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# Fibonacci Riccati Method for KdV Models with Dependent Space Time Coefficient

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**Abstract:** In this paper a number of findings present various ways of new research with novel class of trigonometric functions which unifies the properties of the well-known standard functions. From our previous knowledge, the important and significant role which investigated by the trigonometric functions in simplifying results in astronomy, physics and engineering, therefore, we may naturally predict that these new studies of trigonometric functions can lead to interpretations and results that have not appeared before and are new in mathematics, physics, biology, engineering and other branches of science. By introducing the variable-coefficient Riccati Fibonacci procedure, we investigate explicit solutions for some Kortewege de Vries models with variable coefficient (vcKdV). The main and basic idea of this procedure is based on finding solutions to desired models as a series in terms of solutions of the quadratic Riccati differential equation which are satisfied by Fibonacci trigonometric symmetric functions.

**Keywords:** Fibonacci Riccati method, variable-coefficient KdV equations, symmetrical Fibonacci functions.

## 1 Introduction

In the last three decades, research and study have been intensified in nonlinear differential models, which are largely represented in different fields of modern science and technology. The investigation and search for explicit and numerical solutions to nonlinear partial differential models represent a vital and major role in nonlinearity research of natural phenomena. In earlier years, great developments have been done in analyzing and finding accurate solutions to nonlinear models. Accordingly, various important procedures have been investigated, for example, Darboux transformation, inverse scattering [1], Hirota [2], Cole-Hopf transformation, Bäcklund transformation [2, 3], Painlevé [2, 4], homogeneous balance [5, 6, 7, 8, 9], tanh and coth [10, 11], the generalized hyperbolic [12, 13] and so on. As a result of the emergence of various scientific computational programs for example Maple, Matlab and Mathematica that give us opportunity to carry out some complex or long algebraic calculations as well as differential calculations on computer which help to investigate other explicit solutions of desired nonlinear differential models.

The effective and important procedures for finding explicit solutions of nonlinear differential models is the tanh procedure [10, 11]. In the previous few years, Fan [13] proposed an extension for tanh method. Furthermore, it is further extended by many authors for example Fan [14, 15], Yan [16] and Chen et-al [17, 18]. Also, the extended tanh method is modified by Elwakil et-al [19] and investigated another explicit solution for some nonlinear models. It has become important to find new mathematical algorithms to find explicit solutions to nonlinear differential models which may have a significant impact and a major role on future research.

Recently, much attention has been paid to the vcKdV models which is mostly used in describing different natural phenomena in modern physics, plasma and engineering [20, 21]. The generalized vcKdV model can be written with dissipative, perturbative, and force extrinsic terms as

$$u_t+\mu_1(t)\,u\,u_x+\mu_2(t)\,u_{xxx}+\mu_3(t)\,u_x+\mu_4(t)\,u=\mu_5(t), \tag{1}$$

with $u(x,t)$ function in $x$ and $t$ called wave amplitude, and the coefficient $\mu_1(t)$ is the nonlinear, $\mu_2(t)$ is dispersive, $\mu_3(t)$ is dissipative, $\mu_4(t)$ is perturbed and $\mu_5(t)$ is external-force terms [20, 21].
The generalized vcKdV [22, 23]

$$u_t+[\alpha(t)+\beta(t)x]\,u_x-3c\,\gamma(t)\,u\,u_x+2\beta(t)\,u+\gamma(t)\,u_{xxx}=0, \tag{2}$$

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The generalized modified vcKdV (vcmKdV) model reads as

$$u_t-[4\alpha(t)-\beta(t)x]\,u_x+6\,\gamma(t)\,u^2\,u_x+\beta(t)\,u-\gamma(t)\,u_{xxx}=0, \quad (3)$$

which plays an important role in the field of mathematical physics and its applications. The well-known modified KdV and cylindrical mKdV are considered special cases of model (3) [24, 25, 26, 27, 28].

