Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170621

A Novel Formula for Solving Integral Transforms

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Received: 2 Sep. 2023, Revised: 12 Oct. 2023, Accepted: 29 Oct. 2023
Published online: 1 Nov. 2023

Abstract: In this paper, we present a novel formula to solve well-known integral transforms (Laplace andFourier) as well as new integral transform (Sumudu) and their inverses in a clear and practicalmanner. This formula is restricted to integrals that include a derivable function multiplied by anexponential function. The proposed methodology is presented gradually in this article to dealwith these integrals. Moreover, we give provide examples to illustrate the effectiveness of thenew formula.

Keywords: Integral Transforms , Laplace Transform , Fourier Transform , Sumudu Transform, Differential Opertator.

1 Introduction

Integral transform methods are powerful techniques for solving differential equations (DEs).These methods can be modeled as mathematical equations expressed in terms of DEs[1,2,3,4,5] allowing for the transformation of DEs into algebraic equations and obtaining exact solutions. Scholars have developed and perfected these methods to deal with various mathematical problems. The widely-recognized examples comprise the Fourier, Laplace, and numerous other transforms[6,7,8,9,10].

The Fourier Transform is a well-established and effective tool used to transform signals between two different domains, such as transforming a function of time into a function of frequency domain with applications in engineering, physics, signal processing, and RADAR [11, 12,13].

The Laplace transform is a mathematical tool that changes one signal into another according to some fixed set of rules or equations. Thus, it can be seen as a converting means between time and the frequency domain[14,15,16].The Laplace transform is the most effective method for converting differential equations into algebraic ones, and it also provides insight into many other types of equations in the field of mathematics[17, 18].

Sumudu transforms is recent contributions to the literature of integral transform[19]. In principle, this transform can be applied to solve mathematical problems in a manner that is similar to more established older transforms[20,21,22]. Yet, it remains to be illustrated how this transform can in fact solve problems that cannot be solved through more well-known approaches[23].

This paper provides some examples of Fourier, Laplace, and Sumudu transforms solved both in the conventional manner as well as our novel formula presented here. Although no examples of special functions are furnished,such cases can be solved through minor mathematical modifications either to the variable used,or the technique itself.

2 Basic concepts

In this section, we introduce some concepts which are used throughout this paper.

Definition 1. *Exponential order[24]: A function $f(t)$ is called an exponential order c if there exist constants c, $M>0$, $T>0$ such that for all $t>T$, $|f(t)|\leq Me^{ct}$.*

Definition 2. *Laplace transform [25]: The Laplace transform is defined by*

$$\mathscr{L}[f(t)] = \phi(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

where s is a complex variable and $f(t)$ a real or complex function of the real variable t, sets up a transformation between functions $\phi(s)$, $f(t)$.

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Definition 3. *Sumudu transform[26]: Let $f(t)$ be a function in the set $\mathscr{B}$ such as:*

$$\mathscr{B} = \{f(t) : \exists \xi_1, \xi_2 > 0, |f(t)| < Me^{\frac{|t|}{\xi_i}}, if \quad t \in (-1)^i \times [0,\infty)\}$$

The Sumudu transform (ST) of $f(t)$ is defined by:

$$S[f(t)] = \mathscr{G}(u) = \frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u}}dt \tag{2}$$

for all $t \geq 0$ and $u \in (\xi_1, \xi_2)$

Definition 4. *An Operator D is the symbol that indicates a differential operator that acts on a function and returns another function[27]. Let*

$$w = f(t) \tag{3}$$

Then, the derivatives of the function w in a general way, D,D^2,....,D^k can be shown as:

$$D = \frac{dw}{dt}, D^2 = \frac{d^2w}{dt^2}, D^k = \frac{d^kw}{dt^k} \tag{4}$$

Algebraic rules :

