Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/12S104

Quadratic Transmuted Modified Size-Biased Lehmann Type-II Distribution

M. Ibrahem[1,], M. M. Nassar[2], S. S. Radwan[3] and A. Elmasry[2]*

[1]Department of Basic Sciences, Faculty of Computer and Information Sciences, Ain Shams University, Abbassia, Cairo, Egypt
[2]Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt
[3]Department of Mathematics, Faculty of Science Girls section, AlAzhar University, Cairo, Egypt

Received: 9 Jun. 2023, Revised: 10 Nov. 2023, Accepted: 19 Nov. 2023.
Published online: 1 Dec. 2023.

Abstract: In this paper, a two-parameter generalization of Modified Size-Biased Lehmann Type-II distribution is obtained, with the purpose of obtaining a more flexible model relative to the behaviour of hazard rate functions. Various statistical properties of this distribution including the density, hazard rate functions, quantile function, mode, moments, incomplete moments, moment generating functions, Lorenz, Bonferroni and Zenga curves, Rényi entropy and distribution of r^{th} order statistics have been derived. The method of maximum likelihood estimation has been used to estimate the parameters of the Quadratic Transmuted-Modified Size-Biased Lehmann Type-II distribution and its performance is discussed by following a simulation study. Real data sets are presented to demonstrate the effectiveness of the new model.

Keywords: Quadratic Transmutation; Transmuted Distribution, A new generalization of Lehmann type-II distribution, moment generating function, entropy and maximum likelihood estimation.

1 Introduction

The procedure of adding new shape parameters to a family of distributions to generate new distributions that are more flexible is a well-known technique in the statistical literature, but our main focus in this paper is to present a transmuted -G class of distribution that increases the flexibility of the distribution. We first present some well-known generators such as the exponentiated G distributions due to Gupta and Kundu [1], Nassar and Eissa [2], [3] and others, the Beta-G distributions by Eugene et al. [4], Jones [5], Nadarajah and Kotz [6], [7], Nassar and Nada [8], [9], [10], Nassar and Elmasry [11] and Mahmoud et al. [12], Kumaraswamy-G by Cordeiro et al. [13], [14], Cordeiro and de Castro [15], Nassar [16], Topp–Leone-G family distributions proposed by Al-Shomrani et al. [17], Nassar and Ibrahem [18], Gamma-G distributions by Zografos and Balakrishnan [19] and Transmuted Family of Distributions by Shaw and Buckley [20], introduced an interesting method of adding new parameter to an existing distribution and named the family as quadratic transmuted family (QT-G) of distributions. Many authors considered this transmutation map to generalize some existing distributions.

The quadratic transmuted family distributions proposed by Cordeiro et al. [21], with its cumulative distribution function (cdf) and probability density function (pdf) are given by,

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \qquad ,\lambda \in [-1,1]. \tag{1}$$

$$f(x) = \big(1+\lambda-2\lambda G(x)\big)g(x) \qquad ,\lambda \in [-1,1]. \tag{2}$$

Where $G(x)$and g(x) are the cdf and the pdf of the base distribution respectively.

Many authors considered this transmutation map to generalize some existing distributions. For example, Merovci [22] used the transmuted family to introduce the transmuted Lindley distribution, Ashour and Eltehiwy [23] proposed the transmuted Lomax distribution, transmuted Ishita distribution by Gharaibeh and Al-Omari [24], transmuted Generalized Gamma distribution by Cordeiro et al. [25] and others.

In this paper, we introduce a two-parameter model, called the Quadratic Transmuted-Modified Size-Biased Lehmann Type-II, to extend the Modified Size-Biased Lehmann Type-II model for its importance and usefulness in many practical applications.

*Corresponding author e-mail: mohamedibrahem1993@yahoo.com

© 2023 NSP
Natural Sciences Publishing Cor.

The Modified Size-Biased Lehmann Type-II distribution is developed by Arshad et al. [26] with its (cdf) and (pdf) are given by,

$$G(x) = 1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}. \quad (3)$$

$$g(x) = \alpha(\alpha+1)\frac{(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}}. \quad (4)$$

where $0 < x < 1$ and $\alpha > 0$ is a shape parameter.

The only generalization of the modified size-biased Lehmann type-II (MSBL-II) distribution was the three - parameters Kumaraswamy of MSBL-II (Kum–MSBL-II) distribution proposed by Arshad et al. [26].

2 The Quadratic Transmuted-Modified Size-Biased Lehmann Type-II Distribution

In this section, we introduce the Quadratic Transmuted-Modified Size-Biased Lehmann Type-II (QT- MSBL-II) distribution. Some reliability functions corresponding to the QT- MSBL-II distribution are also discussed. the QT-MSBL-II distribution is obtained simply by inserting Equation (3) in Equation (1). Hence, the associated cdf of the QT-MSBL-II distribution with two shape parameters takes the form;

$$F(x) = (1+\lambda)\left(1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right) - \lambda\left(1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{2}, 0 < x < 1, \alpha > 0, \lambda \in [-1,1]. \quad (5)$$

And The pdf corresponding to Equation (5) is given by

$$f(x) = \alpha(\alpha+1)\left(1 + \lambda - 2\lambda\left(1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)\right)\frac{(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}},\ 0 < x < 1, \alpha > 0, \lambda \in [-1,1]. (6)$$

Plots of the (cdf) (5) for selected values of the QT-MSBL-II distribution are given in Figure 1. It is illustrated that when the value of $\alpha > 1$, the graphs of $F(x)$ are increasing then constant but when $0 < \alpha < 1$ *and* $-1 < \lambda < 0$, the graphs of $F(x)$ are concave up graph.