From the known facts that there are two mathematical natural constants $e-$ and $\pi$-numbers, which have a major role in mathematics, astronomy and all branches of physics, their extreme importance is due to the fact that they form and establish the classical functions: hyperbolic, exponential, logarithmic, and trigonometric functions. For instance, the role played by standard hyperbolic functions in geometry and in astronomical research. Furthermore, there is another mathematical constant that appears in modeling process in the interpretation of phenomena of physical nature called the Golden Mean, Section, Proportion and Ratio [29,30, 31, 32, 33, 34, 35]. However, we may say that this constant did not take its real role in explaining phenomena in mathematics and modern physics. One of the famous and well-known facts is that the principal symbols of esotericism are related to golden section. Furthermore, in modern physics perspective and attitude toward applications of the golden section and the interconnected Lucas and Fibonacci numbers seemed to changed rapidly. The new discoveries and recent researches of modern science, which were built on the use of golden section, are of great importance and made a breakthrough in the development of science. It should be noted that harmony and chaos are opposites of each other. Referring to Euclid's Elements book, we find that he referred to the geometric problem that is a division of a segment line in the extreme and middle ratio which called the golden section problem [29,30, 31, 32, 33, 34, 35]. When finding a solution to the problem of the golden section, we arrive at an algebraic equation of the second degree in the form, $x^2=x+1$. It is known that this equation has two roots. The positive root $\alpha=(1+\sqrt{5})/2$ is called golden section, or golden proportion or golden mean or golden ratio.

This manuscript is arranged as: Section 2 explores and introduces some axioms of the symmetric Fibonacci trigonometric functions. The Fibonacci Riccati method for finding explicit solutions for nonlinear differential models is introduced in Section 3. In sections 4, 5 and 6, we use the desired method to solve the vcKdV equation (1), the generalized vcKdV (2) and the generalized vcmKdV (3). Section (6) is discussion and summery.

## 2 Definitions, properties, and laws induced by Fibonacci symmetric trigonometric functions

Using the definitions of trigonometric functions, we can define the following which depend on the golden ratio, Fibonacci hyperbolic symmetric sine sFs, Fibonacci hyperbolic symmetric cosine cFs and Fibonacci hyperbolic symmetric tangent tFs functions as

$$\mathrm{sFs(y)}=\frac{\alpha^y-\alpha^{-y}}{\sqrt{5}},\quad \mathrm{cFs(y)}=\frac{\alpha^y+\alpha^{-y}}{\sqrt{5}},\quad \mathrm{tFs(y)}=\frac{\alpha^y-\alpha^{-y}}{\alpha^y+\alpha^{-y}}.$$

That have been presented by considering symmetric representation of Fibonacci hyperbolic functions that contribute to modern natural sciences, taking into account the vital role played by the golden ratio in the results of modern research. In addition, we may define three functions corresponding to the three functions that were defined before, Fibonacci hyperbolic symmetric cotan as $\mathrm{cotFs}=1/\mathrm{tFs}$ Fibonacci hyperbolic symmetric sec as $\mathrm{secFs}=1/\mathrm{cFs}$ Fibonacci hyperbolic symmetric cosec as $\mathrm{cscFs}=1/\mathrm{sFs}$ which satisfies the following

$$\mathrm{cFs}^2(y)-\mathrm{sFs}^2=\frac{4}{5},\quad 1-\mathrm{tFs}^2(y)=\frac{4}{5}\mathrm{secFs}^2,\quad \mathrm{cotFs}^2(y)-1=\frac{4}{5}\mathrm{cscFs}^2.$$

Also, based on the above definitions, we can compute differential formulas for Fibonacci hyperbolic symmetric functions as :

$$\frac{d}{dy}\,\mathrm{sFs}(y)=\mathrm{cFs}(y)\ln(\alpha),\quad \frac{d}{dy}\,\mathrm{cFs}(y)=\mathrm{sFs}(y)\ln(\alpha),\quad \frac{d}{dy}\,\mathrm{tFs}(y)=\frac{4}{5}\mathrm{secFs}^2(y)\ln(\alpha).$$

We may also give the corresponding definitions of Fibonacci trigonometric symmetric sTFs, Fibonacci trigonometric symmetric Cosine cTFs, and Fibonacci trigonometric symmetric tan tTFs, functions on the form [32]

$$\mathrm{sTFs(y)}=\frac{\alpha^{iy}-\alpha^{-iy}}{i\sqrt{5}},\quad \mathrm{cTFs(y)}=\frac{\alpha^{iy}+\alpha^{-iy}}{\sqrt{5}},\quad \mathrm{tTFs(y)}=\frac{\alpha^{iy}-\alpha^{-iy}}{\alpha^{iy}+\alpha^{-iy}}.$$

