Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/12S102

Statistical Inference of Power Hazard Rate Distribution in the Presence of Competing Risks Model with Application

Hassan M. Aljohani

Department of Mathematics & Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

Received: 12 Sep. 2023, Revised: 22 Nov. 2023, Accepted: 27 Nov. 2023
Published online: 1 Dec. 2023

Abstract: Over the last decades, several authors have studied competing strategies to analyse the risk. This article investigates the competing risks model when the failure times have power failure rate distribution. The model is formulated considering that causes of failure are independent, and a sample is obtained in complete form. The parameters are studied for the classical maximum likelihood and Baye's methods. Also, we constructed confidence intervals concerning asymptotic maximum likelihood (ML) confidence intervals and bootstrap confidence intervals. Under the Bayesian approaches, we adopted the Markov chain Monte Carlo (MCMC) method to formulate credible intervals. The performances of estimators are discussed with a real data set and assessed through an extensive simulation study. Finally, the numerical results are examined in the conclusion section.

Keywords: Power hazard rate distribution, Competing risks model, Maximum likelihood estimators, Bootstrap confidence interval, Bayesian estimators, Parameters estimation

Based on the age reached units, the hazard rate function (HRF) in reliability studies measure the propensity to fail or to die. Also, HRF is used to classifying the lifetime distribution and characterizing the process of aging. The FRF in extreme value theory is presented as intensity function describe the force of decrement in actuarial work or force of mortality. In vital statistics, we used term age-specific death rate and Mill's rate in economics to express of FRF. As given by Rinne [1] the FRF at any time in the applications of reliability present the instantaneous failure rate. The power HRF can be selected to characteriz different lifetime distributions, see Mugdadi [2]. Let X be a random variable that has power FR probability density function (PDF) defined as

$$f_X(x) = \beta x^{\alpha} \exp\left\{\frac{-\beta}{\alpha+1} x^{\alpha+1}\right\},\ x>0; \alpha>-1, \beta>0, \tag{1}$$

where α and β are shape and scal parameters, respectively and the notation $f_X(x)$ will be written as $f(x)$ where X will be omitted. The corresponding FRF of power FR distribution is defined as

$$h(x) = \beta x^{\alpha},\ x>0; \alpha>-1, \beta>0. \tag{2}$$

The two-parameters power FR probability density function denoted by PFRD(α, β) has some properties described as

1. The HFR of PFRD is increasing, constant or decreasing depend on the value of the parameter α, Figures 1 present different representation of PDFs and FRFs for $\beta = 2$ and different values of α.
2. The PFRD reduced to, exponential distribution when $\alpha = 0$, Rayleigh distribution when $\alpha = 1$ and $\beta = \frac{1}{\theta}$, linear fialure rate distribution when $\alpha = 1$ and Weibull distribution when $\alpha = \beta - 1$.

Different works presented about PFRD, Mugdadi [2] used least-squares method and Mugdadi and Min [3] used Bayesian approach to estimate the parameters of PFRD. Ismail [4] have discussed the parameter of stress-strength reliability of PFRD. Recently, different modification of PFRD was considered, weighted PFRD discussed by Khan and Mustafa [5] and length-biased PFRD Mustafa and Khan [6].

* Corresponding author e-mail: hmjohani@tu.edu.sa

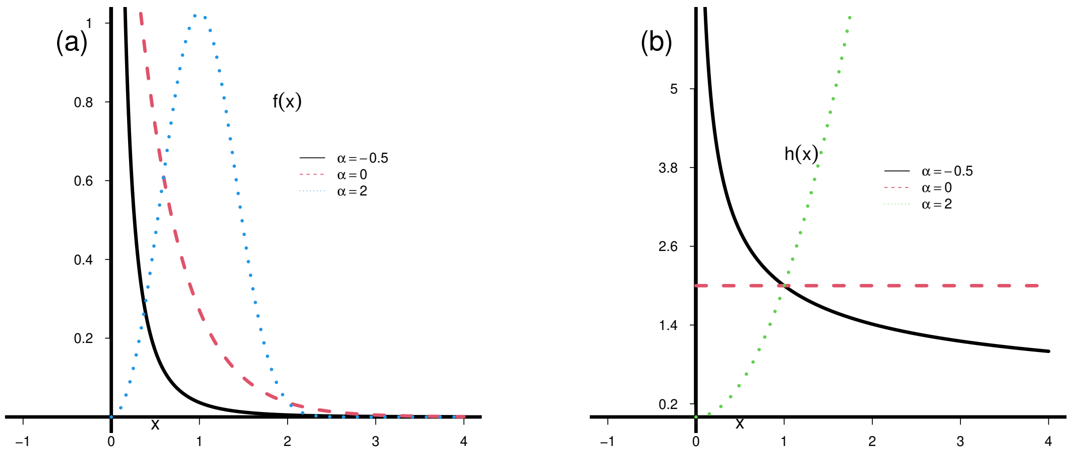

Fig. 1: Plots of PDF and FRFs, when $\beta = 2$.

In reliability analysis, units commonly fail with respect to other causes of failure, which the competing risks model knows. In this model, we aim to measure the effect of the cause respected to the other causes. This model was considered early by Cox [7] and Crowder [8]. The competing risks model discussed for exponential lifetime product by Balakrishnan and Han [9] and Algarni et al. [10] for Nadarajaha and Haghighi lifetime distribution. Also, the competing risks representative discussed under accelerated life tests principal by Ganguly and Kundu [11], Samanta et al [12], Aljohani et al. [13], Almarashi and Abd-Elmougod [14] and Tahani et al. [15].

Developing the statistical inference of competing risks PFR models under complete sample is the major objective of this paper. The results obtained in this paper can be specialized for exponential, Rayleigh, linear failure rate and Weibull distribution. The proposed model and its properties are investigated using different statistical tools. The parameters are estimated using the maximum likelihood and Baye's approaches. The confidence interval estimators of the parameters are formulated with asymptotic maximum likelihood properties, Bootstrap technique, and Baye's credible intervals.

This article is planned as follows: The proposed methodology and its properties are drafted in Section 2. The point estimate with maximum likelihood and Baye's methods is in Section 3. Interval estimation under the asymptotic distribution of maximum likelihood of the parameters estimate, bootstrap technique, and Baye's approach with the MCMC procedure is formulated in Section 4. An application is analysed in Section 5, an investigation of the quality of estimators through the Monte Carlo analysis in Section 6. The conclusion and comment are provided in Section 6.

