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Marshall–Olkin Bivariate Omega Model for Modeling Failure of Paired Organs

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Abstract: Bivariate Marshall Olkin distribution methods are very useful for modelling failure of paired organs, such as the eyes, kidneys, and lungs. Although there are inevitable relations between the components of such organs, these organs may possibly fail one after the other or at the same time. In this paper, a new model using Bivariate Marshall-Olkin distribution methods, namely Bivariate Omega Model (BOM) is introduced and applied for modeling time of two eyes blindness in diabetic retinopathy patients. Some probabilistic properties of the bivariate Omega distribution are derived and studied. The dependence properties for bivariate Omega distribution are proposed using the Marshall-Olkin copula. Parameters estimators are investigated using the maximum likelihood method. Two data sets are illustrated to show the usefulness of the new model for fitting such data.

Keywords: Bivariate distribution; Diabetic Retinopathy Study; Failure rate; Marshall-Olkin copula; Maximum-likelihood estimators; Omega distribution.

1 Introduction

The Omega probability (OM) distribution created by [1] and established based on the omega function. The behaviors of the hazard rate function for OM distribution make it more fitting for modeling bathtub-shaped failure rate curves. The probability density function (pdf), cumulative distribution function (cdf), survival function, and hazard rate function of OM distribution are expressed as follows,

$$f(x) = \frac{\alpha\beta d^{2\beta} x^{\beta-1}}{d^{2\beta} - x^{2\beta}} \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{-\alpha d^\beta}{2}}, \tag{1}$$

$$F(x) = 1 - \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{-\alpha d^\beta}{2}}, \tag{2}$$

$$S(x) = \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{-\alpha d^\beta}{2}}, \tag{3}$$

$$h(x) = \frac{\alpha\beta d^{2\beta} x^{\beta-1}}{d^{2\beta} - x^{2\beta}}. \tag{4}$$

where $\alpha, \beta, d > 0$ are the parameters and a random variable $0 < x < d$.

Omega probability distribution is famous for its versatility and ease of usage. The fact that the cdf and the hazard rate function of OM distribution are power functions rather than exponential functions helps to the distribution apparent ease of use. While the exponential function tends to infinity over an unbounded domain, the omega function does so over a bounded domain (0, d). This allows the omega hazard rate function to be more appropriately follow sudden changes (d > 0). The properties of OM distribution investigated by [2]. They also showing that the OM is better suited to the data than the other distributions (Exponentiated Weibull, generalized power Weibull, generalized Weibull, modified Weibull, modified Weibull extension, odd Weibull, and reduced modified Weibull distributions) examined by [1] for analysis the number of operating hours between successive failure times of air conditioning systems in Boeing airplanes.

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In numerous applications, multivariate survival analysis is necessary. It is essential to consider the multivariate distributions for modeling the multivariate data. Various methods have been offered for multivariate survival data, see [3]. This paper aims to introduce the Bivariate Omega distribution (BOM), whose marginal probability density functions are OM distribution using Marshall-Olkin formulation [4]. This new bivariate model is constructed from three independent OM distributions using a minimization process. Various articles have introduced Marshall-Olkin type of bivariate distributions, which are widely utilized for applications in the field of failure time, for instance, [5, 6, 7, 8, 9, 10].

This paper is organized as follows: in Section 2, we formulate and describe a new bivariate model established Marshall-Olkin method, called Bivariate Omega (BOM) distribution. Also, some properties of this new bivariate model are studied. The dependence properties for bivariate Omega distribution are proposed using Marshall-Olkin copula are discussed in Section 3. The bivariate hazard rate function of BOM distribution is provided in Section 4. Section 5 is dedicated to studying the reliability stress-strength model. The maximum likelihood estimators of the parameters are provided in Section 6. In Section 7, two real data are analyzed for illustrative purposes. Finally, conclusions are proposed in Section 8.

2. Model Formulation

2.1. Marshall-Olkin Type Distribution in Shock Model and Competing Risks Model

The bivariate Marshall-Olkin type model is used in the shock model or the competing risks model. In these models, the system consists of two components which are exposed to shocks or risk arriving from three sources of events.

Suppose U_1, U_2 *and* U_3 are three independent random variables such that $U_i \sim OM(\alpha_i, \beta, d)$ for $i = 1, 2, 3$, and let $X_1 = \min(U_1, U_3)$, $X_2 = \min(U_2, U_3)$. Hence the bivariate vector (X_1, X_2) has a BOM distribution with parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, d)$, denoted by $(X_1, X_2) \sim BOM(\alpha_1, \alpha_2, \alpha_3, \beta, d)$. The shock and competing risks models for bivariate omega distribution can be described as follows,

Shock model: Suppose two components labelled 1 and 2 according to three types of shocks in a system. If the shock of the first type happens, then component 1 fails. If the shock of the second type happens, then component 2 fails. But when the third type of shock happens, the two components 1 and 2 are failed. Consider that the occurrences of these shocks are controlled by three independent processes with the related inter-arrival times denoted by U_1, U_2 *and* U_3. The lifetime of component 1 is the random variable $X_1 = \min(U_1, U_3)$ and that of the component 2 is $X_2 = \min(U_2, U_3)$. Then, the survival time of (X_1, X_2) follows the BOM distribution.

Competing risks model: Suppose a system with two components labelled 1 and 2 subjects to three independent causes of failures, which may affect the system. Let U_1, U_2 *and* U_3 are the lifetimes of failure causes. The lifetime of component 1 is the random variable $X_1 = \min(U_1, U_3)$ can fail due to cause 1, the lifetime of component 2 is $X_2 = \min(U_2, U_3)$ can fail due to cause 2, while both the components 1 and 2 fail at the same time as a result of cause 3. Let U_1, U_2 *and* U_3 are the lifetimes of failure follow Omega distribution, then (X_1, X_2) follows the BOM distribution.

2.2. Bivariate Omega Distribution

For the independent random variables $U_i \sim OM(\alpha_i, \beta, d)$, $i = 1, 2, 3$, the random variables $X_1 = \min(U_1, U_3)$ and $X_2 = \min(U_2, U_3)$ are dependent due to the common random (latent) variable U_3. Hence the vector (X_1, X_2) has BOM distribution, with parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, d)$. The following result presents the joint survival function of (X_1, X_2).

Theorem 1. If $(X_1, X_2) \sim BOM(\alpha_1, \alpha_2, \alpha_3, \beta, d)$, then the joint survival function of two variables X_1 and X_2 is given by

$$S_{X_1,X_2}(x_1, x_2) = \left(\frac{d^\beta + x_1^\beta}{d^\beta - x_1^\beta}\right)^{\frac{-\alpha_1 d^\beta}{2}} \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-\alpha_2 d^\beta}{2}} \left(\frac{d^\beta + z^\beta}{d^\beta - z^\beta}\right)^{\frac{-\alpha_3 d^\beta}{2}} \quad (5)$$

where $z = \max(x_1, x_2)$.