Further, we may define Fibonacci trigonometric symmetric cotan as $\mathrm{cotTFs}=1/\mathrm{tTFs}$ Fibonacci trigonometric symmetric sec as $\mathrm{secTFs}=1/\mathrm{cTFs}$ and Fibonacci trigonometric symmetric cosec as $\mathrm{cscTFs}=1/\mathrm{sTFs}$ which satisfy the following relations [32]

$$\mathrm{cTFs}^2(y)+\mathrm{sTFs}^2=\frac{4}{5},\quad 1+\mathrm{tTFs}^2(y)=\frac{4}{5}\mathrm{secTFs}^2,\quad \mathrm{cotTFs}^2(y)+1=\frac{4}{5}\mathrm{cscTFs}^2.$$

The differential formulas are given as follows:

$$\frac{d}{dy}\text{sTFs}(y) = \text{cTFs}(y)\ln(\alpha),$$
$$\frac{d}{dy}\text{cTFs}(y) = -\text{sTFs}(y)\ln(\alpha),$$
$$\frac{d}{dy}\text{tTFs}(y) = \frac{4}{5}\text{secTFs}^2(y)\ln(\alpha).$$

## 3 The variable coefficient Riccati Fibonacci technique

The basic and principle idea of this technique is based on finding solutions to desired models as a series in terms of solutions of the quadratic Riccati differential equation that are satisfied by Fibonacci trigonometric symmetric functions.

Taking into consideration a given variable-coefficient nonlinear differential model

$$M(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{4}$$

Assuming that $u(x,t)$ can be expressed as finite series in $F(\xi)$ as

$$u(x,t) = a_0(t) + \sum_{i=1}^{n} a_i(t) F^i(\xi(x,t)); \quad \xi = \xi(x,t) = g(t) + f(t)x, \tag{5}$$

with $n$ represents highest score of finite series, that will be calculated by equating the highest derivational term with nonlinear term(s) in equation (4) and $f(t)$ and $g(t)$ are optimal functions of $t$ to be computed. The function $F(\xi)$ have the differential form

$$F'(\xi) = A + BF(\xi)^2, \quad ' \equiv \frac{d}{d\xi}, \tag{6}$$

with $A$ and $B$ arbitrary constants.

Using (5) with (6) into (4), which enables us to convert the differential equation (4) into an identity or a series in $F(\xi)$. Taking each coefficient of $F(\xi)$ in the series to zero produces system of PDEs in $a_i(t), f(t)$ and $g(t)$. Solving this system, then $a_i(t), f(t)$ and $g(t)$ can be calculated by $A$ and $B$. Using the results in (5), the general solution of model (4) can be constructed. By choosing special and appropriate values for the constants A and B for obtaining the corresponding solution $F(\xi)$ for the desired equation (6) in the form of one of symmetric Fibonacci functions mentioned above.

Case 1: when $A = \ln(\alpha)$, $B = -\ln(\alpha)$, so (6) possesses solutions

$$\text{tFs}(\xi), \quad \text{cotFs}(\xi).$$

Case 2: when $A = \ln(\alpha)$, $B = \ln(\alpha)$, so (6) possesses a solution

$$\text{tTFs}(\xi).$$

Case 3: when $A = -\ln(\alpha)$, $B = -\ln(\alpha)$, so (6) possesses a solution

$$\text{cotTFs}(\xi).$$

Case 4: when $A = \frac{\ln(\alpha)}{2}$, $B = -\frac{\ln(\alpha)}{2}$ so (6) possesses a solution

$$\frac{\text{tFs}(\xi)}{1 \pm \text{secFs}(\xi)}.$$

Case 5: when $A = \ln(\alpha)$, $B = -4\ln(\alpha)$, so (6) possesses a solution

$$\frac{\text{tFs}(\xi)}{1 \pm \text{tFs}(\xi)^2}.$$

Case 6: when $A = \frac{\ln(\alpha)}{2}$, $B = \frac{\ln(\alpha)}{2}$ so (6) possesses a solution

$$\text{tTFs}(\xi) \pm \text{secTFs}(\xi), \quad \frac{\text{tTFs}(\xi)}{1 \pm \text{secTFs}(\xi)}, \quad \text{cscTFs}(\xi) - \text{cotTFs}(\xi).$$

Case 7: when $A = \ln(\alpha)$, $B = 4\ln(\alpha)$, so (6) possesses a solution

$$\frac{\text{tTFs}(\xi)}{1 - \text{tTFs}(\xi)^2}.$$

Case 8: when $A = -\frac{\ln(\alpha)}{2}$, $B = -\frac{\ln(\alpha)}{2}$ so (6) possesses a solution

$$\text{cotTFs}(\xi) \pm \text{cscTFs}(\xi), \quad \frac{\text{cotTFs}(\xi)}{1 \pm \text{cscTFs}(\xi)}, \quad \text{secTFs}(\xi) - \text{tTFs}(\xi).$$

Case 9: when $A = -\ln(\alpha)$, $B = -4\ln(\alpha)$, so (6) possesses a solution

$$\frac{\text{cotFs}(\xi)}{1 - \text{cotFs}(\xi)^2}.$$

Now, we can apply the variable coefficient Fibonacci Riccati method to a class of vcKdV equations.