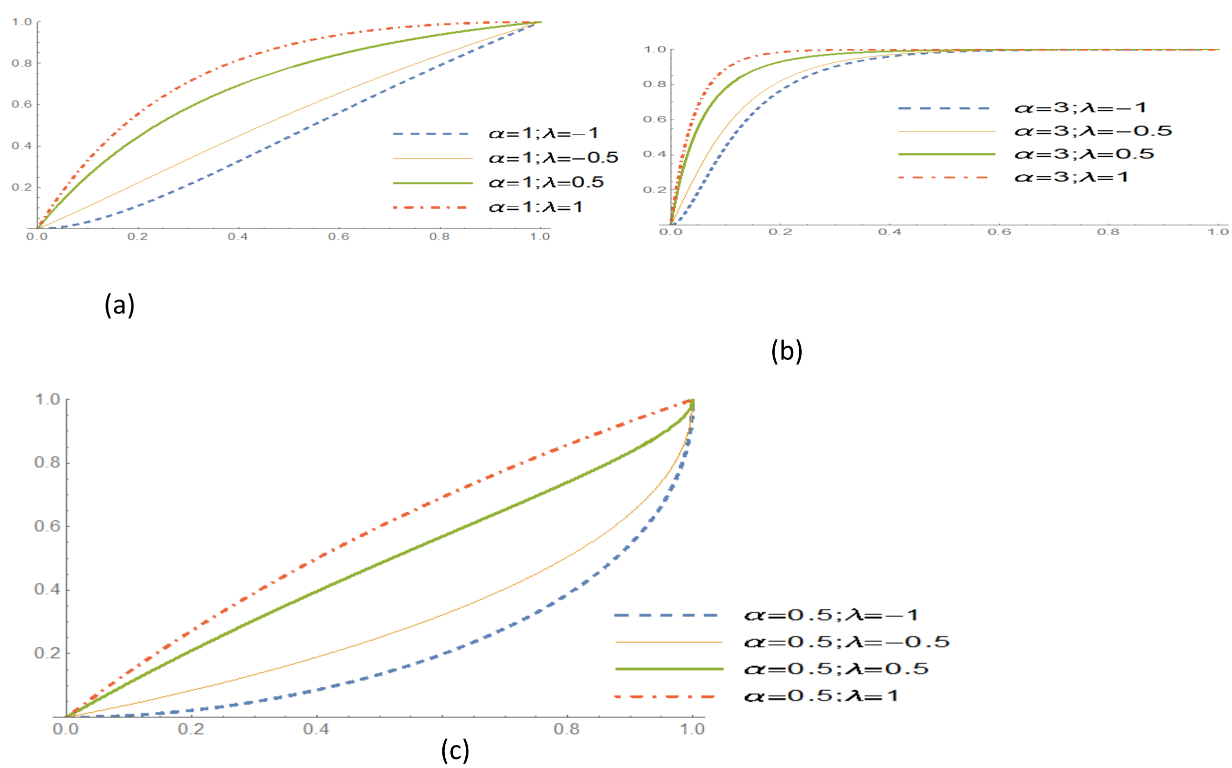

Fig.1: *Plots of the cumulative distribution function of QT-MSBL-II distribution for some parameter values.*

© 2023 NSP
Natural Sciences Publishing Cor.

We define the hazard rate function of the QT-MSBL-II distribution as follow $h(x) = \frac{[1+\lambda-2\lambda G(x)]g(x)}{[1-G(x)][1-\lambda G(x)]}$,

Then the hazard rate function of the QT-MSBL-II distribution (6) is given by

$$h(x) = \frac{\alpha(\alpha+1)}{(1+\alpha x)(1-x)}\left[1 + \frac{\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}}{1-\lambda+\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}}\right] \quad , 0 < x < 1, \alpha > 0, \lambda \in [-1, 1]. \qquad (7)$$

Plots of the hazard rate function (7) of the QT-MSBL-II distribution (6) are given in Figure 2, for selected values of the parameters, where we demonstrate the possible shapes of the hazard rate function which include bathtub shape and increasing hazard rate.

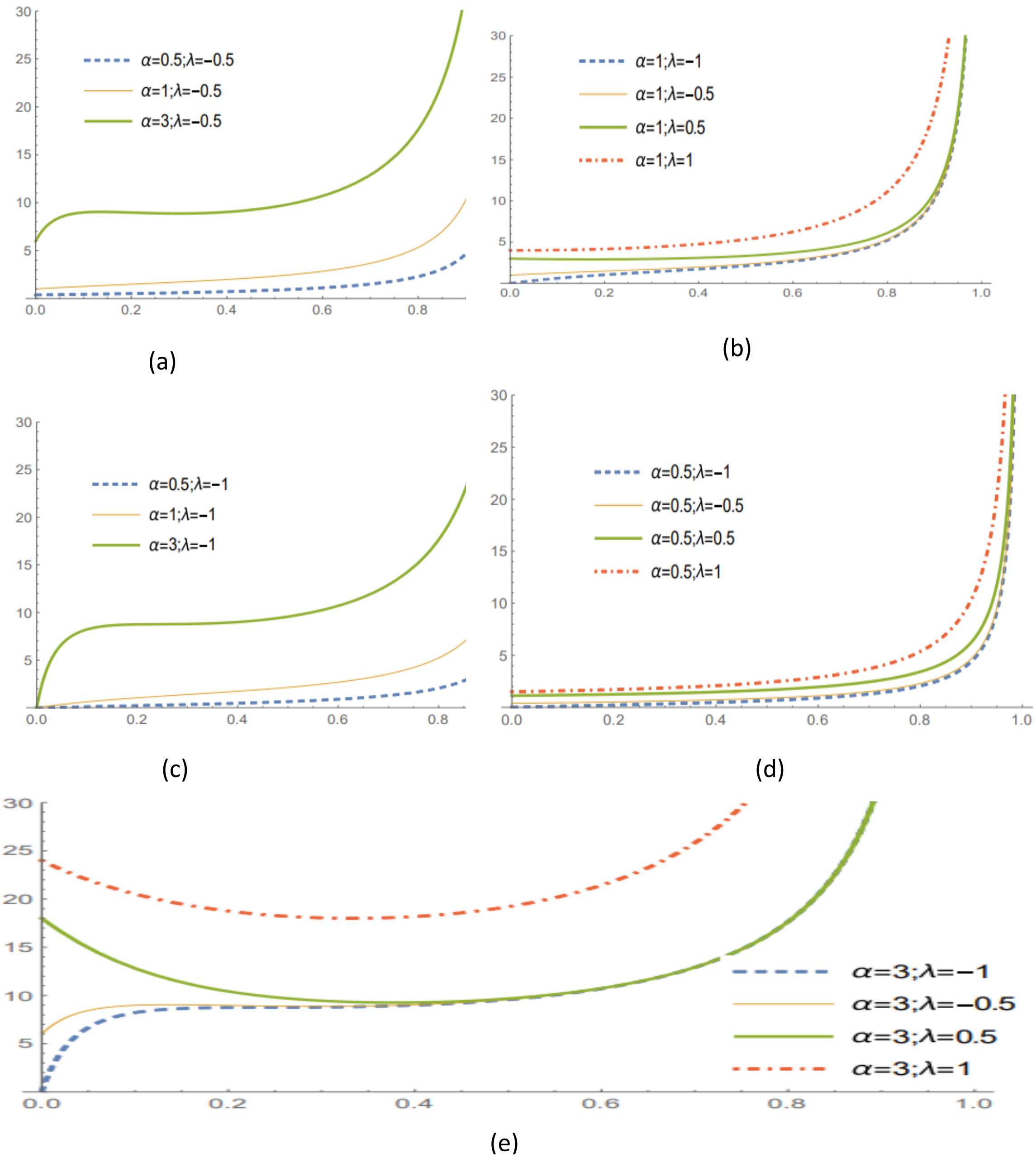

Fig.2: *Plots of the hazard rate function of QT-MSBL-II distribution for some parameter values.*

Plots of the density function (6) for selected values of the QT-MSBL-II distribution are given in Figure 3., It is observed that these plots show great flexibility of the QT-MSBL-II for different values of the shape parameter α and λ . we present some possible shapes of the pdf given in (6) including right-skewed shape, asymmetric shape and when the value of $\alpha \geq 1$ and $1 > \lambda > 0$, the graphs of $f(x)$ are decreasing.