Let q,r and s be the order of the derivative and for constants a,b, and c. The differential operator D satisfies the following rules:

i. $(aD^q + bD^r)u = aD^q u + bD^r u = (bD^r + aD^q)u$
ii. $(aD^q + bD^r)u = aD^q u + bD^r u = (bD^r + aD^q)u$
iii. $aD^q \cdot bD^r u = aD^q(bD^r u)$
iv. $(aD^q) \cdot (bD^r)u = (bD^q) \cdot (aD^r)u$
v. $[aD^q + (bD^r + cD^s)]u = [(aD^q + bD^r) + cD^s]u$
vi. $aD^q(bD^r \cdot cD^s)u = (aD^q \cdot bD^r)cD^s u$
vii. $aD^q \cdot (bD^r + cD^s)u = aD^q \cdot bD^r u + aD^q \cdot cD^s u$

Suppose that $f(t)$ is a continuous and differentiable function, then:

$$f(t)D^n w = f(t)\frac{d^n w}{Dt^n}$$

Now, we use the above rules to construct the following polynomial,

$$a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D^1 + a_n$$

Where $a_0, a_1, ..., a_n$ are constants and

$$\left(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D^1 + a_n\right)w = a_0\frac{d^n w}{dt^n} + a_1\frac{d^{n-1}w}{dt^{n-1}} + \cdots + a_n.$$

3 Methodology

Examples of well-known integral transforms(Laplace, Fourier) as well as new integral transform (Sumudu) that have been solved using both the conventional and our new method are provided below.

Theorem 1. *Let $L(D)$ be a polynomial in D, and $H(t) = e^{\frac{t}{M}}F(t)$ where $M \neq 0$, if $F(t)$ has all derivatives then*

$$L(D)\left(e^{\frac{t}{M}}F(t)\right) = e^{\frac{t}{M}}L\left(D + \frac{1}{M}\right)F(t) \tag{5}$$

Proof
Since $L(D)$ is polynomial, we suppose that

$$L(D) = D^n \tag{6}$$

By substituting Eq.(6) in Eq.(5), we have

$$D^n\left(e^{\frac{t}{M}}F(t)\right) = e^{\frac{t}{M}}\left(D + \frac{1}{M}\right)^n F(t) \tag{7}$$

Now, we will use the mathematical induction to prove Eq.(7).
For $n = 0$

$$D^0\left(e^{\frac{t}{M}}F(t)\right) = e^{\frac{t}{M}}\left(D + \frac{1}{M}\right)^0 F(t)$$
$$\Rightarrow e^{\frac{t}{M}}F(t) = e^{\frac{t}{M}}F(t)$$

For $n = k$

$$D^k\left(e^{\frac{t}{M}}F(t)\right) = e^{\frac{t}{M}}\left(D + \frac{1}{M}\right)^k F(t) \tag{8}$$

Now, by derivative both side of Eq.(8), we get

$$D\left[D^k\left(e^{\frac{t}{M}}F(t)\right)\right] = D\left[e^{\frac{t}{M}}\left(D + \frac{1}{M}\right)^k F(t)\right]$$
$$D^{k+1}\left(e^{\frac{t}{M}}F(t)\right) = D\left[e^{\frac{t}{M}}\left(D + \frac{1}{M}\right)^k F(t)\right] \tag{9}$$

$$\begin{aligned} D^{k+1}\left(e^{\frac{t}{M}}F(t)\right) &= e^{\frac{t}{M}}D\left[\left(D + \frac{1}{M}\right)^k F(t)\right] \\ &+ \left[\left(D + \frac{1}{M}\right)^k F(t)\right]De^{\frac{t}{M}} \\ &= e^{\frac{t}{M}}D\left[\left(D + \frac{1}{M}\right)^k F(t)\right] \\ &+ \left[\left(D + \frac{1}{M}\right)^k F(t)\right]\frac{1}{M}e^{\frac{t}{M}} \end{aligned} \tag{10}$$