Proof. Since the survival function of (X_1, X_2) is as follows

$$S_{X_1,X_2}(x_1, x_2) = P\{X_1 > x_1, X_2 > x_2\},$$

Then, we get

$$\begin{aligned} S_{X_1,X_2}(x_1, x_2) &= P\{\min(U_1, U_3) > x_1, \min(U_2, U_3) > x_2\} \\ &= P\{U_1 > x_1, U_3 > x_1, U_2 > x_2, U_3 > x_2\} \end{aligned}$$

$$= P\{U_1 > x_1, U_2 > x_2, U_3 > \max(x_1, x_2)\}$$
$$= P\{U_1 > x_1, U_2 > x_2, U_3 > z\},$$

where, $z = \max(x_1, x_2)$.

Since $U_i, i = 1, 2, 3$ are independent random variables, then

$$S_{X_1,X_2}(x_1,x_2) = P(U_1 > x_1)P(U_2 > x_2)P(U_3 > z)$$
$$= S_{OM}(x_1;\alpha_1,\beta,d)\, S_{OM}(x_2;\alpha_2,\beta,d)\, S_{OM}(z;\alpha_3,\beta,d)$$
$$= \left(\frac{d^\beta + x_1^\beta}{d^\beta - x_1^\beta}\right)^{\frac{-\alpha_1 d^\beta}{2}} \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-\alpha_2 d^\beta}{2}} \left(\frac{d^\beta + z^\beta}{d^\beta - z^\beta}\right)^{\frac{-\alpha_3 d^\beta}{2}}$$

Corollary 1. The joint survival function of the $BOM(\alpha_1,\alpha_2,\alpha_3,\beta,d)$ can be also written as:

$$S_{X_1,X_2}(x_1,x_2) = \begin{cases} S_{OM}(x_1;\alpha_1,\beta,d)\, S_{OM}(x_2;\alpha_2+\alpha_3,\beta,d), & if \quad x_1 < x_2 \\ S_{OM}(x_1;\alpha_1+\alpha_3,\beta,d)\, S_{OM}(x_2;\alpha_2,\beta,d), & if \quad x_2 < x_1 \\ S_{OM}(x;\alpha_1+\alpha_2+\alpha_3,\beta,d), & if \ \ x_1 = x_2 = x \end{cases} \tag{6}$$

If $(X_1,X_2) \sim BOM(\alpha_1,\alpha_2,\alpha_3,\beta,d)$, then the marginal distributions of X_1 and X_2 and the distribution of the random variable $\min(X_1,X_2)$ are introduced and proved in the following proposition.

Proposition 1. Let $(X_1,X_2) \sim BOM(\alpha_1,\alpha_2,\alpha_3,\beta,d)$. Then it follows that

(i) The marginal distribution function of X_1 and X_2 are as follows

$X_1 \sim OM(\alpha_1 + \alpha_3,\beta,d)$ and $X_2 \sim OM(\alpha_2 + \alpha_3,\beta,d)$.

(ii) $min\,(X_1,X_2) \sim OM(\alpha_1+\alpha_2+\alpha_3,\beta,d)$

Proof. (i) If $X_1 < X_2$, then $Z = max(X_1,X_2) = X_2$. By taking

$$\lim_{x_1 \to 0} S_{X_1,X_2}(x_1,x_2) = \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-\alpha_2 d^\beta}{2}} \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-\alpha_3 d^\beta}{2}} = \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-(\alpha_2+\alpha_3) d^\beta}{2}}$$
$$= S_{OM}(x_2;\alpha_2+\alpha_3,\beta,d).$$

Analogously, if $X_2 < X_1$, we have $Z = X_1$. Therefore,

$$\lim_{x_2 \to 0} S_{X_1,X_2}(x_1,x_2) = \left(\frac{d^\beta + x_1^\beta}{d^\beta - x_1^\beta}\right)^{\frac{-(\alpha_1+\alpha_3) d^\beta}{2}} = S_{OM}(x_1;\alpha_1+\alpha_3,\beta,d)$$

(ii) By using the following fact

$$P(min\,(X_1,X_2) > y) = P(X_1 > y, X_2 > y) = P(U_1 > y, U_2 > y, U_3 > y)$$
$$= P(U_1 > y)\, P(U_2 > y)\, P(U_3 > y)$$
$$= \left(\frac{d^\beta + y^\beta}{d^\beta - y^\beta}\right)^{\frac{-(\alpha_1+\alpha_2+\alpha_3) d^\beta}{2}}, \quad 0 < y < d$$

Thus, result (ii) holds.

The following Theorems will provide the joint cdf and pdf of the BOM distribution.

Theorem 2. If $(X_1,X_2) \sim BOM(\alpha_1,\alpha_2,\alpha_3,\beta,d)$, then the joint cumulative distribution function of (X_1,X_2) is given by

$$F_{X_1,X_2}(x_1,x_2) = \begin{cases} F_{OM}(x_1;\alpha_1+\alpha_3,\beta,d) - F_{OM}(x_1;\alpha_1,\beta,d)[1 - F_{OM}(x_2;\alpha_2+\alpha_3,\beta,d)], & if \quad x_1 < x_2 \\ F_{OM}(x_2;\alpha_2+\alpha_3,\beta,d) - F_{OM}(x_2;\alpha_2,\beta,d)[1 - F_{OM}(x_1;\alpha_1+\alpha_3,\beta,d)], & if \quad x_2 < x_1 \\ 1 - F_{OM}(x;\alpha_1+\alpha_2+\alpha_3,\beta,d), & if \ \ x_1 = x_2 = x \end{cases} \tag{7}$$

Proof. The joint cdf of X_1 and X_2 can be directly obtained from the relationship

$$F_{X_1,X_2}(x_1,x_2) = P(X_1 > x_1, X_2 > x_2) + P(X_1 < x_1) + P(X_2 < x_2) - 1$$
$$= S_{X_1,X_2}(x_1,x_2) + [1 - S_{X_1}(x_1)] + [1 - S_{X_2}(x_2)] - 1 \tag{8}$$

In the case $x_1 < x_2$**:** From Theorem1, the joint survival function of the $BOM(\alpha_1, \alpha_2, \alpha_3, \beta, d)$ can be also written as,

$$S_{X_1,X_2}(x_1, x_2) = S_{OM}(x_1; \alpha_1, \beta, d)\, S_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d)$$
$$= [1 - F_{OM}(x_1; \alpha_1, \beta, d)][1 - F_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d)] \quad (9)$$

