Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180104

Novel Fractional Inequalities of Opial's Type Via Conformable Calculus

M. R. Kenawy

Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt

Received: 2 Sep. 2023, Revised: 2 Oct. 2023, Accepted: 22 Oct. 2023
Published online: 1 Jan. 2024

Abstract: In this study, we employ Hölder's inequality and the chain rule formula to deduce some Opial fractional inequalities. As a result, for $\alpha = 1$, we derive numerous classical Opial type inequalities.

Keywords: Opial inequality, Chain rule, Conformable calculus, Hölder inequality

1 Introduction

Opial in [1], showed that if $\Psi \in C^1[\varsigma_1, \varsigma_2]$ where $\Psi(\xi) > 0$ and $\Psi(\varsigma_1) = \Psi(\varsigma_2) = 0$, then

$$\int_{\varsigma_1}^{\varsigma_2} \left|\Psi(\xi)\Psi'(\xi)\right| d\xi \leq \frac{\varsigma_2 - \varsigma_1}{4} \int_{\varsigma_1}^{\varsigma_2} \left(\Psi'(\xi)\right)^2 d\xi, \quad (1)$$

where the constant $\varsigma_1/4$ is the best one, and if Ψ is real absolutely continuous on $(0, \varsigma_2)$ with $\Psi(0) = 0$, then

$$\int_0^{\varsigma_2} \left|\Psi(\xi)\Psi'(\xi)\right| d\xi \leq \frac{\varsigma_2}{2} \int_0^{\varsigma_2} \left(\Psi'(\xi)\right)^2 d\xi. \quad (2)$$

Also Opial in [1], proved that if Ψ is absolutely continuous on $[0, \varsigma_1]$ with $\Psi(0) = 0$, $\delta \geq 0$ and $\beta \geq 1$, then

$$\int_0^{\varsigma_1} |\Psi(\xi)|^\delta \left|\Psi'(\xi)\right|^\beta d\xi \leq \frac{\beta}{\delta+\beta} \varsigma_1^\delta \int_0^{\varsigma_1} \left|\Psi'(\xi)\right|^{\delta+\beta} d\xi. \quad (3)$$

Wherever if $\Psi(0) = \Psi(\varsigma_1) = 0$, $\delta \geq 0$ and $\beta \geq 1$, then

$$\int_0^{\varsigma_1} |\Psi(\xi)|^\delta \left|\Psi'(\xi)\right|^\beta d\xi \leq \frac{\beta}{\delta+\beta} \left(\frac{\varsigma_1}{2}\right)^\delta \times \int_0^{\varsigma_1} \left|\Psi'(\xi)\right|^{\delta+\beta} d\xi. \quad (4)$$

Hua in [2], proved an extension of the inequality (2) as follows

$$\int_{\varsigma_1}^{\varsigma_2} |\Psi(\xi)|^\delta \left|\Psi'(\xi)\right| d\xi \leq \frac{(\varsigma_2 - \varsigma_1)^\delta}{\delta+1} \int_{\varsigma_1}^{\varsigma_2} \left|\Psi'(\xi)\right|^{\delta+1} d\xi, \quad (5)$$

where δ is a positive integer and Ψ is an absolutely continuous function with $\Psi(\varsigma_1) = 0$, and if $\delta = 1$, then the inequality (5) becomes

$$\int_{\varsigma_1}^{\varsigma_2} |\Psi(\xi)| \left|\Psi'(\xi)\right| d\xi \leq \frac{\varsigma_2 - \varsigma_1}{2} \int_{\varsigma_1}^{\varsigma_2} \left|\Psi'(\xi)\right|^2 d\xi. \quad (6)$$

Yong in [3], generalized the inequality (2) as follows

$$\int_{\varsigma_1}^{\varsigma_2} |\Psi(\xi)|^\delta \left|\Psi'(\xi)\right|^\beta d\xi \leq \frac{\beta}{\delta+\beta} (\varsigma_2 - \varsigma_1)^\delta \int_{\varsigma_1}^{\varsigma_2} \left|\Psi'(\xi)\right|^{\delta+\beta} d\xi, \text{ for } \delta, \beta \geq 1, \quad (7)$$

where Ψ is absolutely continuous on $[\varsigma_1, \varsigma_2]$ with $\Psi(\varsigma_1) = 0$.

Boyd and Wong in [4], proved a generalization of the inequality (5) as follows

$$\int_0^{\varsigma_1} \theta(\xi) |\Psi(\xi)|^\delta \left|\Psi'(\xi)\right| d\xi \leq \frac{1}{\eta_0(\delta+1)} \int_0^{\varsigma_1} \phi(\xi) \left|\Psi'(\xi)\right|^{\delta+1} d\xi, \text{ for } \delta > 0, \quad (8)$$

where ϕ and $\theta \in C^1[0, \varsigma_1]$ are non-negative functions and η_0 is the smallest eigenvalue of the boundary value problem

$$\left(\phi(\xi)\left(\Omega^{\backprime}(\xi)\right)^\delta\right)^{\backprime} = \eta\theta(\xi)\Omega^\delta(\xi),$$

with $\phi(\varsigma_1)\left(\Omega^{\backprime}(\varsigma_1)\right)^\delta = \eta\theta(\varsigma_1)\Omega^\delta(\varsigma_1)$ such that $\Omega^{\backprime} > 0$ in $[0, \varsigma_1]$ and $\Omega(0) = 0$.

* Corresponding author e-mail: mrz00@fayoum.edu.eg

Beesack and Das in [5], obtained the inequality

$$\int_{\varsigma_1}^{\varsigma_2} \theta(\xi) |\Psi(\xi)|^{\delta} \left|\Psi'(\xi)\right|^{\beta} d\xi \leq \mathscr{L}(\delta,\beta) \int_{\varsigma_1}^{\varsigma_2} \phi(\xi) \left|\Psi'(\xi)\right|^{\delta+\beta} d\xi,$$

where θ, ϕ are non-negative measurable functions on $[\varsigma_1,\varsigma_2]$, Ψ is absolutely continuous on $[\varsigma_1,\varsigma_2]$, $\delta\beta > 0$ and $\delta+\beta < 0$ or $\delta+\beta \geq 1$, where $\mathscr{L}(\delta,\beta)$ is a sharp constant which depends on ϕ, θ, δ and β.

Yong in [6], proved that if Ψ is absolutely continuous on $[\varsigma_1,\varsigma_2]$ with $\Psi(\varsigma_1) = 0$ and $\phi(\xi)$ is bounded positive function then for $\delta \geq 0$ and $\beta \geq 1$, then

$$\int_{\varsigma_1}^{\varsigma_2} \phi(\xi) |\Psi(\xi)|^{\delta} \left|\Psi'(\xi)\right|^{\beta} d\xi \leq \frac{\beta}{\delta+\beta} (\varsigma_2-\varsigma_1)^{\delta} \int_{\varsigma_1}^{\varsigma_2} \phi(\xi) \left|\Psi'(\xi)\right|^{\delta+\beta} d\xi. \quad (9)$$

Opial inequality and its extensions have become currently an important tool in proving the uniqueness and existence of initial and boundary value problems for ordinary, partial differential equations and difference equations, for more details about Opial inequality see; ([2], [4], [7], [8], [9], [10], [11], [12]).

During the last few years, by using the conformable calculus, authors proved some integral inequalities such as Hermite-Hadamard type inequalities ([13], [14], [15]), Chebyshev type inequality [16], Hardy type inequality [17] and Steffensen type inequality [18].

In this paper, we will prove some conformable integral inequalities of Opial type. The paper contains two parts, the first part consists of an introduction about fundamentals of fractional calculus and the second part contains the main results.