Simplifying, reordering, and taking the common factor from the right side of Eq.(10), we get

$$D^{k+1}\left(e^{\frac{t}{M}}F(t)\right)=e^{\frac{t}{M}}\left(D+\frac{1}{M}\right)^{k}\left(D+\frac{1}{M}\right)F(t)$$

Finally, we have

$$D^{k+1}\left(e^{\frac{t}{M}}F(t)\right)=e^{\frac{t}{M}}\left(D+\frac{1}{M}\right)^{k+1}F(t)$$

Since Eq.(7) is valid for all $n\geq 0$, thus the theorem has been proved.

3.1 Inverse of operator D:

The inverse of the differential operator D^k is D^{-k}, where they do not change the function $f(t)$ when both operate on it. Hence,

$$D^kD^{-k}f(t)=f(t)\Rightarrow D^kD^{-k}=1,\quad \text{so}\qquad D^{-k}=\frac{1}{D^k}$$

Now, by applying the inverse of operators D to Eq.(8) we get

$$D^{-n}\left(e^{\frac{t}{M}}F(t)\right)=e^{\frac{t}{M}}\left(D+\frac{1}{M}\right)^{-n}F(t)$$

or

$$\frac{\left(e^{\frac{t}{M}}F(t)\right)}{D^n}=\frac{e^{\frac{t}{M}}F(t)}{(D+\frac{1}{M})^n}$$

If $n=1$, then D^1 and $\frac{1}{D}$ indicate the first derivative and first integral respectively. In this case, we have,

$$\frac{\left(e^{\frac{t}{M}}F(t)\right)}{D}=\frac{e^{\frac{t}{M}}F(t)}{(D+\frac{1}{M})}$$

Therefore,

$$\int e^{\frac{t}{M}}F(t)dt=Me^{\frac{t}{M}}\frac{1}{(1+MD)}F(t)$$

$$\int e^{\frac{t}{M}}F(t)dt=Me^{\frac{t}{M}}\left[1-MD+M^2D^2-\cdots+M^nD^n\right]F(t) \tag{11}$$

Where $\frac{1}{(1+MD)}=\left[1-MD+M^2D^2-\cdots+M^nD^n\right]$.
Eq.(11) is our new formula. It can be used to solve integral transforms and their inverses, for example, the Laplace, Fourier, Sumudu, Transforms, as illustrated below.

4 Examples:

4.1 Laplace Transform:

Let $f(t)=e^{wt}$, to find $\mathscr{L}[f(t)]$ we are going to use:
(i) The traditional way Eq.(1):

$$\begin{aligned}\mathscr{L}[f(t)]&=\int_0^\infty e^{\omega t}e^{-s}dt\\&=\int_0^\infty e^{-(s-\omega)t}dt=-\frac{1}{s-\omega}e^{-(s-\omega)t}\Big|_0^\infty\\&=\frac{1}{s-\omega}\end{aligned}$$

(ii) The new formula Eq.(11):

$$\begin{aligned}\mathscr{L}[f(t)]&=\int_0^\infty e^{\omega t}e^{-s}dt\\&=\int_0^\infty e^{-(s-\omega)t}dt\end{aligned}$$

here we have

$$\frac{1}{M}=-(s-\omega)\Longrightarrow M=\frac{-1}{s-\omega}$$

Now we use Eq.(11) with $f(t)=1$, we get

$$\int_0^\infty e^{-(s-\omega)t}dt=\frac{-1}{s-\omega}\left[e^{-(s-\omega)t}\right]_{t=0}^{t=\infty}=\frac{1}{s-\omega}$$