From proposition 1, the marginal distributions of X_1 and X_2 are $OM(\alpha_1 + \alpha_3, \beta, d)$ and $OM(\alpha_2 + \alpha_3, \beta, d)$, respectively. Thus, we have

$$S_{X_1}(x_1) = S_{OM}(x_1; \alpha_1 + \alpha_3, \beta, d) \text{ and } S_{X_2}(x_2) = S_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d) \quad (10)$$

Substituting from (9) and (10) into (8), we obtained

$$F_{X_1,X_2}(x_1, x_2) = S_{X_1,X_2}(x_1, x_2) + [1 - S_{X_1}(x_1)] + [1 - S_{X_2}(x_2)] - 1$$
$$=[1 - F_{OM}(x_1; \alpha_1, \beta, d)][1 - F_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d)]$$
$$+F_{OM}(x_1; \alpha_1 + \alpha_3, \beta, d) + F_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d) - 1$$
$$= F_{OM}(x_1; \alpha_1 + \alpha_3, \beta, d) - F_{OM}(x_1; \alpha_1, \beta, d)[1 - F_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d)]$$

Analogously follows the case $x_1 > x_2$, but for the case of $x_1 = x_2 = x$ is obvious.

Theorem 3. If $(X_1, X_2) \sim BOM(\alpha_1, \alpha_2, \alpha_3, \beta, d)$, then the joint probability density function of (X_1, X_2) is given by

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & if \quad 0 < x_1 < x_2 < d \\ f_2(x_1, x_2), & if \quad 0 < x_2 < x_1 < d \\ f_0(x), & if \quad 0 < x = x_1 = x_2 < d \end{cases} \quad (11)$$

where,

$$f_1(x_1, x_2) = f_{OM}(x_1; \alpha_1, \beta, d) f_{OM}(x_2; \alpha_2 + \alpha_3, \beta, d)$$
$$f_2(x_1, x_2) = f_{OM}(x_1; \alpha_1 + \alpha_3, \beta, d) f_{OM}(x_2; \alpha_2, \beta, d)$$
$$f_0(x) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{OM}(x; \alpha_1 + \alpha_2 + \alpha_3, \beta, d)$$

Proof. Clearly, $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ can be found by taking $\frac{\partial^2 F_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2}$ for $x_1 < x_2$ and $x_2 < x_1$, respectively. For $f_0(x)$, the following relation can be use

$$\int_0^d \int_0^{x_2} f_1(x_1, x_2)\, dx_1 dx_2 + \int_0^d \int_0^{x_1} f_2(x_1, x_2)\, dx_2 dx_1 + \int_0^d f_0(x) dx = 1$$

So,

$$\int_0^d \int_0^{x_2} f_1(x_1, x_2)\, dx_1 dx_2 = \int_0^d \int_0^{x_2} \frac{\alpha_1(\alpha_2 + \alpha_3) d^{4\beta} \beta^2 x_1^{\beta-1} x_2^{\beta-1}}{(d^{2\beta} - x_1^{2\beta})(d^{2\beta} - x_2^{2\beta})} \left(\frac{d^\beta + x_1^\beta}{d^\beta - x_1^\beta}\right)^{\frac{-\alpha_1 d^\beta}{2}} \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-(\alpha_2+\alpha_3) d^\beta}{2}} dx_1 dx_2$$

$$= \int_0^d \frac{(\alpha_2 + \alpha_3) d^{2\beta} \beta x_2^{\beta-1}}{d^{2\beta} - x_2^{2\beta}} \left[\left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-(\alpha_1+\alpha_2+\alpha_3) d^\beta}{2}} + \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-(\alpha_2+\alpha_3) d^\beta}{2}}\right] dx_2$$

$$= \frac{\alpha_1}{(\alpha_1 + \alpha_2 + \alpha_3)}$$

Also,

$$\int_0^d \int_0^{x_1} f_2(x_1, x_2)\, dx_2 dx_1 = \int_0^d \int_0^{x_1} \frac{\alpha_2(\alpha_1 + \alpha_3) d^{4\beta} \beta^2 x_1^{\beta-1} x_2^{\beta-1}}{(d^{2\beta} - x_1^{2\beta})(d^{2\beta} - x_2^{2\beta})} \left(\frac{d^\beta + x_2^\beta}{d^\beta - x_2^\beta}\right)^{\frac{-\alpha_2 d^\beta}{2}} \left(\frac{d^\beta + x_1^\beta}{d^\beta - x_1^\beta}\right)^{\frac{-(\alpha_1+\alpha_3) d^\beta}{2}} dx_2 dx_1$$

$$= \frac{\alpha_2}{(\alpha_1+\alpha_2+\alpha_3)}$$

Thus,

$$\frac{\alpha_1}{(\alpha_1+\alpha_2+\alpha_3)} + \frac{\alpha_2}{(\alpha_1+\alpha_2+\alpha_3)} + \int_0^d f_0(x)dx = 1$$

Hence, we obtain $\int_0^d f_0(x)dx = \frac{\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)}$.

On the other hand,

$$\int_0^d \left(\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}\right)\left[\frac{(\alpha_1+\alpha_2+\alpha_3)d^{2\beta}\beta x^{\beta-1}}{d^{2\beta}-x^{2\beta}}\left(\frac{d^\beta+x^\beta}{d^\beta-x^\beta}\right)^{\frac{-(\alpha_1+\alpha_2+\alpha_3)d^\beta}{2}}\right]dx = \frac{\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)}$$

Therefore,

$$f_0(x) = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{OM}(x;\alpha_1+\alpha_2+\alpha_3,\beta,d)$$

The proof of the theorem is completed.

The conditional probability function of X_i is introduced in the following theorem,

Theorem 4. The conditional probability functions of X_i, given $X_i = x_i$ denoted by

$$f_{X_1|X_2}(x_1|x_2) = \begin{cases} f^{(1)}_{X_1|X_2}(x_1|x_2), & x_1 < x_2 \\ f^{(2)}_{X_1|X_2}(x_1|x_2), & x_1 > x_2 \end{cases}$$

where,

$$f^{(1)}_{X_1|X_2}(x_1|x_2) = \frac{\alpha_1\beta d^{2\beta}x_1^{\beta-1}}{d^{2\beta}-x_1^{2\beta}}\left(\frac{d^\beta+x_1^\beta}{d^\beta-x_1^\beta}\right)^{\frac{-\alpha_1 d^\beta}{2}} = f_{X_1}(x_1)$$

$$f^{(2)}_{X_1|X_2}(x_1|x_2) = \frac{\alpha_2(\alpha_1+\alpha_3)\beta d^{2\beta}x_1^{\beta-1}}{(\alpha_2+\alpha_3)(d^{2\beta}-x_1^{2\beta})}\left(\frac{d^\beta+x_1^\beta}{d^\beta-x_1^\beta}\right)^{\frac{-(\alpha_1+\alpha_3)d^\beta}{2}}\left(\frac{d^\beta+x_2^\beta}{d^\beta-x_2^\beta}\right)^{\frac{\alpha_3 d^\beta}{2}}$$

