a greater trade-off for the losses. This will enable the investor to avoid large losses. These results imply a reduction in the portfolio risk of the losses, and therefore diversification effects/benefits are harvested. This also implies that investors who are invested in the individual financial risky assets with a VaR and ES values greater than the portfolio may consider adding the risky assets used in this portfolio

to reduce exposure to their risk. This is useful information for international investors seeking to include developing countries' market Indices, such as the South African assets, into their portfolios, since they are less correlated with other developed markets/Global stock markets, thereby reducing the risk of contagion. VaR, ES and diversification effects are important in financial risk management and diversification decisions to control exposure to risk [56].

4.11 Discussion

The estimated diversification effects of the portfolio indicate that there is a trade-off between the gains and losses. This diversified portfolio comes with lower returns or gains because of risk mitigation. This will enable the risk averse investor to avoid making huge losses compared to investing in a single asset. International investors who wish to diversify risk may include or add these financial assets to this portfolio. This study has confirmed the potential of diversification opportunities and benefits for investors holding a portfolio consisting of the J520 and J580 Indices. The results are consistent with studies by [5, 6, 7]. who also estimated portfolio VaR and ES using the GPD-Archimedean Gumbel copula methodology in their studies. [47, 48, 47] went on to further estimate diversification effects, and their results are consistent with this study using South African data. The study will provide a framework that will assist investors and practitioners in estimating diversification effects/benefits for use in their investment selection decisions to reduce risk exposure. This EVT-copula model can be applied in any financial/stock market, as confirmed in the literature by [5, 6, 7]

5. Conclusions

This paper estimated financial risk (univariate VaR/ES, portfolio VaR /ES and diversification effects) of a portfolio consisting of two financial assets, viz: J520 and the J580 Indices using the GPD- extreme value Gumbel copula model. The results show that there is a reduction in losses for investors holding the portfolio. The average mean return of the portfolio remains the same and stable when compared to individual returns as the risk is mitigated. The model can be used as a case for reducing exposure to risk through diversification of risk for the same expected returns. This will be appealing to the risk averse investors wishing to avoid making large losses when invested in a single security.
The results are important to international investors as they can mitigate the impact of rising dependence and inflation between Global stock markets by investing in portfolios which include developing countries' markets, e.g., containing the South African stock market assets. This will help investors improve their investment diversification, as developing countries markets are less correlated with Global stock markets. The results will provide some useful and important direction for investors considering cross-diversification into the South African Industrial and Financial markets.

5.2 Areas of future possible research

This study focused on a one parameter copula function which used a bivariate distribution of two financial assets. Future research should include more assets and complex copulas to cater to the complex dependencies. For future research also the authors would consider estimating diversification effects/benefits using mixture models e.g., GPD-Normal-GPD model and comparing the results with the GPD- extreme value Gumbel copula model.

Conflict of Interest

Authors declare no conflict of interest as regards to publication of this paper.

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