## 4 Explicit analytic solution of the vcKdV model (1)

Now, we can apply the variable coefficient Fibonacci Riccati method to the vcKdV equation (1), balancing $u_{xxx}$ with $u u_x$, gives $n = 2$. Thus, the solution takes the firm

$$u(x,t) = a_0(t) + a_1 F(\xi) + a_2 F(\xi)^2, \tag{7}$$

and we get

$$\begin{aligned} u_t = {} & a_{0t} + a_1 A(f_t x + g_t) + [2a_2 A(f_t x + g_t) \\ & + a_{1t}]F(\xi) + [a_1 B(f_t x + g_t) + a_{2t}]F(\xi)^2 \\ & + 2a_2 B(f_t x + g_t)F(\xi)^3, \end{aligned} \tag{8}$$

$$\begin{aligned} u_x = {} & a_1 A f + 2a_2 A f F(\xi) + a_1 B f F(\xi)^2 \\ & + 2a_2 B f F(\xi)^3, \end{aligned} \tag{9}$$

$$\begin{aligned} uu_x = a_0a_1Af + (2a_0a_2 + a_1^2)AfF(\xi) \\ + a_1(a_0B + 3a_2A)fF(\xi)^2 \\ + [(2a_0a_2 + a_1^2)B + 2a_2^2A]fF(\xi)^3 \\ + 3a_1a_2BfF(\xi)^4 + 2a_2^2BfF(\xi)^5, \end{aligned} \quad (10)$$

$$\begin{aligned} u_{xxx} = 2a_1A^2Bf^3 + 16a_2A^2Bf^3F(\xi) \\ +8a_1AB^2f^3F(\xi)^2 + 40a_2AB^2f^3F(\xi)^3 \\ +6a_1B^3f^3F(\xi)^4 + 24a_2B^3f^3F(\xi)^5, \end{aligned} \quad (11)$$

By substituting (7)- (11) into the vcKdV (1) yields a system of PDEs with respect to $F(\xi)$. Solving this system in $a_i(t)$, $f(t)$ and $g(t)$, we find that

$$\begin{aligned} &a_1(t) = 0, \qquad f(t) = c, \\ &g(t) = -\int [c\mu_1(t)a_0(t) + 8c^3\mu_2(t)AB + c\mu_3(t)B]dt + c_0, \\ &a_0(t) = [\int \mu_5(t)e^{\int \mu_4(t)dt}dt + c_1]e^{\int \mu_4(t)dt}, \\ &a_2(t) = -\frac{12c^2\mu_2(t)B^2}{\mu_1(t)}, \end{aligned} \quad (12)$$

with constraint condition

$$\mu_{1t}(t)\mu_2(t) - \mu_{2t}(t)\mu_1(t) - \mu_1(t)\mu_2(t)\mu_5(t) = 0, \quad (13)$$

where $c$, $c_0$, $c_1$ are constants of integration. Thus, we have the general solution of the general vcKdV model (1) in the form

$$\begin{aligned} u = \{\int \mu_5(t)e^{\int \mu_4(t)dt}dt + c_1\}e^{\int \mu_4(t)dt} \\ -\frac{12c^2\mu_2(t)B^2}{\mu_1(t)}F(cx + g(t))^2, \end{aligned} \quad (14)$$

where $g(t)$ is given in (12) and $\mu_i (i = 1,2,5)$ satisfies the constraint condition (13). By taking the specific value of the $A, B$ and $F(\xi)$, we obtain solution of the vcKdV model (1) as

$$u_1 = a_0 - \frac{12c^2\mu_2(t)(\ln\alpha)^2}{\mu_1(t)}\text{tFs}(cx + g(t))^2, \quad (15)$$

$$u_2 = a_0 - \frac{12c^2\mu_2(t)(\ln\alpha)^2}{\mu_1(t)}\text{cotFs}(cx + g(t))^2, \quad (16)$$