2 Preliminaries And Basic Lemmas

In this section, we will present briefly some basic definitions and lemmas on conformable fractional calculus which can help us to obtain our results; for more details see; ([4], [19], [20]).

Definition 1. *The conformable derivative of order α of a function $\Psi : [0, \infty) \to \mathbb{R}$ is defined by*

$$D_\alpha \Psi(\theta) = \lim_{\varepsilon \to 0} \frac{\Psi\left(\theta+\varepsilon\theta^{1-\alpha}\right) - \Psi(\theta)}{\varepsilon}, \; 0 < \alpha \leq 1. \quad (10)$$

Definition 2. *The conformable integrals of order α of a function $\Psi : [0, \infty) \to \mathbb{R}$ is defined as following*

$$\left(I_\alpha^{\varsigma_1}\Psi\right)(\xi) = \int_{\varsigma_1}^{\xi} \Psi(\theta)\, d_\alpha\theta = \int_{\varsigma_1}^{\xi} \theta^{\alpha-1}\Psi(\theta)\, d\theta, \; 0 < \alpha \leq 1, \quad (11)$$

Theorem 1. *Let Ψ and Ω are α-differentiable with respect to $q > 0$, then for $\alpha \in (0,\ 1]$*

1. $D_\alpha(\varsigma_1\Psi + \varsigma_2\Omega)(q) = \varsigma_1 D_\alpha\Psi(q) + \varsigma_2 D_\alpha\Omega(q)$.
2. $D_\alpha\left(q^{\delta}\right) = \delta q^{\delta-\alpha}$, for all $\delta \in \mathbb{R}$.
3. $D_\alpha(\eta) = 0$, for all constant functions $\Psi(q) = \eta$.
4. $D_\alpha(\Psi\Omega)(q) = \Psi D_\alpha\Omega(q) + \Omega D_\alpha\Psi(q)$.
5. $D_\alpha\left(\frac{\Psi}{\Omega}\right)(q) = \frac{\Omega D_\alpha\Psi(q) - \Psi D_\alpha\Omega(q)}{\Omega^2}$.
6. $D_\alpha\Psi(q) = q^{1-\alpha}\frac{d\Psi}{dq}$, if Ψ is differentiable.

Lemma 1. *Let $\Omega(q)$ is α-differentiable with respect to q and Ψ is differentiable with respect to Ω. Then the fractional derivative chain rule is defined as:*

$$D_\alpha\Psi(\Omega(q)) = \Omega^{\alpha-1}(q)\,(D_\alpha\Psi)(\Omega(q))\, D_\alpha\Omega(q). \quad (12)$$

Lemma 2. *Let Ψ and Ω are α-differentiable with respect to q on $[\varsigma_1,\varsigma_2]$, then the fractional integration by parts is defined as*

$$\int_{\varsigma_1}^{\varsigma_2} (D_\alpha\Psi(q))\,\Omega(q)\, d_\alpha q = \Psi(q)\,\Omega(q)\big|_{\varsigma_1}^{\varsigma_2} - \int_{\varsigma_1}^{\varsigma_2} \Psi(q)\,(D_\alpha\Omega(q))\, d_\alpha q. \quad (13)$$

Lemma 3. *Let $0 < \alpha \leq 1$ and $\Psi, \Omega : [\varsigma_1,\varsigma_2] \to \mathbb{R}$. Then the Hölder inequality is defined as*

$$\int_{\varsigma_1}^{\varsigma_2} |\Psi(q)\,\Omega(q)|\, d_\alpha q \leq \left(\int_{\varsigma_1}^{\varsigma_2} |\Psi(q)|^{\delta} d_\alpha q\right)^{\frac{1}{\delta}} \left(\int_{\varsigma_1}^{\varsigma_2} |\Omega(q)|^{\beta} d_\alpha q\right)^{\frac{1}{\beta}}, \quad (14)$$

where $1/\delta + 1/\beta = 1$ and $\delta > 1$.

3 Main Results

Theorem 2. *Let $\varsigma_1,\ \omega \in \mathbb{R}$, δ, β be positive real number with $\delta \geq 1$, ϕ, θ be non-negative continuous functions on $(\varsigma_1,\ \omega)$, if $\int_{\varsigma_1}^{\omega} \phi^{\frac{-\alpha}{\delta+\beta-\alpha}} d_\alpha q < \infty$ and $\Upsilon : [\varsigma_1,\ \omega] \to R^{+}$ is α-differentiable and Υ does not change its sign in (ς_1,ω). Then we have*

$$(i) \int_{\varsigma_1}^{\omega} \theta(\xi)\,|\Upsilon(\xi)|^{\delta}\,|D_\alpha\Upsilon(\xi)|^{\beta}\, d_\alpha\xi \leq \mathscr{L}_1(\varsigma_1,\ \omega,\ \delta,\ \beta) \int_{\varsigma_1}^{\omega} \phi(\xi)\,|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}\, d_\alpha\xi + 2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta} \int_{\varsigma_1}^{\omega} \theta(\xi)\,|D_\alpha\Upsilon(\xi)|^{\beta}\, d_\alpha\xi, \quad (15)$$

where

$$\mathscr{L}_1(\varsigma_1,\ \omega,\ \delta,\ \beta) = 2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}} \times\left[\int_{\varsigma_1}^{\omega}\theta^{\frac{\delta+\beta}{\delta}}(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta}} \times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\delta+\beta-\alpha} d_\alpha\xi\right]^{\frac{\delta}{\delta+\beta}}. \tag{16}$$

(ii) If $\phi=\theta$, *then*

$$\int_{\varsigma_1}^{\omega}\phi(\xi)|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi \leq \mathscr{L}_2(\varsigma_1,\ \omega,\ \delta,\ \beta) \times\int_{\varsigma_1}^{\omega}\phi(\xi)|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi +2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}\phi(\xi)|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi, \tag{17}$$

where

$$\mathscr{L}_2(\varsigma_1,\ \omega,\ \delta,\ \beta) = 2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}}\left[\int_{\varsigma_1}^{\omega}\phi^{1+(1-\alpha)\frac{\beta}{\delta}}(\xi) \times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\delta+\beta-\alpha}d_\alpha\xi\right]^{\frac{\delta}{\delta+\beta}}. \tag{18}$$

(iii) If $\phi=1$, *then*

$$\int_{\varsigma_1}^{\omega}|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi \leq \mathscr{L}_3(\varsigma_1,\ \omega,\ \delta,\ \beta)\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi +2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi, \tag{19}$$

where

$$\mathscr{L}_3(\varsigma_1,\ \omega,\ \delta,\ \beta) = 2^{\delta-1}\frac{\beta^{\frac{\beta}{\delta+\beta}}}{(\delta+\beta)^{\frac{\alpha\delta+\beta}{\delta+\beta}}}(\omega-\varsigma_1)^{\delta}. \tag{20}$$

Proof.(i) Since $D_\alpha\Upsilon$ does not change sign in $(\varsigma_1,\ \omega)$, we have

$$|\Upsilon(\xi)|-|\Upsilon(\varsigma_1)| \leq |\Upsilon(\xi)-\Upsilon(\varsigma_1)| = \left|\int_{\varsigma_1}^{\xi}D_\alpha\Upsilon(q)d_\alpha q\right| \leq \int_{\varsigma_1}^{\xi}|D_\alpha\Upsilon(q)|d_\alpha q. \tag{21}$$