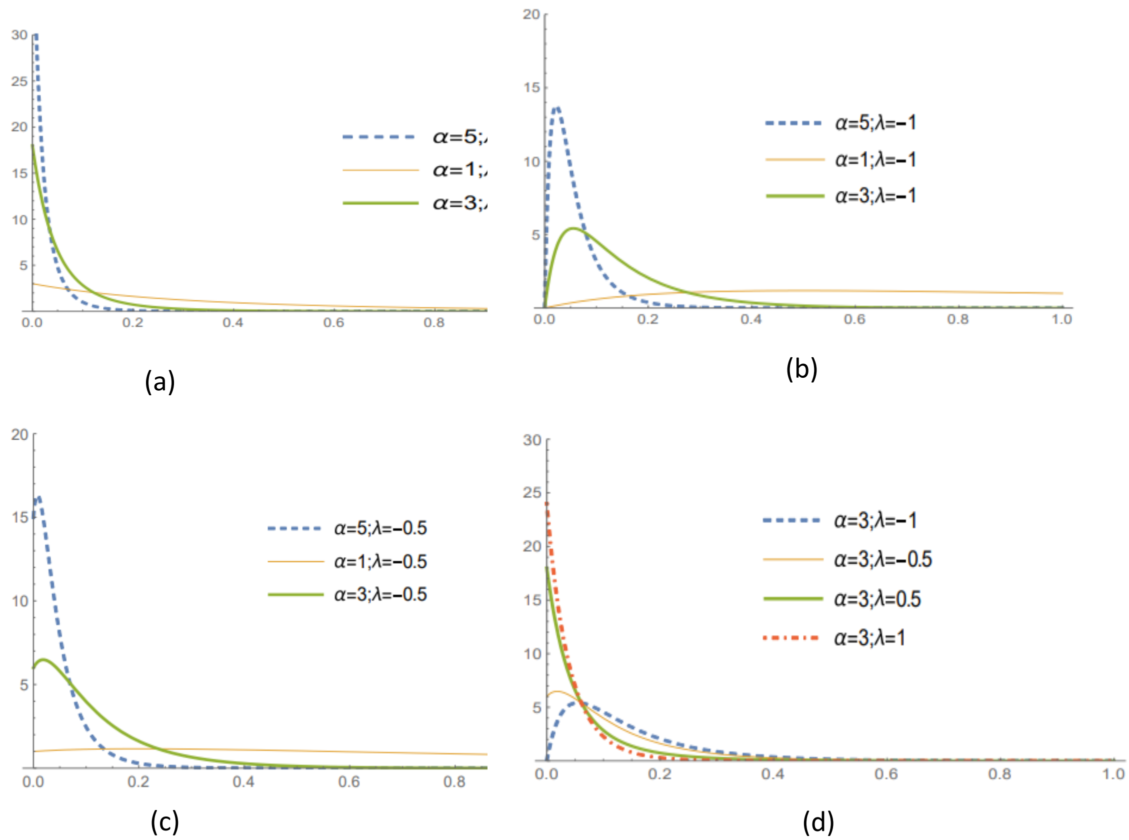

Fig. 3: *Plots of the density function of QT-MSBL-II distribution for some parameter values.*

3 Expansions for the QT-MSBL-II distribution

When determining the mathematical properties, linear combination provides a much more informal method of discussing the cdf and pdf than does traditional integral computation. The following binomial expansions is considered:

$$(1-z)^{\beta} = \sum_{i=0}^{\infty}(-1)^{i}\binom{\beta}{i}(z)^{i}\,, \; |z| < 1 \qquad (8)$$

The pdf of QT-MSBL-II in Equation (6) can be expressed as

$$f(x) = \alpha(\alpha+1)(1-\lambda)\frac{(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}} + 2\lambda\alpha(\alpha+1)\frac{(1-x)^{2\alpha-1}}{(1+\alpha x)^{2\alpha+1}}. \qquad (9)$$

Using the binomial expansion (8) in Equation (9), the infinite linear combinations of pdf is given as follows

$$f(x) = \alpha(\alpha+1)(1-\lambda)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\alpha^{i}(1-x)^{\alpha-1}x^{i} + 2\lambda\alpha(\alpha+1)\sum_{i=0}^{\infty}\binom{-2\alpha-1}{i}\alpha^{i}(1-x)^{2\alpha-1}x^{i}. \qquad (10)$$

The cdf of QT-MSBL-II in Equation (5) can be expressed as

$$F(x) = \left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)\left(1+\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right) \qquad (11)$$

© 2023 NSP
Natural Sciences Publishing Cor.

4 Statistical Properties of the QT-MSBL-II distribution

In this section, we discuss some statistical properties of the proposed distribution such as

quantile function, mode, r^{th} moment, moment generating functions, incomplete moment, Lorenz, Bonferroni and Zenga curves and Rényi of entropy.

4.1 Quantile function

Theorem 1:

Let X be a random variable following QT-MSBL-II distribution and let $u \in (0,1)$ where $F(x) = u$ and F(x) is the cdf of the QT-MSBL-II distribution. Then the quantile function is given by

$$x = \frac{1-\left(\frac{\lambda-1}{2\lambda}-\sqrt{\frac{(1+\lambda)^2}{4\lambda^2}-\frac{u}{\lambda}}\right)^{1/\alpha}}{1+\alpha\left(\frac{\lambda-1}{2\lambda}-\sqrt{\frac{(1+\lambda)^2}{4\lambda^2}-\frac{u}{\lambda}}\right)^{1/\alpha}} \tag{12}$$

Proof:

The quantile function of QT-MSBL-II distribution $x = F^{-1}(u), u \in (0,1)$ can be obtained by inverting Equation(5) as

$$F(x) = (1+\lambda)\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right) - \lambda\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^2 = u$$

Then

$$-\frac{u}{\lambda} = \left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^2 - \frac{(1+\lambda)}{\lambda}\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)$$

Using complete squares, the last equation can be expressed as

$$\frac{(1+\lambda)^2}{4\lambda^2} - \frac{u}{\lambda} = \left[\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right) - \frac{(1+\lambda)}{2\lambda}\right]^2$$

Therefore,

$$\left(\frac{1-x}{1+\alpha x}\right)^{\alpha} = \frac{\lambda-1}{2\lambda} - \sqrt{\frac{(1+\lambda)^2}{4\lambda^2} - \frac{u}{\lambda}}$$

Then

$$1-x = \left(\frac{\lambda-1}{2\lambda} - \sqrt{\frac{(1+\lambda)^2}{4\lambda^2} - \frac{u}{\lambda}}\right)^{1/\alpha} + \alpha\left(\frac{\lambda-1}{2\lambda} - \sqrt{\frac{(1+\lambda)^2}{4\lambda^2} - \frac{u}{\lambda}}\right)^{1/\alpha} x$$

Therefore, the quantile function of order u of the QT-MSBL-II distribution is the solution of Equation (12).