with

$$g(t) = -\int [c\mu_1(t)a_0(t) - 8c^3\mu_2(t)(\ln\alpha)^2 - c\mu_3(t)\ln\alpha]dt + c_0$$

and

$$u_3 = a_0 - \frac{12c^2\mu_2(t)(\ln\alpha)^2}{\mu_1(t)}\text{tTFs}(cx + g(t))^2, \quad (17)$$

with $g(t) = -\int [c\mu_1(t)a_0(t) + 8c^3\mu_2(t)(\ln\alpha)^2 + c\mu_3(t)\ln\alpha]dt + c_0$ and

$$u_4 = a_0 - \frac{12c^2\mu_2(t)(\ln\alpha)^2}{\mu_1(t)}\text{cotTFs}(cx + g(t))^2, \quad (18)$$

with $g(t) = -\int [c\mu_1(t)a_0(t) + 8c^3\mu_2(t)(\ln\alpha)^2 - c\mu_3(t)\ln\alpha]dt + c_0$. We omitted the reminder solutions for simplicity.

## 5 Explicit solutions of the generalized vcKdV model (2)

With the aim of obtaining the explicit solution of the generalized vcKdV equation (2), we first assume that the form of solution to equation (2) is the same as equation (7). By substituting (7)- (11) into the vcKdV (2) yields a system of PDEs with respect to $F(\xi)$. Solving this system of equations for $a_i(t)$, $f(t)$ and $g(t)$, we find that

$$\begin{aligned} &a_1(t) = 0, \\ &g(t) = -\int \alpha(t)\beta(t)[1 + 8f(t)^2AB - 3ca_0(t)]dt + c_0, \\ &f(t) = c_2e^{-\int \beta(t)dt}, \qquad a_0(t) = c_0e^{-2\int \beta(t)dt}, \\ &a_2(t) = -\frac{4f(t)^2B^2}{c}, \end{aligned} \quad (19)$$

with $c$, $c_0$, $c_1$ are integration constants. Thus, we have the general solution of general vcKdV model (2)

$$u = c_0e^{-2\int \beta(t)dt} - \frac{4f(t)^2B^2}{c}F(f(t)x + g(t)), \quad (20)$$

With $f(t)$ and $g(t)$ are given in (19). By choosing the different values of the constants $A, B$ and the consistent function $F(\xi)$, we obtain the subsequent solutions of generalized vcKdV equation (2):

$$u_1 = c_0e^{-2\int \beta(t)dt} - \frac{4f(t)^2(\ln\alpha)^2}{c}\text{tFs}(f(t)x + g(t)), \quad (21)$$

$$u_2 = c_0e^{-2\int \beta(t)dt} - \frac{4f(t)^2(\ln\alpha)^2}{c}\text{cotFs}(f(t)x + g(t)), \quad (22)$$

with $g(t) = -\int \alpha(t)\beta(t)[1 - 8f(t)^2(\ln\alpha)^2 - 3ca_0(t)]dt + c_0$, $f(t) = c_2e^{-\int \beta(t)dt}$, and

$$u_3 = c_0e^{-2\int \beta(t)dt} - \frac{4f(t)^2(\ln\alpha)^2}{c}\text{tTFs}(f(t)x + g(t)), \quad (23)$$

$$u_4 = c_0e^{-2\int \beta(t)dt} - \frac{4f(t)^2(\ln\alpha)^2}{c}\text{tTFs}(f(t)x + g(t)), \quad (24)$$

with $g(t) = -\int \alpha(t)\beta(t)[1 + 8f(t)^2(\ln\alpha)^2 - 3ca_0(t)]dt + c_0$, $f(t) = c_2e^{-\int \beta(t)dt}$. We omitted the reminder solutions for simplicity.

## 6 Discussion and Summary

In conclusion, using the variable coefficient Fibonacci Riccati, we present explicit solutions of different types of vcKdV models. These solutions are expressed by the Fibonacci symmetrical function. Our hopefulness is that in forthcoming experimental educations these new solutions will be realized in some fields. In fact, this current short manuscript is just a commencement study, due to a wide variety of applications of KdV equations. The Fibonacci Riccati procedure can be study further differential nonlinear models.

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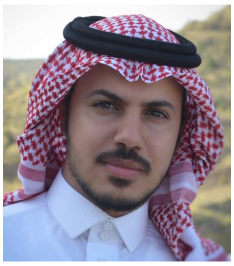

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