From (21), we get

$$|\Upsilon(\xi)| \leq \int_{\varsigma_1}^{\xi}|D_\alpha\Upsilon(q)|d_\alpha q+|\Upsilon(\varsigma_1)| = \int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta}}(q)}\phi^{\frac{\alpha}{\delta+\beta}}(q)|D_\alpha\Upsilon(q)|d_\alpha q+|\Upsilon(\varsigma_1)|.$$

Since ϕ is non-negative on $(\varsigma_1,\ \omega)$, and using Hölder inequality (14) with indices $\frac{\delta+\beta}{\delta+\beta-\alpha}$ and $\frac{\delta+\beta}{\alpha}$ and where

$$\Psi(q)=\frac{1}{\phi^{\frac{\alpha}{\delta+\beta}}(q)},\ \text{and}$$
$$\Omega(\xi)=\phi^{\frac{\alpha}{\delta+\beta}}(q)|D_\alpha\Upsilon(q)|,$$

then

$$|\Upsilon(\xi)| = \left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\frac{\delta+\beta-\alpha}{\delta+\beta}} \times\left(\int_{\varsigma_1}^{\xi}\phi(q)|D_\alpha\Upsilon(q)|^{\frac{\delta+\beta}{\alpha}}d_\alpha q\right)^{\frac{\alpha}{\delta+\beta}}+|\Upsilon(\varsigma_1)|. \tag{22}$$

Since $\delta\geq 1$, by taking the power δ for both sides of (22), we deduce

$$|\Upsilon(\xi)|^{\delta} \leq \left[\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\frac{\delta+\beta-\alpha}{\delta+\beta}} \times\left(\int_{\varsigma_1}^{\xi}\phi(q)|D_\alpha\Upsilon(q)|^{\frac{\delta+\beta}{\alpha}}d_\alpha q\right)^{\frac{\alpha}{\delta+\beta}}+|\Upsilon(\varsigma_1)|\right]^{\delta}. \tag{23}$$

Applying the inequality

$$\varsigma_1^{\delta}+\varsigma_2^{\delta}\leq(\varsigma_1+\varsigma_2)^{\delta}\leq 2^{\delta-1}\left(\varsigma_1^{\delta}+\varsigma_2^{\delta}\right),\ \text{if}\ \varsigma_1,\ \varsigma_2\geq 0,\ \delta\geq 1.$$

on the right hand side of (23), we deduce

$$|\Upsilon(\xi)|^{\delta} \leq 2^{\delta-1}\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\frac{\delta(\delta+\beta-\alpha)}{\delta+\beta}} \times\left(\int_{\varsigma_1}^{\xi}\phi(q)|D_\alpha\Upsilon(q)|^{\delta+\beta}d_\alpha q\right)^{\frac{\alpha\delta}{\delta+\beta}} +2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}.$$

Setting

$$z(\xi):=\int_{\varsigma_1}^{\xi}\phi(q)|D_\alpha\Upsilon(q)|^{\frac{\delta+\beta}{\alpha}}d_\alpha q, \tag{24}$$

we see that $z(\varsigma_1)=0$ and

$$D_\alpha z(\xi)=\phi(\xi)\left|D_\alpha \Upsilon(\xi)\right|^{\frac{\delta+\beta}{\alpha}}>0. \tag{25}$$

From (25), we have

$$\left|D_\alpha \Upsilon(\xi)\right|^{\frac{\delta+\beta}{\alpha}}=\frac{D_\alpha z(\xi)}{\phi(\xi)}, \text{ and}$$

$$\left|D_\alpha \Upsilon(q)\right|^{\beta}=\left(\frac{D_\alpha z(\xi)}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}. \tag{26}$$

From (26), and since θ is non-negative on $(\varsigma_1,\ \omega)$, we get

$$\begin{aligned}
&\theta(\xi)\left|\Upsilon(\xi)\right|^{\delta}\left|D_\alpha \Upsilon(\xi)\right|^{\beta}\\
&\leq 2^{\delta-1}\theta(\xi)\left|D_\alpha \Upsilon(\xi)\right|^{\beta}\\
&\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\frac{\delta(\delta+\beta-\alpha)}{\delta+\beta}}\\
&\times\left(\int_{\varsigma_1}^{\xi}\phi(q)\left|D_\alpha \Upsilon(q)\right|^{\frac{\delta+\beta}{\alpha}}d_\alpha q\right)^{\frac{\alpha\delta}{\delta+\beta}}\\
&+2^{\delta-1}\theta(\xi)\left|D_\alpha \Upsilon(\xi)\right|^{\beta}\left|\Upsilon(\varsigma_1)\right|^{\delta}\\
&=2^{\delta-1}\theta(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}\\
&\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\frac{\delta(\delta+\beta-\alpha)}{\delta+\beta}}\\
&\times z^{\frac{\alpha\delta}{\delta+\beta}}(\xi)\left(D_\alpha z(\xi)\right)^{\frac{\alpha\beta}{\delta+\beta}}\\
&+2^{\delta-1}\theta(\xi)\left(\frac{D_\alpha z(\xi)}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}\left|\Upsilon(\varsigma_1)\right|^{\delta}.
\end{aligned}$$

Integrating the above inequality from ς_1 to ω, we get

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega}\theta(\xi)\left|\Upsilon(\xi)\right|^{\delta}\left|D_\alpha \Upsilon(\xi)\right|^{\beta}d_\alpha\xi\\
&\leq 2^{\delta-1}\int_{\varsigma_1}^{\omega}\left[\theta(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}\right.\\
&\times z^{\frac{\alpha\delta}{\delta+\beta}}(\xi)\left(D_\alpha z(\xi)\right)\\
&\left.\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\frac{\delta(\delta+\beta-\alpha)}{\delta+\beta}}\right]^{\frac{\alpha\beta}{\delta+\beta}}d_\alpha\xi\\
&+2^{\delta-1}\left|\Upsilon(\varsigma_1)\right|^{\delta}\int_{\varsigma_1}^{\omega}\theta(\xi)\left(\frac{D_\alpha z(\xi)}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}d_\alpha\xi.
\end{aligned}$$

Applying Hölder inequality (14) with indices $(\delta+\beta)/\delta$ and $(\delta+\beta)/\beta$ on the right side of the above integral inequality, then

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega}\theta(\xi)\left|\Upsilon(\xi)\right|^{\delta}\left|D_\alpha \Upsilon(\xi)\right|^{\beta}d_\alpha\xi\\
&\leq 2^{\delta-1}\left[\int_{\varsigma_1}^{\omega}\theta^{\frac{\delta+\beta}{\delta}}(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta}}\right.\\
&\left.\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\delta+\beta-\alpha}d_\alpha\xi\right]^{\frac{\delta}{\delta+\beta}}\\
&\times\left[\int_{\varsigma_1}^{\omega}z^{\frac{\alpha\delta}{\beta}}(\xi)\left(D_\alpha z(\xi)\right)^{\alpha}d_\alpha\xi\right]^{\frac{\beta}{\delta+\beta}}\\
&+2^{\delta-1}\left|\Upsilon(\varsigma_1)\right|^{\delta}\int_{\varsigma_1}^{\omega}\theta(\xi)\left(\frac{D_\alpha z(\xi)}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}d_\alpha\xi. \qquad (27)
\end{aligned}$$

Using chain rule (12), we obtain

$$\begin{aligned}
&D_\alpha\left(z^{\frac{\delta+\beta}{\beta}}(\xi)\right)\\
&=D_\alpha\left(z^{\frac{\delta+\beta}{\beta}}\right)(z(q))D_\alpha(z(\xi))z^{\alpha-1}(\xi)\\
&=\frac{\delta+\beta}{\beta}z^{\frac{\delta+\beta}{\beta}-\alpha}(\xi)D_\alpha(z(\xi))z^{\alpha-1}(\xi)\\
&=\frac{\delta+\beta}{\beta}z^{\frac{\delta}{\beta}}(\xi)D_\alpha(z(\xi)). \qquad (28)
\end{aligned}$$

