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Multiple soliton solutions of the Korteweg-de Vries in the frame of the time-dependent variable coefficient

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Abstract: This paper aims to investigate exact multiple soliton solutions for Korteweg-de Vries (KdV) with variable coefficients. Based on the similarity transformation and some hyperbolic function methods, multiple wave kink and wave bell shape solutions of the KdV with time dependent variable coefficient are analyzed. Results of this study show that the shapes of multiple kink-type and bell-type solitons can be effectively controlled by selecting some specific form of self-similar variables.

Keywords: KdV equations, tanh function method, sech function method, similarity transformation.

1 Introduction

In past decades, specifically in 1965 when two scientists Zabusky and Kruskal discussed the meaning of soliton and explained that most of the physical phenomena in several areas of our life, for instance problems in engineering, plasma physics, mathematical physics, quantum mechanics, biomathematics, hydrodynamics, optics, and chemistry can be described by means of using nonlinear partial differential equations or systems. Many researchers, mathematicians and physicists have raced to discover good and new methods of studying these models and to obtain accurate numerical and exact wave solutions. These studies play a very important role in clarifying the physical meaning of each model. One of such models is called Korteweg-de Vries (KdV), very important in the field of waves on shallow water surfaces. It is the best example of the type of partial differential equations that can be integrated and whose solutions can be determined accurately. The KdV model is an integrable model which can be solved by the inverse scattering transform method. The KdV model has many connections to a large number of natural phenomena examples including quantum mechanics, plasma, and soliton theory [1-39].

The optical solitons research is nice flourishing, since it can be applied to the new developments of optical communication models and data transmissions [1–30], which include dynamics of electron in semiconductors, metal phase changes induced by light, and chemical reactions. Optical soliton is special form of ultrashort pulses, which enable us to keep its shape and velocity unchanged in transmission long-distance.

There are abundant types of linear and nonlinear partial differential models that have been introduced in the last decades to obtain new analytical and numerical solutions. Different methods have been introduced to study a large variety of such models. Since the description of these linear and nonlinear systems being to supply several structures to the solutions. Examples include tanh method, Hirota method, Backlund transformation, Miura's transformation, inverse scattering method and Adomian decomposition method [1-39].

One of the major goals of the present article is to provide an effective procedure to test developments in a direct fashion, the verification of multiple solitons for the KdV equation with time-dependent coefficient. Several analytical solutions are determined with the aid of symbolic arithmetic calculating software, such as Maple.

2 Model and Technique of Solution

The generalized KdV equation with time-dependent coefficient, describes the dynamics of the following system:

$$U_t - 6\,\alpha\,(t)\,U\,U_x + \beta\,(t)\,U_{xxx} = 0, \qquad (1)$$

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with $\alpha(t) = a\beta(t)$, where $U = U(x,t)$ is a scalar-valued function t is time and x is the spatial coordinate, and $\alpha(t)$, $\beta(t)$ are time dependent coefficients.
By setting $\alpha(t) = \beta(t) = 1$ in (1), it is simplified as the standard KdV equation.
Suppose that

$$U(x,t) = u(x,T) \tag{2}$$

where $T = T(t)$ is a real function of variable t. If we choose $\beta(t) = \frac{\partial T}{\partial t}$, and using the transformation (2) in (1), we see that it takes the following form:

$$u_T - 6auu_x + u_{xxx} = 0 \tag{3}$$

Based on Malfliet technique we obtain multiple soliton solutions of the KdV equation (3) which was suggested in [10]. This procedure gives the multiple soliton solutions easily and conveniently simpler than the inverse scattering technique [11]. Abdel-Rahman [12] discussed the multiple solitons of the mKdV, the regularized long-wave, the Boussinesq, and the modified Boussinesq equations in a slightly modified manner. Moreover, the combined KdV and mKdV have been studied by Zhang et al. [13]. In this manuscript we describe in detail the Malfliet's procedure and calculate multiple soliton solutions to the KdV equation with time dependent variables. We suppose the solitary wave of the suggested nonlinear equation to be as follows:

$$u(x,T) = u(\xi), \quad \xi = k(x - \omega T) + \varphi, \tag{4}$$

where ξ is the travelling wave k, ω are arbitrary constants represent the wave number and wave velocity respectively and φ arbitrary constant.