Proof. By using the joint pdf and the marginal of (X_1, X_2) in Theorem 4 and substituting by it in the following expression, the theorem follows immediately,

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}.$$

3. Copula and Dependence Properties

One of the popular methods for constructing bivariate distribution is the copula type. The importance of copula models is converged by [11] who explained in his theorem "Sklar theorem" the relation between bivariate distribution functions and its related univariate marginals with a variety of dependence structures. For every bivariate distribution function $F_{X_1,X_2}(x_1,x_2)$ with continuous marginals $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ there exists a copula with uniform margins $C:[0,1]\times[0,1]\to[0,1]$, such that: $C(F_1(x_1),F_2(x_2)) = F_{X_1,X_2}(x_1,x_2)$. In this section, the dependence properties for bivariate Omega distribution are proposed using Marshall-Olkin copula type [4], which can be written by:

$$C(u,v) = u^{1-\theta_1}v^{1-\theta_2}\min\left(u^{\theta_1},v^{\theta_2}\right) \tag{12}$$

or

$$C(u,v) = \begin{cases} u^{1-\theta_1}v, & u^{\theta_1} \ge v^{\theta_2} \\ uv^{1-\theta_2}, & u^{\theta_1} \le v^{\theta_2} \end{cases}$$

For all $0 < \theta_i < 1$, $i = 1,2$. For $u = F_{X_1}(x_1)$ and $v = F_{X_2}(x_2)$ where $X_i \sim BOM(\alpha_i+\alpha_3,\beta,d)$ and $\theta_i = \frac{\alpha_3}{\alpha_i+\alpha_3}$, $i =$

$1,2$, then the function $C(u,v)$ gives the same bivariate distribution function $F_{X_1,X_2}(x_1,x_2)$ in (7). The Marshall -Olkin survival copula is as follows

$$\hat{C}(u,v) = uv\min\left(u^{-\theta_1}, v^{-\theta_2}\right) = \min\left(u^{1-\theta_1}v, uv^{1-\theta_2}\right)$$

The density function associated with copula $C(u,v)$ is defined by $c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v)$, so the density function for Marshall-Olkin copula (12) is denoted by

$$c(u,v) = \begin{cases} (1-\theta_1)\, u^{-\theta_1}, & u^{\theta_1} > v^{\theta_2} \\ (1-\theta_2)\, v^{-\theta_2}, & u^{\theta_1} < v^{\theta_2} \end{cases}$$

One of the most concepts in statistics is the dependency or association between variables using copula. Now several properties for $F_{X_1,X_2}(x_1,x_2)$ in terms of $C(u,v)$ related to dependence measures are presented.

3.1. Measures of Association

There are several measures of association between a continuous pair (X_1, X_2) will be proposed as Kendall's tau (τ), Spearman's rho (ρ), Blomqvist medial correlation coefficient (β) and Spearman's footrule coefficient (φ_C) which depends only on the copula $C(u,v)$, see [12].

- *Kendall's tau (τ):*

Kendall's tau has several expressions the following is more tractable

$$\tau = 1 - 4\int_0^1 \int_0^1 \frac{\partial C(u,v)}{\partial u} \frac{\partial C(u,v)}{\partial v} du\, dv$$

Marshall-Olkin copula has Kendall's tau as $\frac{\theta_1\theta_2}{\theta_1+\theta_2-\theta_1\theta_2}$, if $(X_1,X_2) \sim BOM(\alpha_1+\alpha_2+\alpha_3,\beta,d)$ then Kendall's tau is given by

$$\tau = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}$$

for $\theta_i = \frac{\alpha_3}{\alpha_i+\alpha_3}$, $i = 1,2$, and $\alpha_1, \alpha_2, \alpha_3$ various from 0 to ∞.

- *Spearman's rho (ρ):*

Spearman's rho can be expressed by copula as:

$$\rho = 12\int_0^1 \int_0^1 C(u,v)\, du\, dv - 3$$

Marshall-Olkin copula has Spearman's rho as $\frac{3\theta_1\theta_2}{2\theta_1+2\theta_2-\theta_1\theta_2}$, if $(X_1,X_2) \sim BOM(\alpha_1+\alpha_2+\alpha_3,\beta,d)$ then the Spearman's rho is given by

$$\rho = \frac{3\alpha_3}{2\alpha_1+2\alpha_2+3\alpha_3}$$

- *Blomqvist medial correlation coefficient (β):*

The medial correlation coefficient introduced by [13] for a random pair (X_1,X_2) by using the medians of X_1 and X_2. Also, used the copula function to propose Blomqvist medial correlation coefficient (β) which is defined by:

$$\beta = 4C\left(\tfrac{1}{2},\tfrac{1}{2}\right) - 1$$

Therefore, if $(X_1,X_2) \sim BOD(\alpha_1+\alpha_2+\alpha_3,\beta,d)$, the copula $C\left(\frac{1}{2},\frac{1}{2}\right) = (\frac{1}{2})^{2-\min(\theta_1,\theta_2)}$, then

$$\beta = \begin{cases} 4\left(\frac{1}{2}\right)^{2-\theta_1} - 1, & \theta_1 < \theta_2 \\ 4(\frac{1}{2})^{2-\theta_2} - 1 & \theta_1 > \theta_2 \end{cases}$$

where $\theta_i = \frac{\alpha_3}{\alpha_i+\alpha_3}$, $i = 1,2$. The minimum value of β=0 at $\min(\theta_1,\theta_2) = 0$ and the maximum value of β=1 at $\min(\theta_1,\theta_2) = 1$.

- *Spearman's footrule coefficient (φ_C):*

Another measure of association can be calculated using copula function is Spearman's footrule coefficient, see [12], as follows

$$\varphi_C = 6\int_0^1 C(u,u)du - 2$$

Hence,

$$\varphi_C = \frac{6}{3-\min(\theta_1,\theta_2)} - 2$$

Therefore, if $(X_1, X_2) \sim BOD(\alpha_1+\alpha_2+\alpha_3, \beta, d)$, then

$$\varphi_C = \begin{cases} \frac{2\alpha_3}{3\alpha_1+2\alpha_3}, & \theta_1<\theta_2 \\ \frac{2\alpha_3}{3\alpha_2+2\alpha_3}, & \theta_1>\theta_2 \end{cases}$$

3.2. Dependence Structure

Several dependence structures of random variables discussed by [14, 12] using the copula theory.