4.2 Fourier Transform:

Let $t\in R, i\omega\in C, f(t)$ a real function ($f(t)=0$ at $t<0$), $F(i\omega)$ is a frequency function. Then to find $F(i\omega)$ we are going to use:
(i) The traditional way:

$$F(i\omega)=\int_{-\infty}^{\infty}e^{i\omega t}f(t)dt$$

$$f(t)=\begin{cases}0 & t<0\\ e^{-t} & t\geq 0\end{cases}$$

$$\begin{aligned}F(i\omega)&=\int_{-\infty}^{0}e^{i\omega t}(0)dt+\int_0^\infty e^{i\omega t}e^{-t}dt\\&=\int_0^\infty e^{-(1+i\omega)t}dt\\&=\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_0^\infty=\frac{1}{1+i\omega}\end{aligned} \tag{12}$$

(ii) The new formula Eq.(11): From Eq.(12) we have

$$\frac{1}{M}=-(1+i\omega)\Rightarrow M=\frac{-1}{(1+i\omega)},\qquad f(t)=1$$

Hence

$$F(i\omega)=\frac{-1}{1+i\omega}\left[e^{-(1+i\omega)t}\right]_{t=0}^{t=\infty}=\frac{1}{1+i\omega}$$

4.3 Sumudu Transform

Let $f(t) = e^{\omega t}$, to find $S[f(t)]$ we are going to use:
(i) The traditional way Eq.(2):

$$S[f(t)] = \mathscr{G}(u) = \frac{1}{u}\int_0^{\infty} e^{\omega t} e^{-\frac{1}{u}t}\,dt$$
$$= \frac{1}{u}\int_0^{\infty} e^{\left(\omega-\frac{1}{u}\right)t}\,dt$$
$$= \frac{1}{u}\lim_{T\to+\infty}\left[\frac{1}{\left(\omega-\frac{1}{u}\right)} e^{\left(\omega-\frac{1}{u}\right)t}\right]_0^T = \frac{1}{u}\lim_{T\to+\infty}\frac{e^{\left(\omega-\frac{1}{u}\right)T}-1}{\left(\omega-\frac{1}{u}\right)} \tag{13}$$

Considering that $\frac{1}{u} > 0$ and the integral converges then

$$\mathscr{G}(u) = \frac{-1}{u\left(\omega-\frac{1}{u}\right)} = \frac{1}{1-u\omega}$$

(ii) The new formula Eq.(11):
From Eq.(13) we have

$$\frac{1}{M} = \omega - \frac{1}{u} \quad \Rightarrow \quad M = \frac{1}{\omega-\frac{1}{u}}, \quad f(t) = 1$$

Now by using Eq.(11) on Eq.(13) with $f(t) = 1$, we get

$$\mathscr{G}(u) = \frac{1}{u}\int_0^{\infty} e^{\left(\omega-\frac{1}{u}\right)t}\,dt = \frac{1}{u\left(\omega-\frac{1}{u}\right)} e^{\left(\omega-\frac{1}{u}\right)t}\Bigg|_{t=0}^{t=\infty}$$
$$= \frac{-1}{u\omega-1} = \frac{1}{1-u\omega}$$

Also, by similarity, we can find Sumudu transform of Trigonometric and Hyperbolic functions.

5 Conclusion

The advantage of this methodology is that utilizes only those derivatives related to functions that can be transformed through any integral transform. The new formula is concerned with integrals that involve derivable functions multiplied by exponential functions. An application of this formula is direct and straightforward in the case of polynomial functions. However, an indirect result is produced when the method is applied to function with infinite derivatives. In the case of exponential function with indirect arguments, some change of variables related to the argument is required to apply the formula effectively.

Funding

This research is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1445)

Declaration of Competing Interest

The authors declare that they have no competing interests.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research at Prince sattam bin Abdulaziz University for supporting this work by Grant Code: (PSAU/2023/R/1445).

Contribution

The authors have contributed equally to this work.

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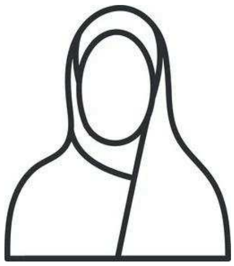

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