Substituting equation (4) into equation (3), we have

$$\omega u' - 6auu' + k^2 u''' = 0. \tag{5}$$

By following the Malfliet's procedure as given in [10], suppose that the solution can be written as product of two functions as follows:

$$u(\xi) = p(\xi)q(\xi), \tag{6}$$

where $p(\xi)$ and $q(\xi)$ are arbitrary functions. Then, we have the following relations

$$u' = (pq') + (p \leftrightarrow q), \quad u''' = (pq''' + 3p''q') + (p \leftrightarrow q), \tag{7}$$

Next, the nonlinear term $-6auu'$ is replaced by

$$\begin{aligned} -6auu' &= -\alpha uu' - \beta uu' \\ &= -\alpha pqu' - \beta u(pq' + qp'); \quad 6a = \alpha + \beta, \end{aligned} \tag{8}$$

Substituting equations (7) and (8) into equation (5), we get

$$p[k^2 q''' + 3k^2 \frac{p''}{p} q' - \alpha(u + \frac{\omega}{\alpha})q' - \frac{\beta}{2} qu'] + q(p \leftrightarrow q) = 0. \tag{9}$$

Since the occurrence of the derivative q' in equation (9), we let $\frac{k^2 p''}{p} = u + \frac{\omega}{\alpha}$ with a similar expression for q. Thus we find that for $\psi = p, q$ the Schrödinger equation obtained is

$$k^2 \psi'' - \left(u + \frac{\omega}{\alpha}\right) \psi = 0, \tag{10}$$

where $\frac{u}{k^2}$ is the scattering potential and $-\frac{\omega}{\alpha k^2}$ is the eigenvalue. Thus equation (9) can be written as

$$p[k^2 q''' + (3 - \alpha)(u + \frac{\omega}{\alpha})q' - \frac{\beta}{2} qu'] + q(p \leftrightarrow q) = 0. \tag{11}$$

Differentiating the Schrödinger equation (10) with respect to ξ, we deduce that the resulting equation coincides with Eq. (9) and gives $\alpha = 4a$ and $\beta = 2a$, since the two functions p, q satisfy the Schrödinger equation

$$k^2 \psi'' - (u + \frac{\omega}{4a})\psi = 0. \tag{12}$$

Suppose the potential $\frac{u}{k^2}$ is attractive, i.e., $\frac{u}{k^2} < 0$ and we can find N distinct discrete eigenvalues $-\frac{\omega_n}{4ak_n^2}$, $n = 1, 2, ..., N$ associated with it. So, equation (12) can be written as

$$\begin{aligned} &k_n^2 \psi_n'' - \left(u + \frac{\omega_n}{4a}\right) \psi_n = 0, \\ &' \equiv \frac{d}{d\xi}, \quad \xi_n = k_n(x - \omega_n T) + \varphi_n. \end{aligned} \tag{13}$$

At this stage, the Schrödinger equation (13) have the wave functions ψ_n; Thus the general solution of the KdV equation with time-dependent coefficient can be expressed in terms of the wave functions ψ_n as follows:

$$u(\xi) = \sum_{n=1}^{N} \psi_n^2(\xi_n), \quad \xi_n = k_n(x - \omega_n T) + \varphi_n. \tag{14}$$

If the functions ψ_n do not overlap with each other, we can write equation (13) as follows:

$$k_n^2 \psi_n'' - \left(\psi_n^2 + \frac{\omega_n}{4a}\right) \psi_n = 0. \tag{15}$$

To get the functions ψ_n satisfying (15), according to the tanh function method, we balance the highest linear term ψ_n'' with the nonlinear term ψ_n^3 and the balancing gives the degree of the series solution as $s = 1$; thus the solution takes the form

$$\psi_n = a_0 + a_1 \tanh(\xi_n), \quad [\tanh(\xi_n)]' = 1 - \tanh^2(\xi_n). \tag{16}$$

Substituting into (15), we can obtain algebraic system by equating the coefficients of the distinct powers of $\tanh(\xi_n)$ to zero, then solving this system, we have

$$a_0 = 0, \quad a_1 = \pm\sqrt{2a}\, k_n, \quad \omega_n = -8ak_n^2. \tag{17}$$

Then the wave functions ψ_n takes the form

$$\psi_n = \pm\sqrt{2a}\, k_n \tanh(k_n(x + 8ak_n^2 T) + \varphi_n). \tag{18}$$

Thus, we obtain the following multiple kink wave solutions of the KdV equation with time dependent variable

$$u(\xi) = 2a \sum_{n=1}^{N} k_n^2 \tanh^2(k_n(x + 8ak_n^2 T) + \varphi_n), \quad T = \int \beta(t)\, dt. \tag{19}$$