Substituting (28) into (27) and since $z(\varsigma_1)=0$, we have

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega}\theta(\xi)\left|\Upsilon(\xi)\right|^{\delta}\left|D_\alpha \Upsilon(\xi)\right|^{\beta}d_\alpha\xi\leq\\
&2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}}\left[\int_{\varsigma_1}^{\omega}\theta^{\frac{\delta+\beta}{\delta}}(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta}}\right.\\
&\left.\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\delta+\beta-\alpha}d_\alpha\xi\right]^{\frac{\delta}{\delta+\beta}}\\
&\times\left[\int_{\varsigma_1}^{\omega}D_\alpha\left(z^{\frac{\delta+\beta}{\beta}}(\xi)\right)d_\alpha\xi\right]^{\frac{\beta}{\delta+\beta}}\\
&+2^{\delta-1}\left|\Upsilon(\varsigma_1)\right|^{\delta}\int_{\varsigma_1}^{\omega}\theta(\xi)\left(\frac{D_\alpha z(\xi)}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}d_\alpha\xi\\
&=2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}}\\
&\times\left[\int_{\varsigma_1}^{\omega}\theta^{\frac{\delta+\beta}{\delta}}(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta}}\right.\\
&\left.\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)}d_\alpha q\right)^{\delta+\beta-\alpha}d_\alpha\xi\right]^{\frac{\delta}{\delta+\beta}}z(\omega)\\
&+2^{\delta-1}\left|\Upsilon(\varsigma_1)\right|^{\delta}\int_{\varsigma_1}^{\omega}\theta(\xi)\left(\frac{D_\alpha z(\xi)}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta+\beta}}d_\alpha\xi.
\end{aligned}$$

From the above inequality, (24) and (25) , we get

$$\int_{\varsigma_1}^{\omega}\theta(\xi)|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi$$
$$\le\mathscr{L}_1(\varsigma_1,\ \omega,\ \delta,\ \beta)\int_{\varsigma_1}^{\omega}\phi(\xi)|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi$$
$$+2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}\theta(\xi)|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi,$$

which is the desired inequality (15).

(ii) The proof follows from (i) at $\phi=\theta$.

(iii) From the chain rule (12), we have

$$D_\alpha\left((q-\varsigma_1)^{\delta+\beta}\right)$$
$$=(\delta+\beta)(q-\varsigma_1)^{\delta+\beta-1}D_\alpha(q-\varsigma_1)$$
$$=(\delta+\beta)(q-\varsigma_1)^{\delta+\beta-1}(q-\varsigma_1)^{1-\alpha}$$
$$=(\delta+\beta)(q-\varsigma_1)^{\delta+\beta-\alpha},$$

then

$$\int_{\varsigma_1}^{\omega}(\xi-\varsigma_1)^{\delta+\beta-\alpha}d_\alpha\xi$$
$$\le\int_{\varsigma_1}^{\omega}\frac{1}{\delta+\beta}D_\alpha\left((\xi-\varsigma_1)^{\delta+\beta}\right)d_\alpha\xi$$
$$=\frac{(\xi-\varsigma_1)^{\delta+\beta}}{\delta+\beta}. \tag{29}$$

From (17) and (18) (where $\phi(q)=1$) and using (29), we get

$$\int_{\varsigma_1}^{\omega}|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi$$
$$\le2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}}$$
$$\times\left[\int_{\varsigma_1}^{\omega}(\xi-\varsigma_1)^{\delta+\beta-\alpha}d_\alpha\xi\right]^{\frac{\delta}{\delta+\beta}}$$
$$\times\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\delta+\beta}d_\alpha\xi$$
$$+2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi$$
$$\le2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}}$$
$$\times\left[\frac{(\omega-\varsigma_1)^{\delta+\beta}}{\delta+\beta}\right]^{\frac{\delta}{\delta+\beta}}$$
$$\times\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\delta+\beta}d_\alpha\xi$$
$$+2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi,$$

then

$$\int_{\varsigma_1}^{\omega}|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi$$
$$\le2^{\delta-1}\frac{\beta^{\frac{\alpha\beta}{\delta+\beta}}}{(\delta+\beta)^{\frac{\alpha\beta+\delta}{\delta+\beta}}}(\omega-\varsigma_1)^{\delta}$$
$$\times\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\delta+\beta}d_\alpha\xi$$
$$+2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi.$$

which the required inequality (20).

Corollary 1. *In Theorem 3.1, if $\Upsilon(\varsigma_1)=0$, the inequality (15), reduces to*

$$\int_{\varsigma_1}^{\omega}\theta(\xi)|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi\le$$
$$\mathscr{L}_1(\varsigma_1,\ \omega,\ \delta,\ \beta)\int_{\varsigma_1}^{\omega}\phi(\xi)|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi,$$

if $\Upsilon(\varsigma_1)=0$ and $\phi=\theta$, the inequality (17), reduces to

$$\int_{\varsigma_1}^{\omega}\phi(\xi)|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi\le$$
$$\mathscr{L}_2(\varsigma_1,\ \omega,\ \delta,\ \beta)\int_{\varsigma_1}^{\omega}\phi(\xi)|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi,$$

and if $\Upsilon(\varsigma_1)=0$ and $\phi=\theta=1$, the inequality (19), reduces to

$$\int_{\varsigma_1}^{\omega}|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi\le$$
$$\mathscr{L}_3(\varsigma_1,\ \omega,\ \delta,\ \beta)\int_{\varsigma_1}^{\omega}|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi,$$

Corollary 2. *In Theorem 3.1, if $\alpha=1$, the inequality (15), reduces to*

$$\int_{\varsigma_1}^{\omega}\theta(\xi)|\Upsilon(\xi)|^{\delta}|\Upsilon'(\xi)|^{\beta}d\xi$$
$$\le\mathscr{L}_4(\varsigma_1,\ \omega,\ \delta,\ \beta)$$
$$\times\int_{\varsigma_1}^{\omega}\phi(\xi)|\Upsilon'(\xi)|^{\delta+\beta}d\xi$$
$$+2^{\delta-1}|\Upsilon(\varsigma_1)|^{\delta}\int_{\varsigma_1}^{\omega}\theta(\xi)|\Upsilon'(\xi)|^{\beta}d\xi,$$

where

$$\mathscr{L}_4(\varsigma_1,\ \omega,\ \delta,\ \beta)=2^{\delta-1}\left(\frac{\beta}{\delta+\beta}\right)^{\frac{\beta}{\delta+\beta}}$$
$$\times\left[\int_{\varsigma_1}^{\omega}\theta^{\frac{\delta+\beta}{\delta}}(\xi)\left(\frac{1}{\phi(\xi)}\right)^{\frac{\beta}{\delta}}\right.$$
$$\left.\times\left(\int_{\varsigma_1}^{\xi}\frac{1}{\phi^{\frac{1}{\delta+\beta-1}}(q)}dq\right)^{\delta+\beta-1}d\xi\right]^{\frac{\delta}{\delta+\beta}}.$$

Based on Theorem (3.1), we obtain the following result by replacing $[\varsigma_1, \omega]$ by $[\omega, \varsigma_2]$.