1 Abbreviations and Model Formulation

1.1 Abbreviations

FRF	Failure rate function.	PFRD	Power failure rate distribution.
PDF	Probability density function	CDF	Cumulative distribution function
$S(.)$	Survival function.	MCMC	Markov chain Monte Carlo.
MH	Metropolis–Hastings.	ACI	Approximate cofidence interval
X_i	The i-th failure time	X_{ij}	The i-th failure time under cause j, $j = 1,2$.
CI	Credible intervals	δ_i	The i-th cause of failure
$F_j(.)$	CDF of X_{ij}.	SEL	Squard error loss.
$S_j(.)$	SF of X_{ij}.	$f_j(.)$	PDF of X_{ij}.

1.2 Model formulation

Suppose that the sample of size n independent units is selected to put under test. Also, the failure is done with respect to two independent causes of failure. Under consideration, only two causes of failure are affected. The failure time X and the corresponding cause of failure δ are recorded during the test, $\delta = \{0,1\}$. The value $\delta = 0$ mean that the failure done with respected the first cause of failure. And the value $\delta = 1$ mean the second failure is observed. Then, data $(X_1,\delta_1) < (X_2,\delta_2) < \cdots < (X_n,\delta_n)$, are called the complete competing risk data. Therefore, the joint concoction between the distribution and the survival function using competing risk data $x = \{(x_1,\delta_1),(x_2,\delta_2),\ldots,(x_n,\delta_n)\}$ is defined as

$$g(x|\theta) = n!\prod_{i=1}^{m}[f_1(x_i)S_2(x_i)]^{\rho(\delta_i=1)}[f_2(x_i)S_1(x_i)]^{\rho(\delta_i=2)}, \tag{3}$$

where $S_j(.) = 1 - F_J(.),\ \ j = 1,2,\ \ 0 < x_1 < x_2 < \cdots < x_m < \infty$, and

$$\rho(\delta_i = j) = \begin{cases} 1, \text{ if } \delta_i = j \\ 0, \text{ if } \delta_i \neq j \end{cases}, j = 1,\ 2, \tag{4}$$

Under consideration that the failure time has PFRD with shape parameter α and scale parameter β, we give the following properties

1.The i-th failure time for $j,\ j = 1,2$ is denoted by X_{ij} has PFRD with parameters α,β_j and PDF defined as

$$f_{j1}(x) = \beta_j x^{\alpha}\exp\left\{\frac{-\beta_j}{\alpha+1}x^{\alpha+1}\right\}, \alpha > -1, \beta_j > 0, \tag{5}$$

and the corresponding CDF defined as

$$F_{j1}(x) = 1 - \exp\left\{\frac{-\beta_j}{\alpha+1}x^{\alpha+1}\right\}, \alpha > -1, \beta_j > 0. \tag{6}$$

Also, the survival functions $S_j(.)$ and HRF $h_{j1}(.)$ at any time $x > 0$, defined as

$$S_{j1}(x) = \exp\left\{\frac{-\beta_j}{\alpha+1}x^{\alpha+1}\right\}\ \ \alpha > -1, \beta_j > 0, \tag{7}$$

and

$$h_{j1}(x) = \beta_j x^{\alpha},\ \alpha > -1, \beta_j > 0. \tag{8}$$

2.The i-th observed failure time $X_i = \min(X_{i1}, X_{i2})$ distributed by $F(.) = F_1(.) + F_2(.) - F_1(.) * F_2(.)$. Therefore, the minimum has PFRD with shape parameter α and scale parameter $\beta_1 + \beta_2$.

3.The number of units fails under the first cause denoted by n_1 and n_2 for the second cause.

1.The integer number n_1 has the binomial distributions with parameters $(n_1, \frac{\beta_2}{\beta_1+\beta_2})$ and n_2 has the binomial distributions with parameters $(n_2, \frac{\beta_1}{\beta_1+\beta_2})$.

2 Parameters Estimation

This section, the maximum likelihood estimation and Baye's methods were assumed to develop the point estimate of model parameters. In Bayesian estimation, we adopted the MCMC method for the parameter estimate.

2.1 Maximum likelihood estimation

The joint likelihood function (3) for given $x = \{(x_1,\delta_1),(x_2,\delta_2),\cdots,(x_n,\delta_n)\}$ and PFRDs given by (5) to (8) can be defined by

$$L(\alpha,\beta_1,\beta_2|x) = \beta_1^{n_1}\beta_2^{n_2}\exp\left\{\alpha\sum_{i=1}^{n}\log x_i - \frac{\beta_1+\beta_2}{\alpha+1}\sum_{i=1}^{n}x_i^{\alpha}\right\}, \tag{9}$$

where $n_1 = \sum_{i=1}^{n}(1-\rho_i)$ and $n_2 = \sum_{i=1}^{n}\rho_i$ be the number of failure under the first and second causes of failure, respectively. By taken the natural logarithm of (9), we obtain to

$$\ell(\alpha,\beta_1,\beta_2|x) = n_1\log\beta_1 + n_2\log\beta_2 + \alpha\sum_{i=1}^{n}\log x_i - \frac{\beta_1+\beta_2}{\alpha+1}\sum_{i=1}^{n}x_i^{\alpha}. \tag{10}$$

The likelihood form received from (10) by taken the first partial derivatives with respect to $\beta_1,\beta_2$and α reduced to

$$\beta_j(\alpha) = \frac{n_j(\alpha+1)}{\sum_{i=1}^{n}x_i^{\alpha}} \tag{11}$$

and

$$\sum_{i=1}^{n}\log x_i + \frac{\beta_1+\beta_2}{(\alpha+1)^2}\sum_{i=1}^{n}x_i^{\alpha} - \frac{\beta_1+\beta_2}{\alpha+1}\sum_{i=1}^{n}x_i^{\alpha}\log x_i = 0. \tag{12}$$

The equation (12) after replacing the value of β_1 and β_2 from (11) is reduced to

$$\sum_{i=1}^{n}\log x_i + (n_1+n_2)\left(\frac{1}{\alpha+1} - \frac{\sum_{i=1}^{n}x_i^{\alpha}\log x_i}{\sum_{i=1}^{n}x_i^{\alpha}}\right) = 0. \tag{13}$$

The maximum likelihood estimate of α, say $\hat{\alpha}$ is obtained from (13) by using Newton Raphson or fixed point iterations. And hence, the maximum likelihood estimate of β_1 and β_2, say $\hat{\beta}_1$ and $\hat{\beta}_2$ are obtained from (11).