Searching for another solution to the wave function ψ_n given in equation (15), using the sech function method, we have

$$\psi_n = b_0 + b_1 \operatorname{sech}(\xi_n), \quad [\operatorname{sech}(\xi_n)]' = \operatorname{sech}(\xi_n)\sqrt{1 - \operatorname{sech}^2(\xi_n)}, \tag{20}$$

substituting into (20) we can obtain algebraic system by equating the coefficients of the distinct powers of $\operatorname{sech}(\xi_n)$ to zero, and solving this system of algebraic equations, we have

$$b_0 = 0, \qquad b_1 = \pm\sqrt{-2a}\,k_n, \qquad \omega_n = 4ak_n^2. \tag{21}$$

Then, the wave functions ψ_n take the form

$$\psi_n = \pm\sqrt{-2a}\,k_n \operatorname{sech}(k_n(x - 4ak_n^2 T) + \varphi_n). \tag{22}$$

Thus, we obtain the following multiple bell wave solutions of the KdV equation with time dependent variable

$$u(\xi) = -2a \sum_{n=1}^{N} k_n^2 \operatorname{sech}^2(k_n(x - 4ak_n^2 T) + \varphi_n); \quad T = \int \beta(t)\, dt. \tag{23}$$

Based on this result, the multiple soliton solutions for the KdV equation with time dependent equation (1) are obtained immediately. The single kink solution for $N = 1$ takes the form

$$u(\xi) = 2ak_1^2 \tanh^2(k_1(x + 8ak_1^2 T) + \varphi_1); \quad T = \int \beta(t)\, dt. \tag{24}$$

The evolutional behaviour of the solution (19) when $N = 1$ represented in figure (1) below, shows one soliton behaviour with choices $a = 1$, $k_1 = 3$ and $\varphi_1 = 0$. Fig (1-a) when $\beta(t) = 1$, Fig (1-b) when $\beta(t) = 10t$, Fig (1-c) when $\beta(t) = \sinh t$, Fig (1-d) when $\beta(t) = \operatorname{sech}^2 t$, Fig (1-e) when $\beta(t) = \cos t$, Fig (1-f) when $\beta(t) = e^t$, and Fig (1-g) when $\beta(t) = \sec^2 t$. We obtain one kink soliton solution represented in figure (1). The evolutional behaviour of the solution (19) when $N = 2$ represented in figure (2) shows two soliton behaviour with the following choices $a = 0.2$, $k_1 = 1, k_2 = -2$, $\varphi_1 = -10$ and $\varphi_2 = 20$. Fig (2-a) when $\beta(t) = 1$, Fig (2-b) when $\beta(t) = 10t$, Fig (2-c) when $\beta(t) = \sinh t$, Fig (2-d) when $\beta(t) = \operatorname{sech}^2 t$, Fig (2-e) when $\beta(t) = \cos t$, Fig (2-f) when $\beta(t) = e^t$, and Fig (2-g) when $\beta(t) = \sec^2 t$,

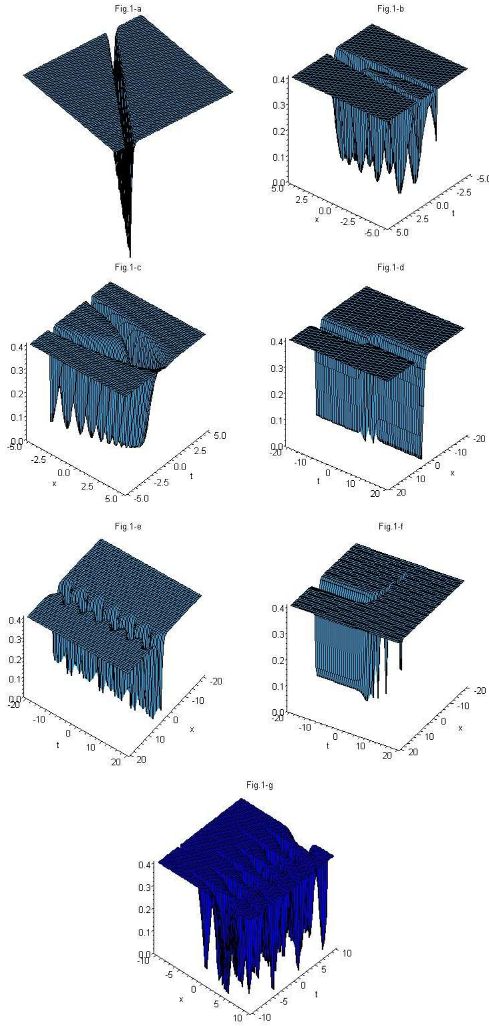

Fig. 1: Structures of one soliton solution (19) with $a = 1$, $k_1 = 3$, and $\varphi_1 = 0$: (a) $\beta(t) = 1$, (b) $\beta(t) = 10t$, (c) $\beta(t) = \sinh(t)$, (d) $\beta(t) = \operatorname{sech}^2(t)$, (e) $\beta(t) = \cos(t)$, (f) $\beta(t) = e^t$, (g) $\beta(t) = \sec^2(t)$.