- *Tail Dependence:*

The idea of tail dependence in a copula type measure is the dependence between the variables in the upper or lower quadrant tail of $[0.1]^2$. [14] introduced the following definition of the upper and lower tail dependence which depend, respectively, on the copula of X_1 and X_2

$$\lambda_U = 2 - \lim_{t\to 1^-}\frac{1-C(t,t)}{1-t} \text{ and } \lambda_L = \lim_{t\to 0^+}\frac{C(t,t)}{t}$$

If $(X_1, X_2) \sim BOD(\alpha_1+\alpha_2+\alpha_3, \beta, d)$, then

$$\lambda_U = \begin{cases} \frac{\alpha_3}{\alpha_1+\alpha_3}, & \theta_1<\theta_2 \\ \frac{\alpha_3}{\alpha_2+\alpha_3}, & \theta_1>\theta_2 \end{cases}$$

Thus, there is no lower tail dependence $\lambda_L = 0$.

- *Quadrant Dependence:*

The two random variables X_1 and X_2 are the positive quadrant dependent (PQD) if

$$P(X_1 \le x_1, X_2 \le x_2) \ge P(X_1 \le x_1)P(X_2 \le x_2)$$

PQD can be written by copula type, see [12], equivalently as

$$C(u,v) \ge uv, \text{ for } u,v \in [0,1]^2 \tag{13}$$

Marshall-Olkin copula is PQD which verified (13). Therefore, if $(X_1, X_2) \sim BOM(\alpha_1+\alpha_2+\alpha_3, \beta, d)$, then they are PQD.

4. The Hazard Rate Function

For BOM, the bivariate hazard rate function, hazard gradients and shape of hazard rate function are provided and discussed in this section.

4.1. Bivariate Hazard Rate

If (X_1, X_2) has joint probability density function $f_{X_1,X_2}(x_1,x_2)$, [15] defined the bivariate failure rate function as follows

$$h_{X_1,X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{S_{X_1,X_2}(x_1,x_2)}$$

Theorem 5. If $(X_1, X_2) \sim BOM(\alpha_1+\alpha_2+\alpha_3, \beta, d)$, then the bivariate hazard rate function is defined by

$$h_{X_1,X_2}(x_1,x_2) = \begin{cases} h_1(x_1,x_2), & if\ x_1 < x_2 \\ h_2(x_1,x_2), & if\ x_2 < x_1 \\ h_0(x), & if\ 0 < x_1 = x_2 = x < d \end{cases}$$

where,

$$h_1(x_1,x_2) = \frac{f_1(x_1,x_2)}{S_1(x_1,x_2)} = \frac{\alpha_1(\alpha_2+\alpha_3)\,\beta^2 d^{4\beta} x_1^{\beta-1} x_2^{\beta-1}}{\left(d^{2\beta}-x_1^{2\beta}\right)\left(d^{2\beta}-x_2^{2\beta}\right)}$$

$$h_2(x_1,x_2) = \frac{f_2(x_1,x_2)}{S_2(x_1,x_2)} = \frac{\alpha_2(\alpha_1+\alpha_3)\,\beta^2 d^{4\beta} x_1^{\beta-1} x_2^{\beta-1}}{\left(d^{2\beta}-x_1^{2\beta}\right)\left(d^{2\beta}-x_2^{2\beta}\right)}$$

$$h_0(x) = \frac{f_0(x)}{S_0(x)} = \frac{d^{2\beta}\beta\alpha_3 x^{\beta-1}}{d^{2\beta}-x^{2\beta}}$$

Proof. By using (6) and (11), it is easy to prove the theorem.

4.2. Hazard Gradients

The hazard rate function measures the failure rate in the univariate state, whereas in the multivariate cases, the failure rate depends on the variable that is changed. Therefore, [16, 17] are defined as the hazard gradients for modeling bivariate and multivariate lifetime data.

The bivariate hazard gradient for continuous random variables X_1 and X_2 is given by

$$\begin{aligned} h_{X_1,X_2}(x_1,x_2) &= \left(h_{X_1}(x_1,x_2), h_{X_2}(x_1,x_2)\right) \\ &= \left(-\frac{\partial}{\partial x_1} log\, S_{X_1,X_2}(x_1,x_2), -\frac{\partial}{\partial x_2} log\, S_{X_1,X_2}(x_1,x_2)\right) \end{aligned}$$

For $(X_1,X_2)\sim BOM(\alpha_1,\alpha_2,\alpha_3,\beta,d)$, the hazard gradients are given by

$$h_{X_1}(x_1,x_2) = \begin{cases} \frac{\alpha_1\beta d^{2\beta} x_1^{\beta-1}}{d^{2\beta}-x_1^{2\beta}}, & if\ \ x_1 < x_2 \\ \frac{(\alpha_1+\alpha_3)\beta d^{2\beta} x_1^{\beta-1}}{d^{2\beta}-x_1^{2\beta}}, & if\ \ x_2 < x_1 \\ \frac{(\alpha_1+\alpha_2+\alpha_3)\beta d^{2\beta} x_1^{\beta-1}}{d^{2\beta}-x_1^{2\beta}}, & if\ \ x_1 = x_2 \end{cases} \tag{14}$$

and

$$h_{X_2}(x_1,x_2) = \begin{cases} \frac{(\alpha_2+\alpha_3)\,\beta d^{2\beta} x_2^{\beta-1}}{d^{2\beta}-x_2^{2\beta}}, & if\ \ x_1 < x_2 \\ \frac{\alpha_2\beta d^{2\beta} x_2^{\beta-1}}{d^{2\beta}-x_2^{2\beta}}, & if\ \ x_2 < x_1 \\ \frac{(\alpha_1+\alpha_2+\alpha_3)\beta d^{2\beta} x_2^{\beta-1}}{d^{2\beta}-x_2^{2\beta}}, & if\ \ x_1 = x_2 \end{cases} \tag{15}$$

4.3. Shape of Hazard Rate Function

Hazard rate function of the omega distribution with parameters α, β and d is given by

$$h(x) = \frac{\alpha\beta d^{2\beta} x^{\beta-1}}{d^{2\beta}-x^{2\beta}} \tag{16}$$

Since $h(x)$ is bathtub shaped for $0 < \beta < 1$ and a monotonic increasing for $\beta \geq 1$. From (14), (15) and (16), we can conclude that for fixed x_2, $h_{X_1}(x_1,x_2)$ has a bathtub shape for $\beta \geq 1$ and monotonic increasing for $\beta \geq 1$. Similarly, the hazard function $h_{X_2}(x_1,x_2)$ hold the same shapes for a fixed x_1.