Remark1:

The initial value needed in the iteration process can be obtained from the profile of the log-likelihood function delivered by

$$z(\alpha) = (n_1+n_2)\log(\alpha+1) - 2\log\sum_{i=1}^{n}x_i^{\alpha} + \alpha\sum_{i=1}^{n}\log x_i + n_1\log n_1 + n_2\log n_2 - (n_1+n_2). \tag{14}$$

Remark 2:

The equations from (11) and (13) have shown, discrete random variable n_1 and n_2 determine the conditional estimators of the model parameters. The estimate value $\hat{\beta}_1$ or $\hat{\beta}_2$ for any zero integer n_1 or n_2 does not exist. Also, as given by Kundu and Joarder [16] the exact distributions for estimators $\hat{\beta}_1$ or $\hat{\beta}_2$ are difficult to obtain which is defined as mixture of discrete and continuous distributions.

2.2 Baye's estimation

Statistical inference under the Bayesian approach depends on prior information about the model parameters and information expressed by the likelihood role. Then, we propose independent gamma priors for all model parameters as follows

$$\alpha \to \text{Gamma}(a,b), \beta_1 \to \text{ Gamma}(a_1,b_1), \text{ and } \beta_2 \to \text{Gamma}(a_2,b_2). \tag{15}$$

Therefore, the joint posterior function obtained from prior distribution (15) and likelihood function (9) is given by

$$\pi(\alpha,\beta_1,\beta_2|x) \propto \beta_1^{n_1+a_1-1}\beta_2^{n_2+a_2-1}\alpha^{a-1}\exp\left\{-b\alpha - b_1\beta_1 - b_2\beta_2 + \alpha\sum_{i=1}^{n}\log x_i - \frac{\beta_1+\beta_2}{\alpha+1}\sum_{i=1}^{n}x_i^{\alpha}\right\}. \tag{16}$$

Also, corresponding to posterior distribution (16) the Baye's estimate under squared error loss function of the function $\varphi(\alpha,\beta_1,\beta_2)$ given by

$$\hat{\varphi}_B(\alpha,\beta_1,\beta_2) = E_\pi(\varphi(\alpha,\beta_1,\beta_2)) = \int_{-1}^{\infty}\int_0^{\infty}\int_0^{\infty}\varphi(\alpha,\beta_1,\beta_2)\,\pi(\alpha,\beta_1,\beta_2|x)\,d\alpha d\beta_1 d\beta_2. \tag{17}$$

Two functions (16) and (17) have shown that, the closed form of posterior distribution and Baye's estimators can not obtained in the closed form. Different approximate methods can be used such numerical integration, Lindely approximations or Markov chen Monte Carlo (MCMC) method. We employed the MCMC method as follows.

The conditional posterior density functions obtained from the joint posterior density function (16) described by

$$\beta_1 \rightarrow \text{Gamma}\left(n_1+a_1, b_1+\frac{1}{\alpha+1}\sum_{i=1}^{n} x_i^{\alpha}\right), \tag{18}$$

$$\beta_2 \rightarrow \text{Gamma}\left(n_2+a_2, b_2+\frac{1}{\alpha+1}\sum_{i=1}^{n} x_i^{\alpha}\right), \tag{19}$$

and

$$\alpha \propto \alpha^{a-1}\exp\left\{-b\alpha+\alpha\sum_{i=1}^{n}\log x_i-\frac{\beta_1+\beta_2}{\alpha+1}\sum_{i=1}^{n}x_i^{\alpha}\right\}. \tag{20}$$

The plot of the function (20) is equivalent to the normal distribution. Therefore, Metropolis–Hastings (MH) algorithms are more appropriate for generation from (20) with normal proposal distribution, Metropolis et al. [17]. In the following, we describe the algorithms used to approximate posterior distribution and hence Baye's estimate of the model parameters under the MCMC approach as follows

Algorithm 1: (Simulate the empirical posterior distribution)

Step 1:Place $\kappa = 1$ and initial guises values $\{\alpha^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}\} = \{\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2\}$.
Step 2:Generate $\beta_1^{(\kappa)}$ from gamma distribution (18).
Step 3:Generate $\beta_2^{(\kappa)}$ from gamma distribution (19).
Step 4:Generate $\alpha^{(\kappa)}$ under MH algorithms with normal proposal distribution.
Step 5:Place $\kappa = \kappa + 1$.
Step 6:Repeated the steps from (2) to (5) MC times.
Step 7:The Baye's estimate of the function $\varphi(\alpha, \beta_1, \beta_2)$, mean the parameter or any function of the parameters by

$$\hat{\varphi}_{\mathrm{B}} = \frac{1}{MC - MC^{*}}\sum_{j=MC^{*}+1}^{MC}\varphi\left(\alpha^{(i)}, \beta_1^{(i)}, \beta_2^{(i)}\right), \tag{21}$$

where MC^{*} is the number of iteration to reached the stationary distribution.
Step 8:The corresponding posterior variance equals

$$\mathrm{V}(\varphi(\alpha, \beta_1, \beta_2)) = \frac{1}{MC - MC^{*}}\sum_{j=MC^{*}+1}^{MC}\left(\varphi\left(\alpha^{(i)}, \beta_1^{(i)}, \beta_2^{(i)}\right) - \hat{\varphi}_{\mathrm{B}}\right)^2. \tag{22}$$

3 Interval Estimations

In this section, we adopt different approaches to obtain the confidence interval of the model parameters. First, we consider asymptotic confidence intervals, and second, bootstrap confidence intervals adopted. Finally, we assume Baye's credible intervals.