The median of the QT-MSBL-II distribution can be defined at u= 0.5 in Equation (12).

The QT-MSBL-II distribution is easily simulated from $F(x)$ in Equation (5) using the form of the quantile function in Equation (12).

4.2 Mode

The density function of QT-MSBL-II distribution given in (6) by solving $\frac{df(x)}{dx} = 0$ for x, to obtain the mode of Quadratic Transmuted-Modified Size-Biased Lehmann Type-II distribution as follows

$$\frac{df(x)}{dx} = \frac{\alpha(\alpha+1)(1-x)^{\alpha-2}}{(1+\alpha x)^{\alpha+2}}\left[-2\lambda\alpha(\alpha+1)\left(\frac{1-x}{1+\alpha x}\right)^{\alpha} + (2\alpha x - 2\alpha - \alpha^2 + 1)\left[1-\lambda+2\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right]\right]$$

© 2023 NSP
Natural Sciences Publishing Cor.

Therefore $\frac{df(x)}{dx} = 0$,

Then $x = 1$.

or

$$\left[-2\lambda\alpha(\alpha+1)\left(\frac{1-x}{1+\alpha x}\right)^{\alpha} + (2\alpha x - 2\alpha - \alpha^2 + 1)\left[1 - \lambda + 2\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right]\right] = 0.$$

But we cannot obtain an explicit Form, so we calculate the mode numerically for different values of α *and* λ.

Table 1: Mode for some chosen different values of α and λ.

The values of α and λ	**Mode**
$\alpha = 5$ and $\lambda = 0.5$	1
$\alpha = 3$ and $\lambda = 0.5$	1
$\alpha = 1$ and $\lambda = 0.5$	1
$\alpha = 5$ and $\lambda = -0.5$	0.00772
$\alpha = 1$ and $\lambda = -0.5$	0.2
$\alpha = 3$ and $\lambda = -0.5$	0.018429
$\alpha = 1$ and $\lambda = 1$	1
$\alpha = 3$ and $\lambda = 1$	1
$\alpha = 5$ and $\lambda = 1$	0.999967
$\alpha = 5$ and $\lambda = -1$	0.022092
$\alpha = 3$ and $\lambda = -1$	0.05521
$\alpha = 1$ and $\lambda = -1$	0.5
$\alpha = 0.5$ and $\lambda = 0.5$	0.634711

4.3 Moments

Theorem 2:

If X follows the QT-MSBL-II distribution given by the pdf (10), then the r$^{\text{th}}$ moment of X is given by

$$\mu'_r = (1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(i+r+1,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)B(i+r+1,2\alpha), \quad (13)$$

where

$$\left.\begin{array}{c} v_i(\alpha) = (\alpha+1)\alpha^{i+1}\binom{-\alpha-1}{i}, \\ \omega_i(\alpha) = (\alpha+1)\alpha^{i+1}\binom{-2\alpha-1}{i} \\ B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \text{ is the beta function} \end{array}\right\} \quad (14)$$

Proof:

The r$^{\text{th}}$ moment of the QT-MSBL-II distribution is given as follows

$$\mu'_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx .$$

From Equation (10), we have

$$\mu'_r = \int_0^1 x^r\left[\alpha(\alpha+1)(1-\lambda)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\alpha^i(1-x)^{\alpha-1}x^i + 2\lambda\alpha(\alpha+1)\sum_{i=0}^{\infty}\binom{-2\alpha-1}{i}\alpha^i(1-x)^{2\alpha-1}x^i\right]dx$$

© 2023 NSP
Natural Sciences Publishing Cor.

Then

$$\mu_r' = \int_0^1 (\alpha+1)(1-\lambda)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\alpha^{i+1}x^{i+r}(1-x)^{\alpha-1}\,dx + \int_0^1 2\lambda(\alpha+1)\sum_{i=0}^{\infty}\binom{-2\alpha-1}{i}\alpha^{i+1}x^{i+r}(1-x)^{2\alpha-1}dx$$

Therefore,

$$\mu_r' = (1-\lambda)\sum_{i=0}^{\infty}v_i(\alpha)\int_0^1 x^{i+r}(1-x)^{\alpha-1}dx + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\int_0^1 x^{i+r}(1-x)^{2\alpha-1}dx$$

This yields the r^{th} moment given in Equation (13).

Putting r =1 in Equation (13), we easily obtain the mean of QT-MSBL-II distribution,

If X follows the QT-MSBL-II distribution given by the pdf (10), the moment generating function (mgf) of X is given by

$$E(e^{tx}) = 1 + \sum_{r=1}^{\infty}\frac{t^r}{r!}\mu_r'$$

4.4 Incomplete moments

Theorem 3:

If X follows the QT-MSBL-II distribution defined in Equation (10), then the n^{th} incomplete moments denoted as $m_n(z)$ is given by

$$m_n(z) = (1-\lambda)\sum_{i=0}^{\infty}v_i(\alpha)B(z,i+n+1,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(z,i+\mathrm{n}+1,2\alpha), \qquad (15)$$

where

$$\left.\begin{array}{c} v_i(\alpha),\omega_i(\alpha) \text{is defined in Equation (14)} \\ \text{and also } \mathrm{B}(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\mathrm{d}t \text{ is the incomplete beta function} \end{array}\right\} \qquad (16)$$

Proof

The n^{th} incomplete moments denoted as $m_n(z)$ can be obtained as follows:

$$m_n(z) = \int_{-\infty}^{z} x^n f(x)dx$$

From Equation (10), we have

$$m_n(z) = \int_0^z x^n\left[\alpha(\alpha+1)(1-\lambda)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\alpha^i(1-x)^{\alpha-1}x^i + 2\lambda\alpha(\alpha+1)\sum_{i=0}^{\infty}\binom{-2\alpha-1}{i}\alpha^i(1-x)^{2\alpha-1}x^i\right]dx$$

Then

$$m_n(z) = \int_0^z (\alpha+1)(1-\lambda)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\alpha^{i+1}x^{i+n}(1-x)^{\alpha-1}\,dx + \int_0^z 2\lambda(\alpha+1)\sum_{i=0}^{\infty}\binom{-2\alpha-1}{i}\alpha^{i+1}x^{i+n}(1-x)^{2\alpha-1}dx$$

Therefore,

$$m_n(z) = (1-\lambda)\sum_{i=0}^{\infty}v_i(\alpha)\int_0^z x^{i+n}(1-x)^{\alpha-1}dx + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\int_0^z x^{i+n}(1-x)^{2\alpha-1}dx$$

This yields the n^{th} incomplete moments given in Equation (15).