5. Stress-Strength Reliability Analysis

The stress–strength measure explains the life of a component that has a strength X_2 and random stress X_1. The component fails at the time that the stress exceeds the strength. In this section, the stress- strength reliability measure,

$R = \text{P}(X_1 < X_2)$ is derived for BOM distribution when X_1 and X_2 are dependent random. The stress-strength reliability measure R for BOM distribution is derived as follows,

Theorem 6. If (X_1, X_2) has a BOM distribution defined in (8), then the stress-strength measure is

$$R = \text{P}(X_1 < X_2) = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}$$

Proof. The stress-strength reliability measure can be derived as follows

$$R = \text{P}(X_1 < X_2) = \int_0^d \int_{x_1}^d f_1(x_1, x_2)\, dx_2 dx_1$$

$$= \int_0^d \int_{x_1}^d \frac{\alpha_1(\alpha_2+\alpha_3)d^{4\beta}\beta^2 x_1^{\beta-1} x_2^{\beta-1}}{\left(d^{2\beta}-x_1^{2\beta}\right)\left(d^{2\beta}-x_2^{2\beta}\right)} \left(\frac{d^\beta+x_1^\beta}{d^\beta-x_1^\beta}\right)^{\frac{-\alpha_1 d^\beta}{2}} \left(\frac{d^\beta+x_2^\beta}{d^\beta-x_2^\beta}\right)^{\frac{-(\alpha_2+\alpha_3)d^\beta}{2}} dx_2 dx_1$$

$$= \int_0^d \frac{\alpha_1 d^{2\beta}\beta x_1^{\beta-1}\left(\frac{d^\beta-x_1^\beta}{d^\beta+x_1^\beta}\right)^{\frac{1}{2}d^\beta(\alpha_1+\alpha_2+\alpha_3)}}{d^{2\beta}-x_1^{2\beta}}\, dx_1 = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}.$$

6. Maximum Likelihood Estimation

The estimation of the unknown parameters for BOM distribution are derived in this section using the maximum likelihood estimation (MLE) method. Let $(x_{11}, x_{21}), (x_{12}, x_{22}), \dots, (x_{1n}, x_{2n})$ are random variables from the $BOM(\alpha_1, \alpha_2, \alpha_3, \beta, d)$ distribution and take into consideration the following notations:

$$I_1 = \{(x_{1i}, x_{2i}) : x_{1i} > x_{2i}, i = 1, \dots, n\},\ I_2 = \{(x_{1i}, x_{2i}) : x_{1i} < x_{2i}, i = 1, \dots, n\},$$

$I_3 = \{(x_{1i}, x_{2i}) : x_{1i} = x_{2i}, i = 1, \dots, n\}, I = I_1 \cup I_2 \cup I_3, n_1 = |I_1|, n_2 = |I_2|, n_3 = |I_3|$ and $n_1 + n_2 + n_3 = n.$

The likelihood function for the parameter vector $\theta = (\alpha_1, \alpha_2, \alpha_3, \beta, d)$ is given as

$$L(\theta|x_1, x_2) = \prod_{i\in I_1} f_1(x_{1i}, x_{2i}) \prod_{i\in I_2} f_2(x_{1i}, x_{2i}) \prod_{i\in I_3} f_3(x_i)$$

where,

$$\prod_{i\in I_1} f_1(x_{1i}, x_{2i}) = \alpha_1^{n_1}(\alpha_2+\alpha_3)^{n_1}\beta^{2n_1}d^{4n_1\beta} \prod_{i\in I_1}\left[\left(\frac{x_{1i}^{\beta-1}}{d^{2\beta}-x_{1i}^{2\beta}}\right)\left(\frac{d^\beta+x_{1i}^\beta}{d^\beta-x_{1i}^\beta}\right)^{\frac{-\alpha_1 d^\beta}{2}}\left(\frac{x_{2i}^{\beta-1}}{d^{2\beta}-x_{2i}^{2\beta}}\right)\left(\frac{d^\beta+x_{2i}^\beta}{d^\beta-x_{2i}^\beta}\right)^{\frac{-(\alpha_2+\alpha_3)d^\beta}{2}}\right]$$

$$\prod_{i\in I_2} f_2(x_{1i}, x_{2i}) = \alpha_2^{n_2}(\alpha_1+\alpha_3)^{n_2}\beta^{2n_2}d^{4n_2\beta} \prod_{i\in I_2}\left[\left(\frac{x_{1i}^{\beta-1}}{d^{2\beta}-x_{1i}^{2\beta}}\right)\left(\frac{d^\beta+x_{1i}^\beta}{d^\beta-x_{1i}^\beta}\right)^{\frac{-(\alpha_1+\alpha_3)d^\beta}{2}}\left(\frac{x_{2i}^{\beta-1}}{d^{2\beta}-x_{2i}^{2\beta}}\right)\left(\frac{d^\beta+x_{2i}^\beta}{d^\beta-x_{2i}^\beta}\right)^{\frac{-\alpha_2 d^\beta}{2}}\right]$$

$$\prod_{i\in I_3} f_3(x_i) = \alpha_3^{n_3}\beta^{n_3}$$

The log-likelihood function is given by

$$\begin{aligned}
L = {} & (2n_1 + 2n_2 + n_3)\ln\beta + n_1 \ln\alpha_1 + n_2\ln\alpha_2 + n_3\ln\alpha_3 + n_1\ln(\alpha_2+\alpha_3) \\
& + n_2 \ln(\alpha_1+\alpha_3) + (4n_1\beta + 4n_2\beta + 2n_3\beta)\ln d + \sum_{i\in I_1}\ln\left(\frac{x_{1i}^{\beta-1}}{d^{2\beta}-x_{1i}^{2\beta}}\right) \\
& - \frac{\alpha_1 d^\beta}{2}\sum_{i\in I_1}\ln\left(\frac{d^\beta+x_{1i}^\beta}{d^\beta-x_{1i}^\beta}\right) + \sum_{i\in I_1}\ln\left(\frac{x_{2i}^{\beta-1}}{d^{2\beta}-x_{2i}^{2\beta}}\right) - \frac{(\alpha_2+\alpha_3)d^\beta}{2}\sum_{i\in I_1}\ln\left(\frac{d^\beta+x_{2i}^\beta}{d^\beta-x_{2i}^\beta}\right) \\
& + \sum_{i\in I_2}\ln\left(\frac{x_{1i}^{\beta-1}}{d^{2\beta}-x_{1i}^{2\beta}}\right) - \frac{(\alpha_1+\alpha_3)d^\beta}{2}\sum_{i\in I_2}\ln\left(\frac{d^\beta+x_{1i}^\beta}{d^\beta-x_{1i}^\beta}\right) + \sum_{i\in I_2}\ln\left(\frac{x_{2i}^{\beta-1}}{d^{2\beta}-x_{2i}^{2\beta}}\right) \\
& - \frac{\alpha_2 d^\beta}{2}\sum_{i\in I_2}\ln\left(\frac{d^\beta+x_{2i}^\beta}{d^\beta-x_{2i}^\beta}\right) + \sum_{i\in I_3}\ln\left(\frac{x_i^{\beta-1}}{d^{2\beta}-x_i^{2\beta}}\right) - \frac{(\alpha_1+\alpha_2+\alpha_3)d^\beta}{2}\sum_{i\in I_3}\ln\left(\frac{d^\beta+x_i^\beta}{d^\beta-x_i^\beta}\right)
\end{aligned}$$