3.1 Asymptotic confidence interval

From the log-likelihood function (10), the second partial derivative with respected to model parameters given by

$$\frac{\partial^2 \ell(\alpha, \beta_1, \beta_2|x)}{\partial \alpha^2} = \frac{\beta_1+\beta_2}{\alpha+1}\left[\frac{-2}{(\alpha+1)^2}\sum_{i=1}^{n}x_i^{\alpha} + \frac{2}{(\alpha+1)}\sum_{i=1}^{n}x_i^{\alpha}\log x_i - \sum_{i=1}^{n}x_i^{\alpha}(\log x_i)^2\right], \tag{23}$$

$$\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_1^2} = \frac{-n_1}{\beta_1^2}, \tag{24}$$

$$\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_2^2} = \frac{-n_2}{\beta_2^2}, \tag{25}$$

$$\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha \partial \beta_1} = \frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_1 \partial \alpha} = \frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_2 \partial \alpha} = \frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha \partial \beta_2} = \frac{1}{(\alpha+1)^2}\sum_{i=1}^{n} x_i^{\alpha} - \frac{1}{\alpha+1}\sum_{i=1}^{n} x_i^{\alpha}\log x_i, \tag{26}$$

and

$$\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_2 \partial \beta_1} = 0. \tag{27}$$

Generally, the minus expectation of the second partial derivative of the log-likelihood function represented the Fisher information matrix. In several situations, especially cases where the model has a high length of the parameters vector replaced by an approximate information matrix. Hence, we use the approximate information matrix defined by

$$\Omega(\alpha,\beta_1,\beta_2) = \begin{pmatrix} -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha^2} & -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha \partial \beta_1} & -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha \partial \beta_2} \\ -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha \partial \beta_1} & -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_1^2} & -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_1 \partial \beta_2} \\ -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \alpha \partial \beta_2} & -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_1 \partial \beta_2} & -\frac{\partial^2 \ell(\alpha,\beta_1,\beta_2|x)}{\partial \beta_2^2} \end{pmatrix}. \tag{28}$$

The value of approximate information matrix at the maximum likelihood estimate $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$ is denoted by $\Omega_0(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$. Under the property that, the maximum likelihood estimate $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$ have bivariate normal distribution given by

$$\left(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2\right) \longrightarrow \mathrm{N}\left((\alpha_1,\alpha_2,\beta,\ \gamma), \Omega_0^{-1}(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)\right). \tag{29}$$

The 100(1-γ)% approximate confidence intervals of the model parameters (α,β_1,β_2) are defined as

$$\begin{cases} \hat{\alpha} \mp z_{\gamma/2}\sqrt{e_{11}}, \\ \hat{\beta}_1 \mp z_{\gamma/2}\sqrt{e_{22}}, \\ \hat{\beta}_2 \mp z_{\gamma/2}\sqrt{e_{33}}, \end{cases} \tag{30}$$

where, e_{11}, e_{22} and e_{33} are the element of diagonal for Ω_0^{-1} and the notation $z_{\gamma/2}$ is the standard normal with tail $\gamma/2$.

3.2 Percentile bootstrap confidence intervals

Bootstrap techniques are commonly used in statistical publications, not only in parameter assessment. However, it can be used to calibrate hypothesis tests or to estimate bias and variance for the estimator. Early, the parametric and non-parametric bootstrap methods introduced by Davison and Hinkley [18] and Efron and Tibshirani[19]. Moreover, the parametric percentile bootstrap approach adopted to formulate the confidence intervals of the proposed method parameters, see Hall [20] as follows

Algorithm 2: (Percentile bootstrap confidence intervals)

Step 1:Let the competing risks data $x = \{(x_1,\delta_1),(x_2,\delta_2),\cdots,(x_n,\delta_n)\}$, compute the maximum likelihood estimate $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$.

Step 2:Generate a random sample of size n from PFRD with parameters $\hat{\alpha}$ and $\hat{\beta}_1 + \hat{\beta}_2$ denoted by $x^* = x_1^*, x_2^*, \ldots, x_n^*\}$.

Step 3:Generate n_1^* and n_2^* from the binomial distributions with parameters $(n_1, \frac{\beta_2}{\beta_1+\beta_2})$ and $(n_2, \frac{\beta_1}{\beta_1+\beta_2})$, respectively.

Step 4:For given $x^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ and the integers n_1^* and n_2^* compute $\hat{\alpha}^*, \hat{\beta}_1^*$ and $\hat{\beta}_2^*$.

Step 5:Steps from 2 to 4 are repeated M times, we obtain
$(\hat{\alpha}^{*(1)}, \hat{\beta}_1^{*(1)}, \hat{\beta}_2^{*(1)}), (\hat{\alpha}^{*(2)}, \hat{\beta}_1^{*(2)}, \hat{\beta}_2^{*(2)}), \ldots, (\hat{\alpha}^{*(M)}, \hat{\beta}_1^{*(M)}, \hat{\beta}_2^{*(M)})$.

Table 1: Time-to failure of male mice which received a radiation dose at age 5-6 weeks.

Thymic lymphoma	158	192	193	194	195	202	212	215	229	230	237
	240	244	247	259	300	301	321	337	415	434	444
	485	496	529	537	624	707	800				
Reticulum cell sarcoma	430	590	606	638	655	679	691	693	696	747	752
	760	778	821	986							
Other cases	136	246	255	376	421	565	616	617	652	655	658
	660	662	675	681	734	736	737	757	769	777	800
	807	825	855	857	864	868	870	873	882	895	910
	934	942	1015	1019							

Table 2: Points estimated and 95% interval using ML, boot, and Baye's estimate.

Pa.	$(.)_{\text{ML}}$	$(.)_{\text{B-MCMC}}$	95% ACI	95% boot	95% CI
α	1.9137	1.9092	(1.438, 2.3894)	(1.3624, 2.5421)	(1.542, 2.3472)
β_1	3.746	3.7663	(2.1284, 5.3644)	(2.0745, 5.5412)	(2.1124, 5.3264)
β_2	3.2356	3.2518	(1.7875, 4.6836)	(1.7054, 4.8452)	(1.7942, 4.6564)

Step 6:The bootstrap sample estimate put in order
$(\hat{\alpha}^*_{(1)},\hat{\beta}^*_{1(1)},\hat{\beta}^*_{2(1)}),(\hat{\alpha}^*_{(2)},\hat{\beta}^*_{1(2)},\hat{\beta}^*_{2(2)}),\ldots,(\hat{\alpha}^*_{(M)},\hat{\beta}^*_{1(M)},\hat{\beta}^*_{2(M)})$.