© 2023 NSP
Natural Sciences Publishing Cor.

4.5 Lorenz, Bonferroni, and Zenga curves

Lorenz, Bonferroni and Zenga curves are important applications for the first incomplete moments. These curves are useful in many fields such as medicine, insurance, reliability, demography and economics. The Lorenz, Bonferroni and Zenga curves are defined, respectively, as follows:

$$L(F(z)) = \frac{1}{E(z)}\int_{-\infty}^{z} x f(x)dx = \frac{m_1(z)}{\mu_1'} \;,\quad B\big(F(z)\big) = \frac{L(F(z))}{F(z)}$$

and $A(z) = 1 - \frac{M^-(z)}{M^+(z)}$

where $M^-(z) = \frac{1}{F(z)}\int_{-\infty}^{z} x f(x)dx \,,\; M^+(z) = \frac{1}{1-F(z)}\int_{z}^{\infty} x f(x)dx$

Therefore, using Equations (15) and (13), we obtain the Lorenz curve as follows

$$L\big(F(z)\big) = \frac{(1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(z,i+2,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(z,i+2,2\alpha)}{(1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(i+2,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(i+2,2\alpha)}, \tag{17}$$

From Equations (17) and (5), we find the Bonferroni curve as

$$B\big(F(z)\big) = \frac{(1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(z,i+2,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(z,i+2,2\alpha)}{\left[(1+\lambda)\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right) - \lambda\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right)^2\right](1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(i+2,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(i+2,2\alpha)}, \tag{18}$$

Hence, the Zenga curve can be defined as follows

$$A(z) = 1 - \left\{\frac{\left\{1-\left[(1+\lambda)\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right) - \lambda\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right)^2\right]\right\}(1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(z,i+2,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(z,i+2,2\alpha)}{\left[(1+\lambda)\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right) - \lambda\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right)^2\right](1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)[B(i+2,\alpha) - B(z,i+2,\alpha)] + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)[\mathrm{B}(i+2,2\alpha) - \mathrm{B}(z,i+2,2\alpha)]}\right\}, \tag{19}$$

where

$$M^-(z) = \frac{(1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)B(z,i+2,\alpha) + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)\mathrm{B}(z,i+2,2\alpha)}{\left[(1+\lambda)\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right) - \lambda\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right)^2\right]}$$

and $M^+(z) = \frac{(1-\lambda)\sum_{i=0}^{\infty} v_i(\alpha)[B(i+2,\alpha) - B(z,i+2,\alpha)] + 2\lambda\sum_{i=0}^{\infty}\omega_i(\alpha)[\mathrm{B}(i+2,2\alpha) - \mathrm{B}(z,i+2,2\alpha)]}{\left\{1-\left[(1+\lambda)\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right) - \lambda\left(1-\left(\frac{1-z}{1+\alpha z}\right)^{\alpha}\right)^2\right]\right\}}.$

4.6 Rényi entropy

The entropy of a random variable represents the amount of variation of the uncertainty. The Rényi entropy has broad applications in different areas such as statistics, physics and ecology as the index of diversity. Rényi [27] entropy of X is described by;

$$J_R(\xi) = \frac{1}{1-\xi}\log\big(I(\xi)\big),$$

where $I(\xi) = \int f^{\xi}(x)dx \,,\; \xi > 0, and\ \xi \neq 1.$

Using this notion, we deduce the Rényi entropy of a random variable following the QT-MSBL-II pdf (6), in Theorem 4.

Theorem 4:

Let X be a continuous random variable following the QT-MSBL-II distribution given by Equation (6). The Rényi entropy of X is given by

$$J_R(\xi) = (1-\xi)^{-1}\left\{\xi\log(\alpha) + \xi\log(\alpha+1) + \log\left[\sum_{i=0}^{\xi}\sum_{j=0}^{\infty}\nabla_{i,j}(\alpha,\lambda,\xi)B(j+1,\xi\alpha+\alpha i-\xi+1)\right]\right\} \tag{20}$$

© 2023 NSP
Natural Sciences Publishing Cor.

where

$$\nabla_{i,j}(\alpha,\lambda,\xi) = (1-\lambda)^{\xi-1}(2\lambda)^i\alpha^j\binom{\xi}{i}\binom{-(\xi\alpha+\alpha i+\xi)}{j}.$$

Proof:

Setting the pdf of QT-MSBL-II (6) in the definition of Rényi entropy given above, we have

$$f^{\xi}(x) = \alpha^{\xi}(\alpha+1)^{\xi}\left(1-\lambda+2\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{\xi}\frac{(1-x)^{\xi\alpha-\xi}}{(1+\alpha x)^{\xi\alpha+\xi}}$$

The last equation can be written by following the binomial expansion in Equation (8) and it given as follows

$$f^{\xi}(x) = \alpha^{\xi}(\alpha+1)^{\xi}\sum_{i=0}^{\xi}\sum_{j=0}^{\infty}(1-\lambda)^{\xi-1}(2\lambda)^i\alpha^j\binom{\xi}{i}\binom{-(\xi\alpha+\alpha i+\xi)}{j}x^j(1-x)^{\xi\alpha+\alpha i-\xi}$$

$$I(\xi) = \alpha^{\xi}(\alpha+1)^{\xi}\sum_{i=0}^{\xi}\sum_{j=0}^{\infty}(1-\lambda)^{\xi-1}(2\lambda)^i\alpha^j\binom{\xi}{i}\binom{-(\xi\alpha+\alpha i+\xi)}{j}\int_0^1 x^j(1-x)^{\xi\alpha+\alpha i-\xi}dx$$

$$I(\xi) = \alpha^{\xi}(\alpha+1)^{\xi}\sum_{i=0}^{\xi}\sum_{j=0}^{\infty}(1-\lambda)^{\xi-1}(2\lambda)^i\alpha^j\binom{\xi}{i}\binom{-(\xi\alpha+\alpha i+\xi)}{j}B(j+1,\xi\alpha+\alpha i-\xi+1)$$

Then

$$I(\xi) = \alpha^{\xi}(\alpha+1)^{\xi}\sum_{i=0}^{\xi}\sum_{j=0}^{\infty}\nabla_{i,j}(\alpha,\lambda,\xi)\,B(j+1,\xi\alpha+\alpha i-\xi+1)$$

Finally, the Rényi entropy can be expressed as in Equation (20).