The MLEs for the parameters α_1, α_2, α_3, β and d are obtained by computing the first partial derivatives of the log-likelihood function with respect to α_1 , α_2, α_3, β, d and equating these first partial derivatives by zero. The likelihood equations are in the following form

$$\frac{\partial L}{\partial \alpha_1}=\frac{n_1}{\alpha_1}+\frac{n_2}{\alpha_1+\alpha_3}-\frac{1}{2}d^{\beta}\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\sum_{i\in I_3}ln\left(\frac{d^{\beta}+x_{i}^{\beta}}{d^{\beta}-x_{i}^{\beta}}\right)$$

$$\frac{\partial L}{\partial \alpha_2}=\frac{n_2}{\alpha_2}+\frac{n_1}{\alpha_2+\alpha_3}-\frac{1}{2}d^{\beta}\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\sum_{i\in I_3}ln\left(\frac{d^{\beta}+x_{i}^{\beta}}{d^{\beta}-x_{i}^{\beta}}\right)$$

$$\frac{\partial L}{\partial \alpha_3}=\frac{n_3}{\alpha_3}+\frac{n_2}{\alpha_1+\alpha_3}+\frac{n_1}{\alpha_2+\alpha_3}-\frac{1}{2}d^{\beta}\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\sum_{i\in I_3}ln\left(\frac{d^{\beta}+x_{i}^{\beta}}{d^{\beta}-x_{i}^{\beta}}\right)$$

$$\frac{\partial L}{\partial \beta}=\frac{2n_1+2n_2+n_3}{\beta}+(4n_1+4n_2+2n_3)\,ln(d)-\frac{1}{2}d^{\beta}\alpha_1 ln(d)\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta}(\alpha_1+$$
$$\alpha_3)ln(d)\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta}(\alpha_2+\alpha_3)ln(d)\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta}\alpha_2 ln(d)\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta}(\alpha_1+$$
$$\alpha_2+\alpha_3)ln(d)\sum_{i\in I_3}ln\left(\frac{d^{\beta}+x_{i}^{\beta}}{d^{\beta}-x_{i}^{\beta}}\right)+d^{2\beta}\alpha_1\sum_{i\in I_1}\frac{[ln(d)-ln(x_{1i})]x_{1i}^{\beta}}{d^{2\beta}-x_{1i}^{2\beta}}+d^{2\beta}(\alpha_1+\alpha_3)\sum_{i\in I_2}\frac{[ln(d)-ln(x_{1i})]x_{1i}^{\beta}}{d^{2\beta}-x_{1i}^{2\beta}}+$$
$$\sum_{i\in I_1}\frac{d^{2\beta}[-2ln(d)+ln(x_{1i})]x_{1i}^{\beta}ln(x_{1i})}{d^{2\beta}-x_{1i}^{2\beta}}+\sum_{i\in I_2}\frac{d^{2\beta}[-2ln(d)+ln(x_{1i})]x_{1i}^{\beta}ln(x_{1i})}{d^{2\beta}-x_{1i}^{2\beta}}+d^{2\beta}(\alpha_2+\alpha_3)\sum_{i\in I_1}\frac{[ln(d)-ln(x_{2i})]x_{2i}^{\beta}}{d^{2\beta}-x_{2i}^{2\beta}}+$$
$$d^{\beta}\alpha_2\sum_{i\in I_2}\frac{[ln(d)-ln(x_{2i})]x_{2i}^{\beta}}{d^{2\beta}-x_{2i}^{2\beta}}+\sum_{i\in I_1}\frac{d^{2\beta}[-2ln(d)+ln(x_{2i})]x_{1i}^{\beta}ln(x_{2i})}{d^{2\beta}-x_{2i}^{2\beta}}+\sum_{i\in I_2}\frac{d^{2\beta}[-2ln(d)+ln(x_{2i})]x_{1i}^{\beta}ln(x_{2i})}{d^{2\beta}-x_{2i}^{2\beta}}+\frac{1}{2}d^{\beta}(\alpha_1+\alpha_2+$$
$$\alpha_3)\sum_{i\in I_3}\frac{2d^{\beta}[ln(d)-ln(x_i)]x_i^{\beta}}{d^{2\beta}-x_i^{2\beta}}+\sum_{i\in I_3}\frac{2d^{\beta}[-2ln(d)+ln(x_i)]x_i^{2\beta}ln(x_i)}{d^{2\beta}-x_i^{2\beta}}$$

$$\frac{\partial L}{\partial d}=\frac{4\beta n_1+4\beta n_2+2\beta n_3}{d}-\frac{1}{2}d^{\beta-1}\beta\alpha_1\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta-1}\beta(\alpha_1+\alpha_3)\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{1i}^{\beta}}{d^{\beta}-x_{1i}^{\beta}}\right)-\frac{1}{2}d^{\beta-1}\beta(\alpha_2+$$
$$\alpha_3)\sum_{i\in I_1}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta-1}\beta\alpha_2\sum_{i\in I_2}ln\left(\frac{d^{\beta}+x_{2i}^{\beta}}{d^{\beta}-x_{2i}^{\beta}}\right)-\frac{1}{2}d^{\beta-1}\beta(\alpha_1+\alpha_2+\alpha_3)\sum_{i\in I_3}ln\left(\frac{d^{\beta}+x_{i}^{\beta}}{d^{\beta}-x_{i}^{\beta}}\right)-$$
$$2d^{2\beta-1}\beta\sum_{i\in I_1}\frac{1}{d^{2\beta}-x_{1i}^{2\beta}}-2d^{2\beta-1}\beta\sum_{i\in I_2}\frac{1}{d^{2\beta}-x_{1i}^{2\beta}}+d^{2\beta-1}\beta\alpha_1\sum_{i\in I_1}\frac{x_{1i}^{\beta}}{d^{2\beta}-x_{1i}^{2\beta}}+d^{2\beta-1}\beta(\alpha_1+\alpha_3)-$$
$$2d^{2\beta-1}\beta\sum_{i\in I_1}\frac{1}{d^{2\beta}-x_{2i}^{2\beta}}-2d^{2\beta-1}\beta\sum_{i\in I_2}\frac{1}{d^{2\beta}-x_{2i}^{2\beta}}+d^{2\beta-1}\beta(\alpha_2+\alpha_3)\sum_{i\in I_1}\frac{x_{2i}^{\beta}}{d^{2\beta}-x_{2i}^{2\beta}}+d^{2\beta-1}\beta\alpha_2\sum_{i\in I_2}\frac{x_{2i}^{\beta}}{d^{2\beta}-x_{2i}^{2\beta}}-$$
$$2d^{2\beta-1}\beta\sum_{i\in I_3}\frac{1}{d^{2\beta}-x_{i}^{2\beta}}+d^{2\beta-1}\beta(\alpha_1+\alpha_2+\alpha_3)\sum_{i\in I_3}\frac{x_{i}^{\beta}}{d^{2\beta}-x_{i}^{2\beta}}.$$

Since the above system of non-linear equations cannot be solved analytically, a numerical technique is needed to get the MLEs.