Step 7:The $100(1-\gamma)\%$ percentile bootstrap confidence intervals of the proposed method parameters are given by

$$(\hat{\alpha}^*_{\text{boot}(\frac{\gamma}{2})},\ \hat{\alpha}^*_{\text{boot}(1-\frac{\gamma}{2})}),\ (\hat{\beta}^*_{1\text{boot}(\frac{\gamma}{2})},\ \hat{\beta}^*_{1\text{boot}(1-\frac{\gamma}{2})}),$$

and

$$(\hat{\beta}^*_{2\text{boot}(\frac{\gamma}{2})},\ \hat{\beta}^*_{2\text{boot}(1-\frac{\gamma}{2})}). \tag{31}$$

3.3 Bayesian Credible interval

From the Algorithms 1,

Step 1:The value
$(\alpha^{(MC^*+1)},\beta_1^{(MC^*+1)},\beta_2^{(MC^*+1)}),(\alpha^{(MC^*+2)},\ \beta_1^{(MC^*+2)},\beta_2^{(MC^*+2)}),\ldots,(\alpha^{(MC)},\beta_1^{(MC)},\beta_2^{(MC)})$ generated by MCMC method put in a ascending order as
$(\alpha_{(1)},\beta_{1(1)},\beta_{2(1)}),(\alpha_{(2)},\beta_{1(2)},\beta_{2(2)}),\ldots,(\alpha_{(MC-MC^*)},\beta_{1(MC-MC^*)},\beta_{2(MC-MC^*)})$.

Step 2:$100(1-\gamma)\%$ credibal intervals of the model parameters are obtained by

$$(\alpha_{(MC-MC^*)(\frac{\gamma}{2})},\ \alpha_{(MC-MC^*)(1-\frac{\gamma}{2})}), \tag{32}$$

$$(\beta_{1(MC-MC^*)(\frac{\gamma}{2})},\ \beta_{1(MC-MC^*)(1-\frac{\gamma}{2})}), \tag{33}$$

and

$$(\beta_{2(MC-MC^*)(\frac{\gamma}{2})},\ \beta_{2(MC-MC^*)(1-\frac{\gamma}{2})}). \tag{34}$$

4 Numerical Discussion

In this section, we analysis the set of real data obtained from the laboratory for illustration purposes. Also, we consider the Monte Carlo simulation study to test and compare the estimators.

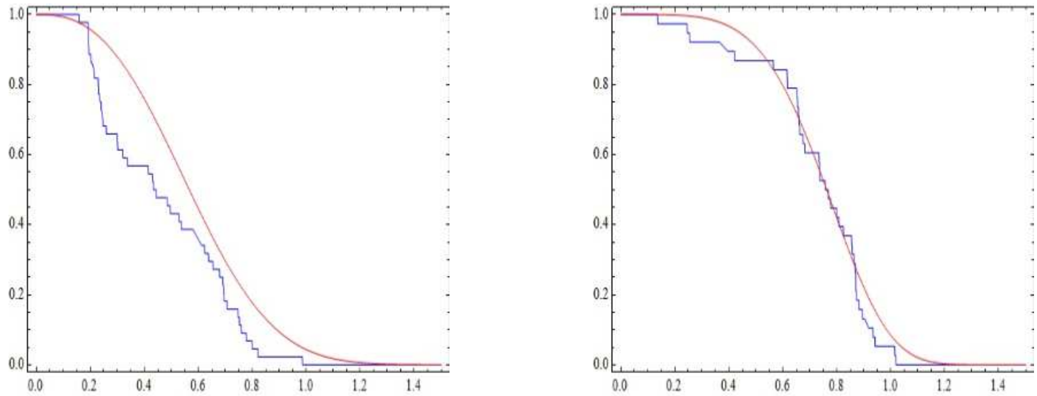

Fig. 2: Plots of the empirical survival functions with fitted survival functions.

4.1 Data analysis

We selected the real data set given by Hoel [21] from a laboratory experiment. The data obtained from male mice lived in a germ-free environment–see Table 1. Two independent causes of failure are considered; the first cause is Thymic Lymphoma with Reticulum cell sarcoma, and the second is other causes. The developed results in this paper are applied to this data for illustration purposes; for more detail, see Koley and Kundu [22]. For simplicity, the data are separated by 1000. The fact of modeling these data by PFRD is checked by plotting the empirical survival functions with fitted survival functions; see Figure 2. The Kolmogorov-Smirnov (K-S) for the fit data satisfies distances between the practical distribution functions and the fitted distribution functions 0.3981 and 0.102. The transformed data are accepted to be from PFRDs. The ML estimate and Baye's estimate with non-informative prior $a = a_1 = a_2 = b = b_1 = b_2 = 0.0001$, and he results demonstrated in Table 2 for point and 95% interval estimate. For the MCMC method in the Baye's approach, the chan is run at 11000 with the first 1000 value as brun-in. The observed posterior distribution under the MCMC method is illustrated in Figures 3 and 5.

4.2 Monte Carlo Studying

The generated results in this article are tested and compared through formulating the Monte Carlo simulation investigation. Throughout this study, we adopt different sample sizes and different parameter values. In the Bayesian approach, we adopt informative and non-informative prior information. The simulation results depend on 1000 different samples, and each sample computes the point and interval estimate. Therefore, we adopted two sets of the parameter values, $(\alpha, \beta_1, \beta_2) = \{(2.0, 2.0, 2.5), (0.5, 0.5, 0.8)\}$. The non-informative prior information denoted by P^0=$\{0.0001, 0.0001, 0.0001\}$. For the first set of the values of the parameters, consider prior information P^1=$\{(3,3), (2, 2.5), (3, 2)\}$and for the second set of the values of the parameters, consider the prior information $P^2 = \{(2, 1.5), (1.5, 2), (2, 3)\}$. Through this study, we study the effect of change, sample size, parameter values, and prior information. The point estimate was tested by computing the mean estimate (ME) and mean squared error (MSE). Interval estimate tested under computing mean interval length (MIL) and coverage percentage (CP). The algorithm used to describe the Monte Carlo study is described as follows

Algorithm 3: (Monte Carlo studying)

Step 1:Create a random sample of size n from PFRD with parameters α and $\beta_1 + \beta_2$.

Step 2:Generate n_1 and n_2 from the binomial distributions with parameters $(n_1, \frac{\beta_2}{\beta_1+\beta_2})$ and $(n_2, \frac{\beta_1}{\beta_1+\beta_2})$, respectively.