5 Order statistics

In reliability analysis and life testing of a *component* in quality control, order statistics and its moments are considered worthy measures. Let $X_{1:m} \le X_{2:m} \le \cdots \le X_{m:m}$ be the order sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k:m}$, the k^{th} order statistic is given by

$$f_{X_{k:m}}(x) = \frac{m!}{(k-1)!(m-k)!}f(x)[F(x)]^{k-1}[1-F(x)]^{m-k} \quad ; k = 1,2,\ldots,m. \tag{21}$$

The pdf of the k^{th} order QT-MSBL-II random variable $X_{k:m}$ can be obtained using Equations (9) and (11) in (21),

$$f_{X_{k:m}}(x) = \frac{m!\ \alpha(\alpha+1)}{(k-1)!\,(m-k)!}\left(\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)\left(1+\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)\right)^{k-1}$$
$$\left(1-\left[\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)\left(1+\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)\right]\right)^{m-k}\left[(1-\lambda)\frac{(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}}+2\lambda\frac{(1-x)^{2\alpha-1}}{(1+\alpha x)^{2\alpha+1}}\right]$$

Using the binomial expansion (8), we obtain

$$f_{X_{k:m}}(x) = \frac{m!\ \alpha(\alpha+1)}{(k-1)!\,(m-k)!}\sum_{i=0}^{\infty}\binom{m-k}{i}(-1)^i\left(1-\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{k+i-1}\left(1+\lambda\left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{k+i-1}$$
$$\left[(1-\lambda)\frac{(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}}+2\lambda\frac{(1-x)^{2\alpha-1}}{(1+\alpha x)^{2\alpha+1}}\right]$$

Therefore by using the binomial expansion (8), we have the pdf of the k^{th} order QT-MSBL-II random variable $X_{k:m}$ is as follows

$$f_{X_{k:m}}(x) = \frac{m!\ \alpha(\alpha+1)}{(k-1)!\,(m-k)!}\sum_{i,j,s=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}(-1)^{i+j}\lambda^s\left(\frac{1-x}{1+\alpha x}\right)^{\alpha(j+s)}$$
$$*\left[(1-\lambda)\frac{(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}}+2\lambda\frac{(1-x)^{2\alpha-1}}{(1+\alpha x)^{2\alpha+1}}\right] \tag{22}$$

Then, the pdf of the k^{th} order QT-MSBL-II random variable $X_{k:m}$ in Equation (22) can written as

© 2023 NSP
Natural Sciences Publishing Cor.

$$f(x)=\frac{m!\,\alpha(\alpha+1)}{(k-1)!(m-k)!}\left[\sum_{i,j,s=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}(-1)^{i+j}\lambda^{s}(1-\lambda)\frac{(1-x)^{\alpha(j+s+1)-1}}{(1+\alpha x)^{\alpha(j+s+1)+1}}+\right.$$
$$\left.2\sum_{i,j,s=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}(-1)^{i+j}\lambda^{s+1}\frac{(1-x)^{\alpha(j+s+2)-1}}{(1+\alpha x)^{\alpha(j+s+2)+1}}\right] \quad (23)$$

Also, the n^{th} moment for the k^{th} order statistic with pdf $f_{X_{k:m}}(x)$ is given by

$$\mu_{k:m}^{(n)}=\int_{-\infty}^{\infty}x^{n}\,f_{k:m}(x)dx$$

Then , the n^{th} moment for the k^{th} order QT-MSBL-II random variable $X_{k:m}$ can be obtained using Equation (23)

$$\mu_{k:m}^{(n)}=\frac{m!\,\alpha(\alpha+1)}{(k-1)!(m-k)!}\left[\sum_{i,j,s=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}(-1)^{i+j}\lambda^{s}(1-\lambda)\int_0^1 x^{n}\frac{(1-x)^{\alpha(j+s+1)-1}}{(1+\alpha x)^{\alpha(j+s+1)+1}}dx+\right.$$
$$\left.2\sum_{i,j,s=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}(-1)^{i+j}\lambda^{s+1}\int_0^1 x^{n}\frac{(1-x)^{\alpha(j+s+2)-1}}{(1+\alpha x)^{\alpha(j+s+2)+1}}dx\right]$$

By using the binomial expansion (8), we have

$$\mu_{k:m}^{(n)}=\frac{m!\,\alpha(\alpha+1)}{(k-1)!(m-k)!}\left[\begin{array}{l}(1-\lambda)\sum_{i,j,s,l=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}\binom{-\alpha(j+s+1)-1}{l}(-1)^{i+j}\lambda^{s}\alpha^{l}\int_0^1 x^{n+l}(1-x)^{\alpha(j+s+1)-1}dx\\ +2\sum_{i,j,s,v=0}^{\infty}\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}\binom{-\alpha(j+s+2)-1}{v}(-1)^{i+j}\lambda^{s+1}\alpha^{v}\int_0^1 x^{n+v}(1-x)^{\alpha(j+s+2)-1}dx\end{array}\right]$$

which yields the n^{th} moment of $X_{k:m}$ given by

$$\mu_{k:m}^{(n)}=\frac{m!\,\alpha(\alpha+1)}{(k-1)!(m-k)!}\left[\begin{array}{l}(1-\lambda)\sum_{i,j,s,l=0}^{\infty}N_{i,j,s,l}(\lambda,\alpha)\,B\big(n+l+1,\alpha(j+s+1)\big)\\ \quad +2\sum_{i,j,s=0}^{\infty}\gamma_{i,j,s,v}(\lambda,\alpha)B\big(n+v+1,\alpha(j+s+2)\big)\end{array}\right]$$

where $N_{i,j,s,l}(\lambda,\alpha)=\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}\binom{-\alpha(j+s+1)-1}{l}(-1)^{i+j}\lambda^{s}\alpha^{l}$ and

$\gamma_{i,j,s,v}(\lambda,\alpha)=\binom{m-k}{i}\binom{k+i-1}{j}\binom{k+i-1}{s}\binom{-\alpha(j+s+2)-1}{v}(-1)^{i+j}\lambda^{s+1}\alpha^{v}$.