3 Concluding Remarks

This article introduced KdV equation with time dependent coefficient that possesses the multiple soliton solutions. Based on the similarity transformation and tanh and sech function methods, both multiple kink-type as well as bell-type soliton solutions of the desired equation are obtained. We analyzed the emerging multiple soliton structures by a special selection of time self-similar variable. The results showed that the shapes of both multiple kink-type and bell-type solitons can be effectively controlled by the specific form of these self-similar variable. Multiple soliton solutions have been formally derived. We discussed different choices of the time-dependent coefficient $\beta(t)$, and illustrated the obtained soliton solutions by graphs in figures. The

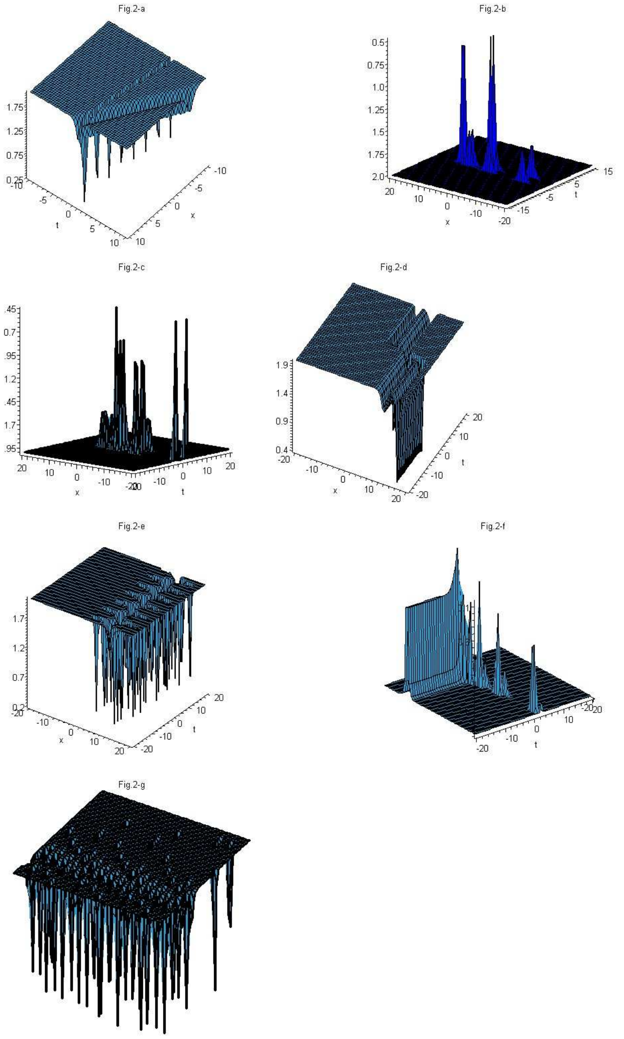

Fig. 2: Structures of one soliton solution (19) with $a=0.2,\ k_1=1,\ k_2=-2,\ \varphi_1=-10$ and$\varphi_2=0$: (a) $\beta(t)=1$(b) $\beta(t)=10t$(c) $\beta(t)=\sinh t$ (d) $\beta(t)=\text{sech}^2 t$ (e) $\beta(t)=\cos t$ (f) $\beta(t)=e^t$ (g) $\beta(t)=\sec^2 t$.

derived results can be helpful to discuss other integrable applications for more findings.

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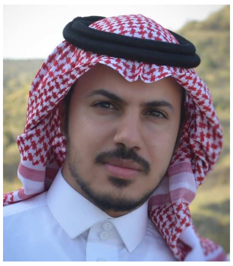

Fadhel Almalki received the PhD degree in Mathematics at Leeds university, the united kingdom. His research interests are in the areas of applied analysis including methods based on (wavelet) covariant and contravariant transforms on Lie groups as well as methods for nonlinear differential equations. He has published research articles in reputed international journals of mathematical and physical sciences.