Theorem 3. *Let $\omega, \varsigma_2 \in \mathbb{R}$, δ, β be positive real number with $\delta \geq 1$, ϕ, θ be non-negative continuous functions on (ω, ς_2), if $\int_\omega^{\varsigma_2} \phi^{\frac{-\alpha}{\delta+\beta-\alpha}} d_\alpha q < \infty$, and $\Upsilon : [\omega, \varsigma_2] \to R^+$ be α-differentiable and Υ does not change its sign in (ω, ς_2). Then*

$$\int_\omega^{\varsigma_2} \theta(\xi) |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi \leq \mathscr{L}_5(\omega, \varsigma_2, \delta, \beta) \int_\omega^{\varsigma_2} \phi(\xi) |D_\alpha \Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}} d_\alpha \xi + 2^{\delta-1} |\Upsilon(\varsigma_2)|^\delta \int_\omega^{\varsigma_2} \theta(\xi) |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi, \tag{30}$$

where

$$\mathscr{L}_5(\omega, \varsigma_2, \delta, \beta) = 2^{\delta-1} \left(\frac{\beta}{\delta+\beta}\right)^{\frac{\alpha\beta}{\delta+\beta}} \times \left[\int_\omega^{\varsigma_2} \theta^{\frac{\delta+\beta}{\delta}}(\xi) \left(\frac{1}{\phi(\xi)}\right)^{\frac{\alpha\beta}{\delta}} \times \left(\int_\xi^{\varsigma_2} \frac{1}{\phi^{\frac{\alpha}{\delta+\beta-\alpha}}(q)} d_\alpha q\right)^{\delta+\beta-\alpha} d_\alpha \xi\right]^{\frac{\delta}{\delta+\beta}}. \tag{31}$$

Corollary 3. *In Theorem 3.2, if $\Upsilon(\varsigma_2) = 0$, the inequality (30) reduces to*

$$\int_\omega^{\varsigma_2} \theta(\xi) |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi \leq \mathscr{L}_5(\omega, \varsigma_2, \delta, \beta) \int_\omega^{\varsigma_2} \phi(\xi) |D_\alpha \Upsilon(\xi)|^{\delta+\beta} d_\alpha \xi$$

Corollary 4. *In Theorem 3.2, if $\alpha = 1$, the inequality (30) reduces to*

$$\int_\omega^{\varsigma_2} \theta(\xi) |\Upsilon(\xi)|^\delta |\Upsilon'(\xi)|^\beta d\xi \leq \mathscr{L}_6(\omega, \varsigma_2, \delta, \beta) \int_\omega^{\varsigma_2} \phi(\xi) |\Upsilon'(\xi)|^{\delta+\beta} d\xi + 2^{\delta-1} |\Upsilon(\varsigma_2)|^\delta \int_\omega^{\varsigma_2} \theta(\xi) |\Upsilon'(\xi)|^\beta d\xi,$$

where

$$\mathscr{L}_6(\omega, \varsigma_2, \delta, \beta) = 2^{\delta-1} \left(\frac{\beta}{\delta+\beta}\right)^{\frac{\beta}{\delta+\beta}} \times \left[\int_\omega^{\varsigma_2} \theta^{\frac{\delta+\beta}{\delta}}(\xi) \left(\frac{1}{\phi(\xi)}\right)^{\frac{\beta}{\delta}} \times \left(\int_\xi^{\varsigma_2} \frac{1}{\phi^{\frac{1}{\delta+\beta-1}}(q)} dq\right)^{\delta+\beta-1} d\xi\right]^{\frac{\delta}{\delta+\beta}}.$$

Let $\mathscr{L}^*(\delta, \beta) = \mathscr{L}_1(\varsigma_1, \omega, \delta, \beta) = \mathscr{L}_5(\omega, \varsigma_2, \delta, \beta) < \infty$ such that $\mathscr{L}_1(\varsigma_1, \omega, \delta, \beta)$ and $\mathscr{L}_5(\omega, \varsigma_2, \delta, \beta)$ are given in Theorems 3.1 and 3.2 and ω is the unique solution of the equation $\mathscr{L}_1(\varsigma_1, \omega, \delta, \beta) = \mathscr{L}_5(\omega, \varsigma_2, \delta, \beta)$. Therefore,

$$\int_{\varsigma_1}^{\varsigma_2} \theta(\xi) |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi = \int_{\varsigma_1}^{\omega} \theta(\xi) |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi + \int_\omega^{\varsigma_2} \theta(\xi) |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi.$$

Now we can combine Theorems (3.1) and (3.2) and obtain the following result.

Theorem 4. *Let $\varsigma_1, \varsigma_2 \in \mathbb{R}$, δ, β be positive real number with $\delta \geq 1$, ϕ, θ be non-negative continuous functions on $(\varsigma_1, \varsigma_2)$ provided that $\int_{\varsigma_1}^{\varsigma_2} \phi^{\frac{-\alpha}{\delta+\beta-\alpha}} d_\alpha q < \infty$ and $\Upsilon : [\varsigma_1, \varsigma_2] \to \mathbb{R}^+$ is α-differentiable and Υ does not change its sign in $(\varsigma_1, \varsigma_2)$. Then*

$$(i) \int_{\varsigma_1}^{\varsigma_2} \theta(\xi) |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi \leq \mathscr{L}^*(\delta, \beta) \int_{\varsigma_1}^{\varsigma_2} \phi(\xi) |D_\alpha \Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}} d_\alpha \xi + 2^{\delta-1} \left(|\Upsilon(\varsigma_1)|^\delta + |\Upsilon(\varsigma_2)|^\delta\right) \times \int_{\varsigma_1}^{\varsigma_2} \theta(\xi) |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi. \tag{32}$$

(ii) If $\phi = \theta = 1$ in (32), we have

$$\int_{\varsigma_1}^{\varsigma_2} |\Upsilon(\xi)|^\delta |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi \leq \mathscr{L}_7(\varsigma_1, \varsigma_2, \delta, \beta) \int_{\varsigma_1}^{\varsigma_2} |D_\alpha \Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}} d_\alpha \xi + 2^{\delta-1} \left(|\Upsilon(\varsigma_1)|^\delta + |\Upsilon(\varsigma_2)|^\delta\right) \int_{\varsigma_1}^{\varsigma_2} |D_\alpha \Upsilon(\xi)|^\beta d_\alpha \xi, \tag{33}$$

where

$$\mathscr{L}_7(\varsigma_1, \varsigma_2, \delta, \beta) = 2^{\delta-1} \frac{\beta^{\frac{\alpha\beta}{\delta+\beta}}}{(\delta+\beta)^{\frac{\alpha\delta+\beta}{\delta+\beta}}} \left(\frac{\varsigma_2 - \varsigma_1}{2}\right)^\delta.$$

(iii) If $\delta = \beta = 1$ in (33), we have

$$\int_{\varsigma_1}^{\varsigma_2} |\Upsilon(\xi)| |D_\alpha \Upsilon(\xi)| d_\alpha \xi \leq \frac{\varsigma_2 - \varsigma_1}{2^{\left(\frac{\alpha+1}{2}\right)}} \int_{\varsigma_1}^{\varsigma_2} |D_\alpha \Upsilon(\xi)| d_\alpha \xi + (|\Upsilon(\varsigma_1)| + |\Upsilon(\varsigma_2)|) \int_{\varsigma_1}^{\varsigma_2} |D_\alpha \Upsilon(\xi)| d_\alpha \xi. \tag{34}$$