6 Estimation of Parameters

In this section, we describe the maximum likelihood estimators (MLEs) and the observed information matrix of the QT-MSBL-II distribution. Let $X_1,X_2,\dots,X_n$ be an independent random sample from the QT-MSBL-II distribution, then the log-likelihood function is given by

$$l=n\mathrm{Log}[\alpha]+n\mathrm{Log}[\alpha+1]+\sum_{i=1}^{n}\mathrm{Log}\left[1-\lambda+2\lambda\left(\frac{1-x_i}{1+\alpha x_i}\right)^{\alpha}\right]+(\alpha-1)\sum_{i=1}^{n}\mathrm{Log}[1-x_i]-(\alpha+1)\sum_{i=1}^{n}\mathrm{Log}[1+\alpha x_i]$$
(24)

Then

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\frac{n}{\alpha+1}+\sum_{i=1}^{n}\mathrm{Log}[1-x_i]-\sum_{i=1}^{n}\mathrm{Log}[1+\alpha x_i]-(\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\alpha x_i}+$$
$$\sum_{i=1}^{n}\frac{2\lambda\left(\frac{1-x_i}{1+\alpha x_i}\right)^{\alpha}\left(\mathrm{Log}\left[\frac{1-x_i}{1+\alpha x_i}\right]-\frac{\alpha x_i}{1+\alpha x_i}\right)}{1-\lambda+2\lambda\left(\frac{1-x_i}{1+\alpha x_i}\right)^{\alpha}} \quad (25)$$

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\frac{-1+2\left(\frac{1-x_i}{1+\alpha x_i}\right)^{\alpha}}{1-\lambda+2\lambda\left(\frac{1-x_i}{1+\alpha x_i}\right)^{\alpha}} \quad (26)$$

The MLEs $(\hat{\alpha},\hat{\lambda})$ of the parameters (α,λ) are obtained by solving the system of nonlinear Equations (25) and (26). These equations cannot be solved analytically, but can be solved using numerical techniques such as Newton-Raphson method.

© 2023 NSP
Natural Sciences Publishing Cor.

7 Applications

In this section, we use real data sets. The data collected from type 2 diabetic patients on oral hypoglycemic medications (ionized Mg levels) of a random sample of 60 patients given by Walaa Reda Badr et al. [28] as: 0.74, 0.93, 0.799, 0.95, 0.84, 0.815, 0.93, 0.96, 0.815, 0.894, 0.86,0.95, 0.85, 0.68, 0.93, 0.95, 0.77, 0.77, 0.894, 0.62, 0.84, 0.71, 0.91, 0.76, 0.76, 0.77, 0.83, 0.894, 0.776, 0.574, 0.78, 0.87, 0.95, 0.934, 0.776, 0.618, 0.85, 0.91, 0.776, 0.894, 0.736, 0.776, 0.89, 0.62, 0.65, 0.776, 0.58, 0.81, 0.94, 0.73, 0.81, 0.77, 0.89, 0.85, 0.89, 0.934, 0.93, 0.776, 0.76, 0.91.

We use this data set to compare the fit of the new distribution, Quadratic Transmuted modified size-biased Lehmann Type-II distribution (QT-MSBL-II) with Lehmann Type-I and II distributions (L-I-II) (Lehmann [29]). First, we obtain the maximum likelihood estimates (MLEs) for the unknown parameters of each distribution and then compare the results of different criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), CAIC (consistent Akaike information criterion) and BIC (Bayesian information criterion),

where,

$AIC = 2K - 2l$

$\text{AICC} = AIC + \frac{2k(k+1)}{n-k-1},$

$CAIC = \frac{2kn}{n-k-1} - 2l,$

$BIC = k\, log\, n - 2l,$

where l denotes the log – likelihood function evaluated at MLEs, k is the number of parameters and n is the sample size.

The best model corresponds to the lowest AIC, AICC, CAIC and BIC values.

Table 2: MLEs for QT-MSBL-II, L-I, L-II models and the statistics AIC, AICC, CAIC, BIC for the data set

Model	$\hat{\alpha}$	$\hat{\lambda}$	$-\ell$	AIC	AICC	CAIC	BIC
QT-MSBL-II	0.329608	0.35967	-7.40141	-10.8028	-10.5922	-10.5922	-11.2465
L-I	4.82299	------	738.345	1478.69	1478.75	1478.75	1478.468
L-II	0.530826	------	37.2163	76.4326	76.536	76.501	76.2107

The minimum value of the goodness-of-fit is the criteria of the better fit mode that QT-MSBL-II distribution perfectly satisfies. Hence, we support that the QT-MSBL-II distribution is a better fit model among all of its competitors.

8 Simulation

In this section, a Monte Carole simulation that evaluates the MLEs of the QT-MSBL-II distribution by the following algorithm is presented.

Step -1 We generate a random sample of sizes n = 50, 100, 200, 400, and 500, respectively.

Step -2 Each sample is simulated 1000 times and results are listed in Tables 3.

Step -3 Table 3 presents the summarized results of bias and Mean squared error.

Step -4 The required results are obtained based on the different combinations of the model parameters place in SET-1 ($\alpha = 1, \lambda = 1$), SET-2 ($\alpha = 1, \lambda = -1$), SET-3 ($\alpha = 3\,, \lambda = 0.5$), and SET-4 ($\alpha = 0.5\,, \lambda = -0.3$), which are shown in Tables 3.

Step -5 It can be observed from Table 3 that there is a gradual decrease in bias and Mean squared error, with the increases in sample size, respectively.

The measures including bias and Mean squared error are given as follows:

$$\text{Bias}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta) \text{ and } \text{M.S.E}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)^2.$$

© 2023 NSP
Natural Sciences Publishing Cor.

Table 3: Bias and Mean squared error.

SET-1	$(\alpha = 1, \lambda = 1)$					
N	$\hat{\alpha}$	$\hat{\lambda}$	Bias($\hat{\alpha}$)	Bias($\hat{\lambda}$)	M.S.E($\hat{\alpha}$)	M.S.E($\hat{\lambda}$)
50	1.13989	0.803667	0.139888	−0.196333	0.0744805	0.184603
100	1.08668	0.868989	0.08668	−0.13101	0.042988	0.0973812
200	1.05446	0.924913	0.0544587	−0.075087	0.0239708	0.0467917
400	1.03986	0.944435	0.0398578	−0.055565	0.0172618	0.0334385
500	1.0447	0.938066	0.0447021	−0.0619339	0.023843	0.0454068
SET-2	$(\alpha = 1, \lambda = -1)$					
50	0.986414	−0.944316	−0.013586	0.0556837	0.0093107	0.020277
100	0.993006	−0.96459	−0.00699399	0.03540589	0.00422739	0.00836895
200	0.9937	−0.975558	−0.00629527	0.0244417	0.002057	0.0025218
400	0.99472	−0.98603	−0.00527763	0.0139698	0.001012	0.00085
500	0.995359	−0.984758	−0.00464141	0.0152419	0.0007997	0.000904529
SET-3	$(\alpha = 3, \lambda = 0.5)$					
50	2.70896	0.747128	−0.291041	0.247128	0.264761	0.131166
100	2.78981	0.673823	−0.210186	0.173823	0.185555	0.103622
200	2.88248	0.594385	−0.117514	0.0943852	0.151014	0.082047
400	2.84906	0.61598	−0.150938	0.11598	0.12095	0.0695848
500	2.88129	0.596686	−0.118708	0.0966863	0.10485	0.0602663
SET-4	$(\alpha = 0.5, \lambda = -0.3)$					
50	0.506003	−0.300229	0.0060034	−0.000229	0.010928	0.182535
100	0.496222	−0.270325	−0.0037779	0.0296753	0.00702077	0.124515
200	0.501236	−0.301525	0.00123577	−0.0015247	0.003172	0.0529802
400	0.502113	−0.305801	0.0021127	−0.00580139	0.00138801	0.0240069
500	0.501223	−0.304957	0.00122311	−0.0049571	0.00106207	0.0160509