Step 3:Compute the point estimate $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$.

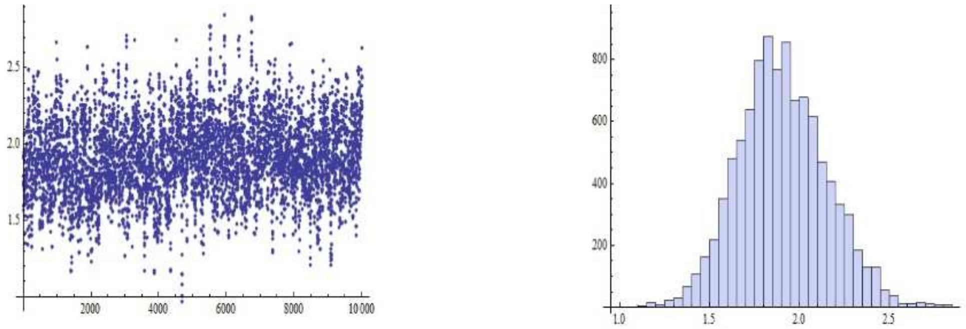

Fig. 3: Plots of monitoring and the corresponding histogram of α created by the MCMC procedure.

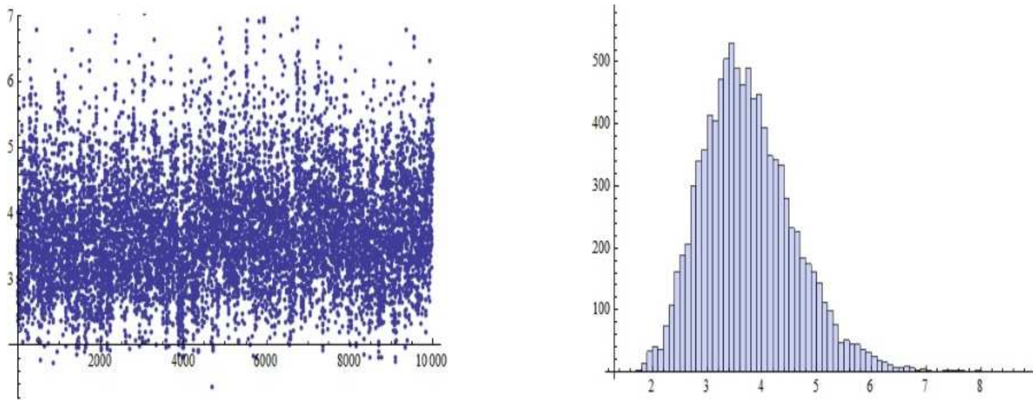

Fig. 4: Plots of monitoring and the corresponding histogram of β_1 created by the MCMC procedure.

Step 4:Compute the interval estimate of the parameters α, β_1 and β_2
Step 5:Repeated the Steps from 1 to 4 by 1000 iteration.
Step 6:Calculate the values of each MEs, MSEs, MILs and PCs and the results reported in Tables 3-6.

5 Conclusions

The competing risk models are one of the most critical problems in life test experiments. In this paper, we are dealing with this problem under consideration of failure for independent causes. We considered the failure time of the units with

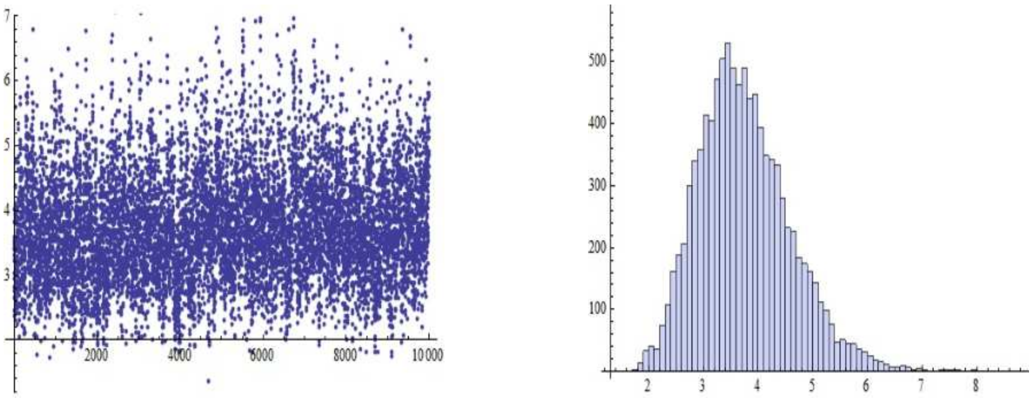

Fig. 5: Plots of monitoring and the corresponding histogram of β_2 created by MCMC procedure.

Table 3: The MEs and MSEs of point estimates with values $(\alpha, \beta_1, \beta_2)$ =(2.0, 2.0, 2.5)

n		MLE			Bayes(P^0)			Bayes(P^1)		
		α	β_1	β_2	α	β_1	β_2	α	β_1	β_2
15	MEs	2.435	2.354	2.754	2.411	2.315	2.724	2.284	2.227	2.664
	MSEs	0.457	0.388	0.487	0.435	0.362	0.461	0.274	0.225	0.288
25	MEs	2.408	2.315	2.718	2.389	2.291	2.703	2.255	2.211	2.625
	MSEs	0.402	0.324	0.414	0.388	0.304	0.411	0.218	0.189	0.242
35	MEs	2.352	2.280	2.671	2.318	2.242	2.651	2.202	2.171	2.591
	MSEs	0.315	0.240	0.311	0.300	0.208	0.325	0.151	0.102	0.182
45	MEs	2.250	2.215	2.624	2.215	2.200	2.601	2.184	2.115	2.525
	MSEs	0.270	0.188	0.260	0.235	0.162	0.237	0.114	0.082	0.132
55	MEs	2.211	2.175	2.600	2.187	2.175	2.588	2.115	2.087	2.511
	MSEs	0.215	0.125	0.195	0.202	0.114	0.189	0.089	0.065	0.107
65	MEs	2.182	2.143	2.589	2.155	2.132	2.551	2.101	2.055	2.495
	MSEs	0.185	0.101	0.182	0.174	0.095	0.165	0.071	0.047	0.093
75	MEs	2.152	2.115	2.555	2.121	2.107	2.522	2.077	2.024	2.496
	MSEs	0.144	0.091	0.155	0.132	0.084	0.142	0.056	0.038	0.077
85	MEs	2.137	2.100	2.536	2.104	2.081	2.502	2.066	2.013	2.504
	MSEs	0.127	0.082	0.133	0.118	0.076	0.125	0.047	0.032	0.071
100	MEs	2.115	2.091	2.522	2.088	2.076	2.503	2.049	2.011	2.498
	MSEs	0.120	0.078	0.124	0.110	0.071	0.114	0.042	0.028	0.069

PFRD with standard shape parameters and different scale parameters. The model parameters are computed by maximum likelihood and Baye's methods for point and interval estimators. Also, we considered interval estimation under the bootstrap technique. The numerical results show that the suggested form and the analysis method serve very well. We observe some points from the results presented in the data analysis and Monte Carlo simulation study.