Corollary 5. *In Theorem 3.3, if $\Upsilon(\varsigma_1)=0$ and $\Upsilon(\varsigma_2)=0$, the inequality (32) reduces to*

$$\int_{\varsigma_1}^{\varsigma_2}\theta(\xi)|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi \le \mathscr{L}^*(\delta,\beta)\int_{\varsigma_1}^{\varsigma_2}\phi(\xi)|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi,$$

if $\Upsilon(\varsigma_1)=\Upsilon(\varsigma_2)=0$, $\phi=\theta=1$, the inequality (33) reduces to

$$\int_{\varsigma_1}^{\varsigma_2}|\Upsilon(\xi)|^{\delta}|D_\alpha\Upsilon(\xi)|^{\beta}d_\alpha\xi \le \mathscr{L}_7(\varsigma_1,\ \varsigma_2,\ \delta,\ \beta)\int_{\varsigma_1}^{\varsigma_2}|D_\alpha\Upsilon(\xi)|^{\frac{\delta+\beta}{\alpha}}d_\alpha\xi,$$

and if $\Upsilon(\varsigma_1)=\Upsilon(\varsigma_2)=0$, then the inequality (34) reduces to

$$\int_{\varsigma_1}^{\varsigma_2}|\Upsilon(\xi)||D_\alpha\Upsilon(\xi)|d_\alpha\xi \le \frac{\varsigma_2-\varsigma_1}{2^{\left(\frac{\alpha+1}{2}\right)}}\int_{\varsigma_1}^{\varsigma_2}|D_\alpha\Upsilon(\xi)|d_\alpha\xi.$$

Corollary 6. *In Theorem 3.3, if $\alpha=1$ and $\Upsilon(\varsigma_1)=\Upsilon(\varsigma_2)=0$, then inequality (34) reduces to*

$$\int_{\varsigma_1}^{\varsigma_2}|\Upsilon(\xi)||\Upsilon'(\xi)|d\xi \le \frac{\varsigma_2-\varsigma_1}{2}\int_{\varsigma_1}^{\varsigma_2}|\Upsilon'(\xi)|^2d\xi$$

which is the inequality (6).

Now, we give some integral inequalities as special cases from Theorems (3.1), (3.2) and (3.3).

Theorem 5. *Let ϕ be a non-negative non-increasing on $[\varsigma_1,\ \omega]$, $\tau>1$, $\varsigma_1,\ \omega\in\mathbb{R}$, and $\Omega:[\varsigma_1,\ \omega]\to\mathbb{R}$ with $\Omega(\varsigma_1)=0$, then for $\alpha\in(0,\ 1]$ and $\eta\ge 0$, we have*

$$\int_{\varsigma_1}^{\omega}\phi(q)|\Omega(q)|^{\eta}|D_\alpha\Omega(q)|^{\tau}d_\alpha q \le \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\omega^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\eta} \times\int_{\varsigma_1}^{\omega}\phi(q)|D_\alpha\Omega(q)|^{\eta+\tau}d_\alpha q. \tag{35}$$

Proof. Let $u(q)=\int_{\varsigma_1}^{q}\phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha\Omega(\theta)|^{\tau}d_\alpha\theta$, for $q\in[\varsigma_1,\ \omega]$. Then

$$D_\alpha u(q)=\phi^{\frac{\tau}{\eta+\tau}}(q)|D_\alpha\Omega(q)|^{\tau}>0 \text{ and } u(\varsigma_1)=0. \tag{36}$$

Applying Hölder's inequality (14) on $|\Omega(q)|$ with indices $\delta=\tau/(\tau-1)$ and $\beta=\tau$, we find that

$$\begin{aligned}|\Omega(q)| &= \left|\int_{\varsigma_1}^{q}D_\alpha\Omega(\theta)d_\alpha\theta\right| \le \int_{\varsigma_1}^{q}|D_\alpha\Omega(\theta)|d_\alpha\theta\\ &= \int_{\varsigma_1}^{q}\phi^{\frac{-1}{\eta+\tau}}(\theta)\phi^{\frac{1}{\eta+\tau}}(\theta)|D_\alpha\Omega(\theta)|d_\alpha\theta\\ &\le \left(\int_{\varsigma_1}^{q}\left(\phi^{\frac{-1}{\eta+\tau}}(\theta)\right)^{\frac{\tau}{\tau-1}}d_\alpha\theta\right)^{\frac{\tau-1}{\tau}} \times\left(\int_{\varsigma_1}^{q}\phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha\Omega(\theta)|^{\tau}d_\alpha\theta\right)^{\frac{1}{\tau}}\\ &\le \phi^{\frac{-1}{\eta+\tau}}(q)\left(\int_{\varsigma_1}^{q}d_\alpha\theta\right)^{\frac{\tau-1}{\tau}} \times\left(\int_{\varsigma_1}^{q}\phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha\Omega(\theta)|^{\tau}d_\alpha\theta\right)^{\frac{1}{\tau}}.\end{aligned} \tag{37}$$

As $D_\alpha\left(\frac{\theta^\alpha}{\alpha}\right)=1$, then

$$\int_{\varsigma_1}^{q}d_\alpha\theta \le \int_{\varsigma_1}^{q}D_\alpha\left(\frac{\theta^\alpha}{\alpha}\right)d_\alpha\theta = \left[\frac{\theta^\alpha}{\alpha}\right|_{\varsigma_1}^{q} = \frac{q^\alpha-\varsigma_1^\alpha}{\alpha}. \tag{38}$$

From (38), we get

$$\begin{aligned}|\Omega(q)| &\le \left(\int_{\varsigma_1}^{q}\left(\phi^{\frac{-1}{\eta+\tau}}(\theta)\right)^{\frac{\tau}{\tau-1}}d_\alpha\theta\right)^{\frac{\tau-1}{\tau}} \times\left(\int_{\varsigma_1}^{q}\phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha\Omega(\theta)|^{\tau}d_\alpha\theta\right)^{\frac{1}{\tau}}\\ &\le \phi^{\frac{-1}{\eta+\tau}}(q)\left(\frac{q^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\tau-1}{\tau}}u^{\frac{1}{\tau}}(q),\end{aligned}$$

then

$$\phi^{\frac{\eta}{\eta+\tau}}(q)|\Omega(q)|^{\eta} \le \left(\frac{q^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}}u^{\frac{\eta}{\tau}}(q). \tag{39}$$

Applying chain rule (12), we obtain

$$\begin{aligned}D_\alpha\left(u^{\frac{\eta+\tau}{\tau}}(q)\right) &= \left(D_\alpha u^{\frac{\eta+\tau}{\tau}}\right)(u(q))(D_\alpha u(q))u^{\alpha-1}(q)\\ &= \left(\frac{\eta+\tau}{\tau}\right)u^{\frac{\eta}{\tau}}(q)D_\alpha u(q).\end{aligned} \tag{40}$$

Therefore, from (36), (39) and (40), we have

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega} \phi(q)|\Omega(q)|^{\eta}|D_\alpha \Omega(q)|^{\tau} d_\alpha q \\
&= \int_{\varsigma_1}^{\omega} \phi^{\frac{\eta}{\eta+\tau}}(q)|\Omega(q)|^{\eta} \phi^{\frac{\tau}{\eta+\tau}}|D_\alpha \Omega(q)|^{\tau} d_\alpha q \\
&\leq \int_{\varsigma_1}^{\omega} \left(\frac{q^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} u^{\frac{\eta}{\tau}}(q) D_\alpha u(q) d_\alpha q \\
&\leq \left(\frac{\omega^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} \int_{\varsigma_1}^{\omega} u^{\frac{\eta}{\tau}}(q) D_\alpha u(q) d_\alpha q \\
&\leq \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\omega^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} \\
&\times \int_{\varsigma_1}^{\omega} D_\alpha\left(u^{\frac{\eta+\tau}{\tau}}(q)\right) d_\alpha q \\
&= \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\omega^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} u^{\frac{\eta+\tau}{\tau}}(\omega). \qquad (41)
\end{aligned}$$