9 Conclusion

In this article, we proposed a new distribution namely the Quadratic Transmuted modified size-biased Lehmann Type-II (QT-MSBL-II) distribution which is considered as a new extension of the modified size-biased Lehmann Type-II distribution. We provide a mathematical treatment of the new distribution including the density, hazard rate functions, quantile function, mode, n^{th} moment, moment generating functions, incomplete moment, Lorenz, Bonferroni and Zenga curves, Rényi entropy and the moments of order statistics. The parameters of the new distribution are estimated by using the method of maximum likelihood. Real data set is applied to demonstrate that the Quadratic Transmuted modified size-biased Lehmann Type-II (QT-MSBL-II) distribution can provide a better fit than the Lehmann Type-I and II distributions.

References

[1] R. D. Gupta and D. Kundu, "Exponentiated exponential family: an alternative to gamma and Weibull" Biometrical Journal, 43, 117-130 (2001).
[2] M. M. Nassar and F. H. Eissa, "On the exponentiated Weibull distribution", Communication in Statistics – Theory and Methods, 32(7),1317-1336 (2003)
[3] M. M. Nassar and F. H. Eissa, "Bayesian estimation for the exponentiated Weibull model", Communications in Statistics – Theory and Methods, 33(10), 2343-2362, (2004)
[4] N. Eugene, C. Lee and F. Famoye, "Beta –normal distribution and its applications", Communications in Statistics – Theory and Methods, 31(4), 497-512, (2002).
[5] M. C. Jones, "Families of distributions arising from distributions of order statistics", Test, 13(1), 1-43, (2004).

© 2023 NSP
Natural Sciences Publishing Cor.

[6] S. Nadarajah and S. Kotz, "The beta Gumbel distribution", Mathematical problems in Engineering, 10, 323-332, (2004).

[7] S. Nadarajah and S. Kotz, "The beta exponential distribution", Reliability Engineering and system Safety, 91, 689-697, (2005).

[8] M. M. Nassar and N. K. Nada, "The beta generalized Pareto distribution", Journal of Statistics - Advances in Theory and Applications, 6, 1-17, (2011).

[9] M. M. Nassar and N. K. Nada, "A new generalization of the exponential geometric distribution", Journal of Statistics - Advances in Theory and Applications, 7, 25-48, (2012).

[10] M. M. Nassar and N. K. Nada, "A new generalization of the Pareto geometric distribution", Journal of Egyption Mathematical Society, 21(2), 148-155, (2013)

[11] M. M. Nassar and A. Elmasry, "A study of generalized Logistic distributions", Journal of Egyption Mathematical Society, 20(2), 126-133, (2012).

[12] M. Mahmoud, M. M. Nassar and M. A. Aefa, "A new generalization of Kumaraswamy Lindley distribution", Journal of Statistics – Advances in Theory and Applications, 14, 69-105, (2015).

[13] G. M. Cordeiro, S. Nadarajah and E. M. M. Ortega, "The Kumaraswamy Gumbel distribution", Statistical Methods & Applications, 21(2), 139-168, (2012).

[14] G. M. Cordeiro, E. M. M. Ortega and S. Nadarajah, "The Kumaraswamy Weibull distribution with application to failure data", Journal of the Franklin Institute, 347(8), 1399-1429, (2010).

[15] G. M. Cordeiro and M. de Castro, "A new family of generalized distribution", Journal of Statistical Computation and Simulation, 81(7), 883-898, (2011).

[16] M. M. Nassar, "The Kumaraswamy Laplace Distribution", Pakistan Journal of Statistics and Operation Research, 12(4), 609-624, (2016).

[17] Al-Shomrani, O. Arif, A. Shawky, S. Hanif and M. Q. Shahbaz, "Topp–Leone family of distributions: some properties and application", Pakistan Journal of Statistics and Operation Research, 12(3), 443-451, (2016).

[18] M. M. Nassar and M. Ibrahem, "A New Generalization of the Moyal Distribution", J.Stat. Appl. Pro. 9, No. 2, 361-378, (2020).

[19] K. Zografos and N. Balakrishnan, "On families of beta-and generalized gamma- generated distributions and associated inference", Statistical Methodology, 6, 344-362, (2009).

[20] W. T. Shaw and I. R. C. Buckley, "The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map", Research report, (2007).

[21] M. Bourguignon, I. Ghosh and G. Cordeiro, "General results for the transmuted family of distributions and new models". Journal of Probability and Statistics, vol. 2016, Article ID 7208425, 12 pages, https://doi.org/10.1155/2016/7208425.(2016).

[22] F. Merovci, "Transmuted Lindley distribution", International Journal of Open Problems in Computer Science and Mathematics, 6, 63-72, (2013).

[23] S. Ashour and M. Eltehiwy, "Transmuted Lomax distribution", American Journal of Applied Mathematics and Statistics, 1(6), 121-127, (2013).

[24] Gharaibeh and Al-Omari, "Transmuted Ishita Distribution and Its Applications", Journal of Statistics Applications & Probability, 8(2), 67-81, (2019).

[25] S. E. F. Lucena, A. H. A. Silva, and G. M. Cordeiro, "The transmuted generalized gamma distribution: Properties and application", Journal of Data Science, 13(2), 409- 420, (2015).

[26] O.S. Balogun, M.Z. Iqbal, A.Z. Afify, P.E. Oguntunde and M.Z. Arshad, "A new generalization of Lehmann type-II distribution: Theory, simulation, and applications to survival and failure rate data", Scientific African, 12, (2021).

[27] Rényi, "On measures of entropy and information, in: Proceedings of the 4th Fourth Berkeley Symposium on Mathematical Statistics and Probability", University of California Press, Berkeley, pp. 547–561, (1961).

[28] Walaa Reda Badr, Nagwa Eid Saad, Marwa Mostafa Ahmed, Moushira H. Ezz El Arab and Samar Fares, "Magnesium Levels In Type 2 Diabetic Patients And Its Correlation With Blood Glycaemia And Inflammation", Journal of Pharmaceutical Negative Results, (2022).

[29] E.L. Lehmann, "The power of rank tests", Ann. Math. Stat., 23–43, (1953).

© 2023 NSP
Natural Sciences Publishing Cor.