1. The proposed approach and the corresponding estimation techniques serve well for all cases.
2. The results get better for the larger value of sample size n.
3. The results under maximum likelihood and non-informative Baye's method are closed.
4. The Baye's and MCMC methods with informative prior information provides more accurate results than the MLE and bootstrap procedures.

Table 4: The MILs and CPs of 95% confidence interval with values $(\alpha, \beta_1, \beta_2)$ =(2.0, 2.0, 2.5)

n		MLE			Boot			Bayes(P^0)			Bayes(P^1)		
		α	β_1	β_2	α	β_1	β_2	α	β_1	β_2	α	β_1	β_2
15	MEs	3.584	3.897	4.214	4.784	3.984	4.587	3.571	3.882	4.201	3.287	3.624	4.044
	MSEs	0.89	0.90	0.88	0.91	0.90	0.89	0.91	0.90	0.92	0.92	0.90	0.91
25	MEs	3.528	3.842	4.061	4.727	3.915	4.531	3.525	3.841	4.151	3.235	3.600	4.007
	MSEs	0.90	0.91	0.91	0.91	0.92	0.91	0.92	0.93	0.92	0.92	0.92	0.94
35	MEs	3.501	3.814	4.035	4.703	3.879	4.502	3.492	3.812	4.118	3.203	3.579	3.982
	MSEs	0.92	0.91	0.92	0.92	0.92	0.93	0.92	0.94	0.94	0.92	0.92	0.96
45	MEs	3.451	3.769	3.981	4.645	3.802	4.452	3.422	3.756	4.071	3.144	3.525	3.929
	MSEs	0.92	0.93	0.93	0.92	0.94	0.93	0.92	0.94	0.93	0.92	0.94	0.94
55	MEs	3.412	3.731	3.947	4.614	3.765	4.417	3.389	3.715	4.028	3.103	3.491	3.900
	MSEs	0.93	0.92	0.93	0.93	0.94	0.92	0.92	0.92	0.94	0.92	0.95	0.97
65	MEs	3.325	3.640	3.900	4.671	3.701	4.362	3.317	3.669	3.981	3.024	3.418	3.824
	MSEs	0.92	0.93	0.94	0.93	0.93	0.92	0.95	0.92	0.94	0.92	0.95	0.93
75	MEs	3.301	3.636	3.882	4.624	3.652	4.315	3.289	3.625	3.968	2.892	3.391	3.755
	MSEs	0.91	0.92	0.94	0.93	0.94	0.92	0.95	0.94	0.95	0.92	0.93	0.95
85	MEs	3.252	3.600	3.841	4.690	3.611	4.289	3.231	3.592	3.919	2.844	3.345	3.718
	MSEs	0.93	0.93	0.94	0.93	0.93	0.92	0.95	0.94	0.93	0.92	0.93	0.93
100	MEs	3.214	3.571	3.811	4.661	3.572	4.244	3.200	3.541	3.865	2.813	3.312	3.688
	MSEs	0.94	0.93	0.93	0.93	0.93	0.93	0.95	0.93	0.93	0.94	0.93	0.95

Table 5: The MEs and MSEs of point estimates with values $(\alpha, \beta_1, \beta_2)$ =(0.5, 0.5, 0.8).

n		MLE			Bayes(P^0)			Bayes(P^2)		
		α	β_1	β_2	α	β_1	β_2	α	β_1	β_2
15	MEs	0.745	0.874	1.153	0.715	0.849	1.135	0.688	0.745	1.032
	MSEs	0.081	0.088	0.125	0.077	0.082	0.115	0.052	0.063	0.081
25	MEs	0.718	0.841	1.119	0.685	0.821	1.108	0.661	0.718	1.005
	MSEs	0.076	0.082	0.119	0.070	0.077	0.104	0.046	0.057	0.075
35	MEs	0.681	0.803	1.069	0.632	0.682	1.070	0.618	0.685	0.950
	MSEs	0.071	0.078	0.113	0.065	0.071	0.092	0.041	0.051	0.069
45	MEs	0.649	0.772	1.035	0.600	0.651	1.038	0.592	0.641	0.921
	MSEs	0.064	0.071	0.104	0.058	0.066	0.084	0.036	0.047	0.064
55	MEs	0.611	0.735	1.001	0.562	0.615	1.000	0.551	0.614	0.886
	MSEs	0.060	0.065	0.098	0.053	0.061	0.078	0.029	0.041	0.055
65	MEs	0.582	0.682	0.981	0.514	0.587	0.947	0.525	0.571	0.844
	MSEs	0.049	0.053	0.075	0.044	0.051	0.072	0.022	0.031	0.043
75	MEs	0.555	0.661	0.945	0.508	0.544	0.918	0.517	0.552	0.812
	MSEs	0.039	0.044	0.062	0.031	0.042	0.061	0.014	0.019	0.035
85	MEs	0.527	0.612	0.902	0.502	0.514	0.887	0.489	0.525	0.798
	MSEs	0.032	0.036	0.049	0.024	0.035	0.055	0.012	0.014	0.028
100	MEs	0.514	0.552	0.884	0.495	0.504	0.852	0.495	0.514	0.821
	MSEs	0.025	0.028	0.041	0.021	0.024	0.045	0.008	0.009	0.019

References

[1] H. Rinne, *The hazard rate: theory and inference*, Justus-Liebig University Press, Germany,(2014).

[2] A. R. Mugdadi, The least squares type estimation of the parameters in the power hazard function, *Applied Mathematics Computation*, 169, 737–748,(2005).