7. Data Analysis

In this section, two real-life data set are used to explain the proposed procedure and show that the importance of BOM distribution. Unfortunately, there is no proper goodness of fit test for bivariate distributions as the univariate case. So, before analyzing the data using BOM distribution, we firstly examined goodness of fit for the marginal X_1, X_2 and min (X_1, X_2) using Kolmogorov-Smirnov (K-S) statistics and its p-value. This gives some indication about fitting of the BOM distribution to the data and it will support to predict the initial values of the parameters. The MLEs, log-likelihood function and goodness of fit criteria are computed for each data set.

First data set:

Bivariate Marshall Olkin distribution methods are very useful for modelling failure of paired organs, such as the eyes, kidneys, and lungs. Although there are inevitable relations between the components of such organs, these organs may possibly fail one after the other or at the same time.

The study has been performed by the National Eye Institute to evaluate the result of laser photocoagulation in delaying the onset of severe vision loss such as blindness in the Diabetic Retinopathy Study (DRS). The study involved 197 high risk patients to investigate the usage of the proposed method. In Table 1, A subset of 38 patients is selected from DRS to explore the usefulness of the proposed BOM distribution. One eye is randomly assigned to each patient to receive treatment using laser and the other eye did not receive any type of treatment. The time from the beginning of treatment to the time when visual acuity dropped below 5/200 is evaluated for each eye. As shown in Table 1, the data is the time of vision loss in months. Let X_1 is the time (in months) to the blindness of the untreated eye and X_2 be the time to the blindness of the eye that received laser treatment. For computational stability with the fitting of the distribution, all the data are divided by 100.

Table 1: Time of Vision Loss for Diabetic Retinopathy Patients

i	X_{1i}	X_{2i}	i	X_{1i}	X_{2i}
1	30.83	38.57	20	13.33	9.60
2	20.17	6.90	21	14.27	7.60
3	10.27	1.63	22	34.57	1.80
4	5.67	13.83	23	4.10	12.20
5	5.77	1.33	24	21.57	9.90
6	5.90	35.53	25	13.77	13.77
7	25.63	21.90	26	33.63	33.63
8	33.90	14.80	27	63.33	27.60
9	1.73	6.20	28	38.47	1.63
10	30.20	22.00	29	10.33	0.83
11	25.80	13.87	30	13.83	1.57
12	5.73	48.30	31	11.07	1.97
13	9.90	9.90	32	2.10	11.30
14	1.70	1.70	33	12.93	4.97
15	1.77	43.03	34	24.43	9.87
16	8.30	8.30	35	13.97	30.40
17	18.70	6.53	36	13.80	19.00
18	42.17	42.17	37	13.57	5.43
19	14.30	48.43	38	42.43	46.63

Second data set:

The data includes 32 claims for compensation from motorcycle accident insurance. In Table 2, X_1 and X_2 represent the cost of property damage and medical expenses, respectively. Before analysing the data, all the data points are divided by 1000.

For Marshall Olkin bivariate distribution, (X_1, X_2) represents the bivariate data with all possibilities as follows (i) $X_1 < X_2$, (ii) $X_1 > X_2$ and (iii) $X_1 = X_2$. First, we fit the Omega distribution for X_1, X_2 and $min\ (X_1, X_2)$. It will support to prediction the initial values of the BOM distribution. The maximum likelihood estimators, the (K-S) distances and p-values are shown in Table 3 for two data sets. Based on the p-values, its shown that the Omega model is fitted for the marginals and for the minimum also.

Table 2: The Cost of Property Damage and The Medical Expenses.

i	X_{1i}	X_{2i}	*i*	X_{1i}	X_{2i}
1	144	793	17	298	271
2	134	945	18	114	489
3	500	500	19	335	807
4	720	400	20	449	499
5	230	784	21	160	542
6	374	881	22	224	349
7	175	175	23	323	103
8	252	252	24	704	522
9	300	417	25	470	470
10	665	456	26	368	368
11	199	243	27	171	999
12	412	198	28	106	974
13	720	183	29	529	202
14	591	784	30	423	375
15	305	222	31	500	198

16	292	214	32	350	707

Table 3: The MLEs, K-S and the p-values

Data Set	*Variable*	α	β	*d*	*K-S*	*p-value*
Diabetic Retinopathy Data	X_1	8.6399	1.3354	1209.42	0.1307	0.5347
	X_2	6.3883	1.0462	4411.35	0.1109	0.7383
	$min\ (X_1, X_2)$	10.7406	1.0360	1.0444	0.1343	0.6110
Motorcycle Accident Insurance Data	X_1	6.7951	2.1418	154.408	0.0687	0.9981
	X_2	3.2524	1.9283	311.185	0.1192	0.7532
	$min\ (X_1, X_2)$	6.1947	2.0453	221.049	0.0802	0.9950

Now, we will fit the BOM distribution. Then, the MLEs and their related log-likelihood for the bivariate data set are presented in Table 4. For model selection, AIC, BIC, CAIC and HQIC are also provided in Table 4. The results show that the BOM distribution is fitted for the two bivariate data sets (diabetic retinopathy data and motorcycle accident insurance Data).

Table 4: The MLEs and Goodness of Fit Criteria for Bivariate Omega Distribution

Data Set	*MLEs*	*- Log(l)*	*AIC*	*BIC*	*CAIC*	*HQIC*
Diabetic Retinopathy Data	$\alpha_1 = 0.6899$ $\alpha_2 = 0.8444$ $\alpha_3 = 0.5381$ $\beta = 0.3431$ $d = 77.197$	46.566	103.132	111.32	105.007	106.045
Motorcycle Accident Insurance Data	$\alpha_1 = 0.6165$ $\alpha_2 = 0.5339$ $\alpha_3 = 0.3975$ $\beta = 0.4269$ $d = 72.637$	94.043	198.086	206.274	199.961	200.999

8. Conclusion

In this paper, we introduced a new model for bivariate distribution using Marshall Olkin methods called Bivariate Omega Model (BOM). Some probabilistic properties and dependence properties of the bivariate Omega distribution are considered. By using the maximum likelihood method parameters estimators are explored. Finally, we proposed the applicability of Bivariate Omega Model (BOM) using two real data for modelling failure of paired organs, such as the eyes, kidneys, and lungs.

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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