Applying (38) and Hölder's inequality (14) on $u(\omega)$ with indices $\delta=(\eta+\tau)/\eta$ and $\beta=(\eta+\tau)/\tau$, we find that

$$\begin{aligned}
u(\omega) &= \int_{\varsigma_1}^{\omega} \phi^{\frac{\tau}{\eta+\tau}}(q)|D_\alpha \Omega(q)|^{\tau} d_\alpha q \\
&\leq \left(\int_{\varsigma_1}^{\omega} d_\alpha q\right)^{\frac{\eta}{\eta+\tau}} \\
&\times \left(\int_{\varsigma_1}^{\omega} \left(\phi^{\frac{\tau}{\eta+\tau}}(q)|D_\alpha \Omega(q)|^{\tau}\right)^{\frac{\eta+\tau}{\tau}} d_\alpha q\right)^{\frac{\tau}{\eta+\tau}} \\
&\leq \left(\frac{\omega^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\frac{\eta}{\eta+\tau}} \\
&\times \left(\int_{\varsigma_1}^{\omega} \phi(q)|D_\alpha \Omega(q)|^{\eta+\tau} d_\alpha q\right)^{\frac{\tau}{\eta+\tau}}. \qquad (42)
\end{aligned}$$

Then from (41) and (42), we get

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega} \phi(q)|\Omega(q)|^{\eta}|D_\alpha \Omega(q)|^{\tau} d_\alpha q \\
&\leq \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\omega^\alpha-\varsigma_1^\alpha}{\alpha}\right)^{\eta} \\
&\times \int_{\varsigma_1}^{\omega} \phi(q)|D_\alpha \Omega(q)|^{\eta+\tau} d_\alpha q,
\end{aligned}$$

which the required inequality (35).

Corollary 7. *In Theorem* 3.4, *if* $\alpha=1$, *then we obtain*

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega} \phi(q)|\Omega(q)|^{\eta}\left|\Omega^{\backprime}(q)\right|^{\tau} dq \leq \\
&\frac{\tau(\omega-\varsigma_1)^{\eta}}{\eta+\tau} \int_{\varsigma_1}^{\omega} \phi(q)\left|\Omega^{\backprime}(q)\right|^{\eta+\tau} dq,
\end{aligned}$$

which is the inequality (9), if $\alpha=\tau=\eta=1$, *then*

$$\begin{aligned}
&\int_{\varsigma_1}^{\omega} \phi(q)|\Omega(q)|\left|\Omega^{\backprime}(q)\right| dq \leq \\
&\frac{\omega-\varsigma_1}{2} \int_{\varsigma_1}^{\omega} \phi(q)\left|\Omega^{\backprime}(q)\right|^{2} dq.
\end{aligned}$$

and if $\phi(q)=\eta=\tau=\alpha=1$, *then we get*

$$\int_{\varsigma_1}^{\omega} |\Omega(q)|\left|\Omega^{\backprime}(q)\right| dq \leq \frac{\omega-\varsigma_1}{2} \int_{\varsigma_1}^{\omega} \left|\Omega^{\backprime}(q)\right|^{2} dq,$$

which is the inequality (6).

Theorem 6. *Let* ϕ *be a non-negative and non-decreasing on* $[\omega,\ \varsigma_2]$, $\omega,\ \varsigma_2 \in \mathbb{R}$, $\tau>1$, *and if* $\Omega:[\omega,\ \varsigma_2] \to \mathbb{R}$ *with* $\Omega(\varsigma_2)=0$, *then for* $\alpha \in (0,\ 1]$ *and* $\eta \geq 0$, *we have*

$$\begin{aligned}
&\int_{\omega}^{\varsigma_2} \phi(q)|\Omega(q)|^{\eta}|D_\alpha \Omega(q)|^{\tau} d_\alpha q \\
&\leq \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\varsigma_2^\alpha-\omega^\alpha}{\alpha}\right)^{\eta} \\
&\times \int_{\omega}^{\varsigma_2} \phi(q)|D_\alpha \Omega(q)|^{\eta+\tau} d_\alpha q. \qquad (43)
\end{aligned}$$

Proof. Let $u(q) = \int_q^{\varsigma_2} \phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha \Omega(\theta)|^{\tau} d_\alpha \theta$, for $q \in [\omega,\ \varsigma_2]$. Then

$$D_\alpha u(q) = -\phi^{\frac{\tau}{\eta+\tau}}(q)|D_\alpha \Omega(q)|^{\tau} < 0 \text{ and } u(\varsigma_2)=0. \qquad (44)$$

Using (38) and Hölder's inequality (14) on $|\Omega(q)|$ with $\delta=\tau/(\tau-1)$ and $\beta=\tau$, we obtain

$$\begin{aligned}
|\Omega(q)| &= \left|\int_q^{\varsigma_2} D_\alpha \Omega(\theta) d_\alpha \theta\right| \\
&\leq \int_q^{\varsigma_2} |D_\alpha \Omega(\theta)| d_\alpha \theta \\
&= \int_q^{\varsigma_2} \phi^{\frac{-1}{\eta+\tau}}(\theta) \phi^{\frac{1}{\eta+\tau}}(\theta)|D_\alpha \Omega(\theta)| d_\alpha \theta \\
&\leq \left(\int_q^{\varsigma_2} \left(\phi^{\frac{-1}{\eta+\tau}}(\theta)\right)^{\frac{\tau}{\tau-1}} d_\alpha \theta\right)^{\frac{\tau-1}{\tau}} \\
&\times \left(\int_q^{\varsigma_2} \phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha \Omega(\theta)|^{\tau} d_\alpha \theta\right)^{\frac{1}{\tau}} \\
&\leq \phi^{\frac{-1}{\eta+\tau}}(q)\left(\int_q^{\varsigma_2} d_\alpha \theta\right)^{\frac{\tau-1}{\tau}} \\
&\times \left(\int_q^{\varsigma_2} \phi^{\frac{\tau}{\eta+\tau}}(\theta)|D_\alpha \Omega(\theta)|^{\tau} d_\alpha \theta\right)^{\frac{1}{\tau}} \\
&\leq \phi^{\frac{-1}{\eta+\tau}}(q)\left(\frac{\varsigma_2^\alpha-q^\alpha}{\alpha}\right)^{\frac{\tau-1}{\tau}} u^{\frac{1}{\tau}}(q).
\end{aligned}$$

Therefore

$$\phi^{\frac{\eta}{\eta+\tau}}(q)|\Omega(q)|^{\eta} \leq \left(\frac{\varsigma_2^\alpha-q^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} u^{\frac{\eta}{\tau}}(q). \qquad (45)$$

Using (44) and chain rule (12), we obtain

$$-D_\alpha\left(u^{\frac{\eta+\tau}{\tau}}(q)\right) = -\left(D_\alpha^{\varsigma_1} u^{\frac{\eta+\tau}{\tau}}\right)(u(q)) \times (D_\alpha u(q))\, u^{\alpha-1}(q) = -\left(\frac{\eta+\tau}{\tau}\right) u^{\frac{\eta}{\tau}}(q)\, D_\alpha u(q). \quad (46)$$