[3] A.R. Mugdadi, Bayes estimation of the power hazard function, *Journal of Interdisciplinary Mathematics*, 12(5), 675-689, (2009).

[4] K. Ismail, Estimation of $P(X < Y)$ for distribution having power hazard function, *Pakistan Journal of Statistics*, 30, 57-70, (2014).

[5] M. I. Khan1 and A. Mustafa, Some properties of the weighted power hazard rate distribution with application, *Pakistan Journal of Statistics*, 38, 219-234, (2022).

[6] A. Mustafa and M. I. Khan1, The length-biased power hazard rate distribution: Some properties and applications, *Statistics in Transition*, 23, 1–16, (2022).

Table 6: The MILs and CPs of 95% confidence interval with values $(\alpha, \beta_1, \beta_2)$ =(0.5, 0.5, 0.8).

n		MLE			Boot			Bayes(P^0)			Bayes(P^2)		
		α	β_1	β_2	α	β_1	β_2	α	β_1	β_2	α	β_1	β_2
15	MEs	1.245	1.356	1.754	1.421	1.542	1.952	1.228	1.341	1.725	1.121	1.157	1.421
	MSEs	0.90	0.88	0.90	0.91	0.92	0.89	0.91	0.91	0.92	0.92	0.92	0.90
25	MEs	1.201	1.312	1.708	1.382	1.497	1.905	1.182	1.303	1.671	1.088	1.114	1.371
	MSEs	0.90	0.91	0.92	0.91	0.90	0.91	0.93	0.92	0.92	0.97	0.90	0.93
35	MEs	1.152	1.281	1.655	1.329	1.442	1.871	1.135	1.261	1.619	1.031	1.061	1.319
	MSEs	0.93	0.91	0.93	0.93	0.90	0.91	0.92	0.94	0.92	0.93	0.95	0.93
45	MEs	1.113	1.251	1.614	1.300	1.409	1.828	1.100	1.232	1.589	0.989	1.025	1.282
	MSEs	0.90	0.93	0.93	0.92	0.90	0.92	0.92	0.94	0.95	0.93	0.94	0.92
55	MEs	1.081	1.174	1.514	1.215	1.374	1.741	1.070	1.165	1.458	0.920	0.974	1.214
	MSEs	0.93	0.91	0.93	0.94	0.92	0.92	0.96	0.95	0.95	0.95	0.94	0.96
65	MEs	1.049	1.131	1.475	1.181	1.328	1.705	1.035	1.118	1.412	0.880	0.932	1.159
	MSEs	0.92	0.92	0.93	0.94	0.94	0.92	0.93	0.95	0.93	0.93	0.94	0.92
75	MEs	1.017	1.101	1.428	1.151	1.281	1.650	1.001	1.071	1.380	0.824	0.889	1.111
	MSEs	0.93	0.90	0.93	0.93	0.94	0.94	0.93	0.95	0.94	0.93	0.94	0.95
85	MEs	0.987	1.024	1.375	1.100	1.224	1.611	0.965	1.024	1.332	0.789	0.841	1.059
	MSEs	0.93	0.92	0.94	0.93	0.93	0.94	0.92	0.94	0.93	0.93	0.92	0.94
100	MEs	0.941	0.987	1.311	1.036	1.171	1.581	0.915	0.974	1.274	0.726	0.800	1.014
	MSEs	0.94	0.93	0.94	0.94	0.96	0.94	0.93	0.94	0.94	0.93	0.94	0.95

[7] D. R. Cox, The analysis of exponentially distributed lifetimes with two types of failures", *Journal of the Royal Statistical Society*, 21, 411-421, (1959).

[8] M. J. Crowder, *Classical competing risks*, Chapman and Hall, London UK, (2001).

[9] N. Balakrishnan and D. Han, Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring, *Journal of Statistical Planning and Inference*, 138, 4172-4186, (2008).

[10] Ali Algarni, Abdullah M. Almarashi, G. A. Abd-Elmougod, Statistical analysis of competing risks lifetime data from Nadarajaha and Haghighi distribution under type-II censoring, *Journal of Intelligent and Fuzzy Systems*, 38, 2591-2601, (2020).

[11] A. Ganguly and D. Kundu, Analysis of simple step-stress model in presence of competing risks, *Journal of Statistical Computation and Simulation*, 86, 1989-2006, (2016).

[12] D. Samanta, A. Gupta and D. Kundu, Analysis of Weibull step-stress sodel in presence of competing risk. *IEEE Transactions on Reliability*, 64, 420-438, (2019).

[13] H. M. Aljohani and N. M. Alfar, Estimations with step-stress partially accelerated life tests for competing risks Burr XII lifetime model under type-II censored data, *Alexandria Engineering Journal*, 59, 1171-1180, (2020).

[14] Abdullah M. Almarashi and G. A. Abd-Elmougod, Accelerated Competing Risks Model from Gompertz Lifetime Distributions with Type-II Censoring Scheme. *Thermal Science*, 24, 165-S175, (2021).

[15] T. A. Abushal, A. A. Soliman and G. A. Abd-Elmougod, Inference of partially observed causes for failure of Lomax competing risks model under type-II generalized hybrid censoring scheme, *Alexandria Engineering Journal*, 61, 5427-5439, (2021).

[16] D. Kundu and A. Joarder, Analysis of type-II progressively hybrid censored data, *Computational Statistics and Data Analysis*, 50(10), 2509–2528, (2006).

[17] N. Metropolis, A. W. Rosenbluth, M.N. Rosenbluth, A. H. Teller and E. Teller, Equations of state calculations by fast computing machines. *Journal Chemical Physics*, 21, 1087–1091, (1953).

[18] A. C. Davison and D. V. Hinkley, *Bootstrap Methods and their Applications*, 2nd, Cambridge University Press, Cambridge United Kingdom, (1997).

[19] B. Efron and R. J. Tibshirani, *An introduction to the bootstrap*, New York Chapman and Hall, (1993).

[20] P. Hall, Theoretical comparison of bootstrap confidence intervals, *Annals of Statistics*, vol. 16, 927-953, (1988).

[21] D. G. Hoel, A Representation of Mortality Data by Competing Risks, *Biometrics*, 28(2), 475-488, (1972).

[22] A. Koley and D. Kundu, On generalized progressive hybrid censoring in presence of competing risks, *Metrika*, 80, 401–426, (2017).