Then, from (44), (45) and (46), we get

$$\begin{aligned}
&\int_\omega^{\varsigma_2} \phi(q)\,|\Omega(q)|^\eta\,|D_\alpha\Omega(q)|^\tau\, d_\alpha q \\
&= \int_\omega^{\varsigma_2} \phi^{\frac{\eta}{\eta+\tau}}(q)\,|\Omega(q)|^\eta\,\phi^{\frac{\tau}{\eta+\tau}}\,|D_\alpha\Omega(q)|^\tau\, d_\alpha q \\
&\le \int_\omega^{\varsigma_2} \left(\frac{\varsigma_2^\alpha - q^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} u^{\frac{\eta}{\tau}}(q)\,(-D_\alpha u(q))\, d_\alpha q \\
&\le \left(\frac{\varsigma_2^\alpha - \omega^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} \int_\omega^{\varsigma_2} \left(-u^{\frac{\eta}{\tau}}(q)\, D_\alpha u(q)\right) d_\alpha q \\
&\le \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\varsigma_2^\alpha - \omega^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} \times \int_\omega^{\varsigma_2} -D_\alpha\left(u^{\frac{\eta+\tau}{\tau}}(q)\right) d_\alpha q \\
&= \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\varsigma_2^\alpha - \omega^\alpha}{\alpha}\right)^{\frac{\eta(\tau-1)}{\tau}} u^{\frac{\eta+\tau}{\tau}}(\omega). \quad (47)
\end{aligned}$$

Applying (38) and Hölder's inequality (14) on $u(\omega)$ with indices $\delta = (\eta+\tau)/\eta$ and $\beta = (\eta+\tau)/\tau$, we have

$$\begin{aligned}
u(\omega) &= \int_\omega^{\varsigma_2} \phi^{\frac{\tau}{\eta+\tau}}(q)\,|D_\alpha\Omega(q)|^\tau\, d_\alpha q \\
&\le \left(\int_\omega^{\varsigma_2} d_\alpha q\right)^{\frac{\eta}{\eta+\tau}} \times \left(\int_\omega^{\varsigma_2} \left(\phi^{\frac{\tau}{\eta+\tau}}(q)\,|D_\alpha\Omega(q)|^\tau\right)^{\frac{\eta+\tau}{\tau}} d_\alpha q\right)^{\frac{\tau}{\eta+\tau}} \\
&\le \left(\frac{\varsigma_2^\alpha - \omega^\alpha}{\alpha}\right)^{\frac{\eta}{\eta+\tau}} \times \left(\int_\omega^{\varsigma_2} \phi(q)\,|D_\alpha\Omega(q)|^{\eta+\tau}\, d_\alpha q\right)^{\frac{\tau}{\eta+\tau}}. \quad (48)
\end{aligned}$$

Then from (47) and (48), we get

$$\int_\omega^{\varsigma_2} \phi(q)\,|\Omega(q)|^\eta\,|D_\alpha\Omega(q)|^\tau\, d_\alpha q \le \left(\frac{\tau}{\eta+\tau}\right)\left(\frac{\varsigma_2^\alpha - \omega^\alpha}{\alpha}\right)^\eta \times \int_\omega^{\varsigma_2} \phi(q)\,|D_\alpha\Omega(q)|^{\eta+\tau}\, d_\alpha q,$$

which the required inequality (43).

Corollary 8.*In Theorem* 3.6*, if* $\alpha = 1$*, then we have*

$$\int_\omega^{\varsigma_2} \phi(q)\,|\Omega(q)|^\eta\,|\Omega'(q)|^\tau\, dq \le \frac{\tau(\varsigma_2-\omega)^\eta}{\eta+\tau}\int_\omega^{\varsigma_2} \phi(q)\,|\Omega'(q)|^{\eta+\tau}\, dq,$$

which is the inequality (9), if $\tau = \alpha = 1$*, then we have*

$$\int_\omega^{\varsigma_2} \phi(q)\,|\Omega(q)|^\eta\,|\Omega'(q)|\, dq \le \frac{(\varsigma_2-\omega)^\eta}{\eta+1}\int_\omega^{\varsigma_2} \phi(q)\,(|\Omega'(q)|)^{\eta+1}\, dq,$$

and if $\phi(q) = \tau = \alpha = 1$*, then we have*

$$\int_\omega^{\varsigma_2} |\Omega(q)|\,|\Omega'(q)|\, dq \le \frac{\varsigma_2-\omega}{\eta+1}\int_\omega^{\varsigma_2} |\Omega'(q)|^2\, dq,$$

which is the inequality (5).

References

[1] Z. Opial, Sur une inégalité, Ann. Polon. Math., **8 (1)**, (1960).
[2] L. K. Hua, On an inequality of Opial, Sci. Ser., **14**, 789-790, (1965).
[3] G. S. Yang, On a certain result of Z. Opial, Proc. Japan Acad., **42 (2)**,78-83, (1966).
[4] D. W. Boyd and J. S. W. Wong, An extension of Opial's inequality, J. Math. Anal. Appl., **19 (1)**, 100-102, (1967).
[5] P. R. Beesack, On an integral inequality of Z. Opial, Trans. Amer. Math. Soc., **104 (3)**, 470-475, (1962).
[6] G. S. Yang, A note on some integrodifferential inequalities, Soochow J. Math., **9**, 231-236, (1983).
[7] R. P. Agarwal and P. Y. Pang, *Opial inequalities with applications in differential and difference equations*, Springer Science & Business Media, 320, (1995).
[8] P. Beesack and K. Das, Extensions of Opial's inequality, Pacific J. Math., **26 (2)**, 215-32, 1968.
[9] N. Levinson, On an inequality of Opial and Beesack, Proc. Amer. Math. Soc., **15 (4)**, 565-566, (1964).
[10] C. L. Mallows, An even simpler proof of Opial's inequality, Proc. Amer. Math. Soc., **16 (1)**, 173, (1965).
[11] C. Olech, A simple proof of a certain result of Z. Opial, Ann. Polon. Math., **8 (1)**, 61-63, (1960).
[12] R. N. Pederson, On an inequality of Opial, Beesack, and Levinson, Proc. Amer.Math. Soc., **16 (1)**, 174, (1965).
[13] T. Abdeljawad, P. O. Mohammed and A. Kashuri, New modified conformable fractional integral inequalities of Hermite–Hadamard type with applications, J. Funct. Spaces, (2020).
[14] M. A. Khan, Y. M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations, J. Funct. Spaces, (2018).
[15] Y. Khurshid, M. A. Khan and Y. M. Chu, Conformable fractional integral inequalities for GG-and GA-convex function, AIMS Math, **5 (5)**, 5012-30, (2020).

[16] A. Akkurt, M. E. Yildirim and H. Yildirim, On some integral inequalities for conformable fractional integrals. Asian J. Math. Comput.Res., **15**, 205-212, (2017).
[17] S. H. Saker, D. O' Regan, M. R. Kenawy and R. P. Agarwal, Fractional Hardy Type Inequalities via Conformable Calculus, Mem. Differential Equations Math. Phys., **73**, 131-140, (2018).
[18] M. Z. Sarikaya, H. Yaldiz and H. Budak, Steffensen's integral inequality for conformable fractional integrals, Int. J. Anal. Appl., **15**, 23-30, (2017).
[19] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math., **279**, 57-66, (2015).
[20] R. Khalil, M. Al Horani and A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., **264**, 65-70, (2014).

M. R. Kenawy is a Lecturer of Pure Mathematics, Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt. He received his PhD in Fractional analysis in 2019, from Faculty of Science, Fayoum University, Fayoum, Egypt. His research interests include: Fractional analysis, Differential equations and Inequalities. He published more than 15 papers in reputed international journals. He has reviewed many scientific papers for many international journals.