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New Class of Trivariate Copula: A Case Study for Water Quality Measurements

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Abstract: In this paper, we provide a new method to construct trivariate copula which generalizes a class of bivariate copula. Based on the new method, two trivariate copulas are introduced using Product, Ali-Mikhail-Haq and Farlie-Gumbel-Morgenstern bivariate copulas. The selection of a bivariate copulas is adapted to make that the new trivariate copula satisfy the mathematical properties of the copula. Some properties concerning dependence concepts of the two new classes of trivariate copulas are discussed. Finally, an application for water quality measurements is presented to show applicability of the proposed copulas. We compare the new trivariate copula to the trivariate nested hierarchical copula using goodness-of-fit criteria to ensure that the proposed trivariate copula are the best fitted.

Keywords: Kendall's tau; Nested Hierarchical Copula; Spearman's rho; Trivariate Copula; Water Quality Measurements.

1 Introduction

Copulas are interesting which provide to construct more flexible models for multivariable random vectors. Moreover, copula function is a relevant tool for describing the pattern of dependencies between random variables. Firstly, [1] describe the relation between the joint distribution function and its marginal. Many authors considered copula function in different fields, [2] is one of the most important authors that collected copula concepts, properties, theorems, examples and applications. Copulas have been influential in a number of statistical fields with more applications. Copula with applications to the energy, forestry and environmental sciences is introduced in [3]. Also, copula modelling is important for economic and financial data, see, [4].

In the case of trivariate copulas, [5] proposed a new class of 3-copulas with some of its properties. A fully nested hierarchical method which creating copulas proposed by [6]. The trivariate copulas are applied using a full nested hierarchical method to design coastal structures in [7]. The nested hierarchical method used to fit trivariate copula model in drought analysis, see [8].

This paper structured as follows. Section 2 introduces general definitions for copula function and their properties. Section 3 is devoted to presentation of trivariate nested hierarchical copula. Moreover, two new kinds of trivariate nested hierarchical copula are introduced and studied. The main novel idea presented in Section 4 by constructs a new method for building trivariate copula, called Trivariate mixed Copula. Based on the new method, two new mixed trivariate copulas are introduced and their dependence structure are calculated. An application for water quality analysis is introduced in Section 5, to show the efficiency for the new trivariate copulas.

2 Preliminary Concepts

2.1 Copulas

Copula function is one of the popular ways to express and generate families of bivariate distributions and study dependence properties. Theorem 2.1 is a main role in the whole theory of copula called "Sklar theorem" introduced by [1]. This theorem shows the relation between bivariate distributions with their related marginal functions.

Theorem 2.1. Let H be a k-dimensional distribution function with marginal distribution functions $F(x_i), i = 1, \ldots, k$. Then there exists an k-dimensional copula C with uniform marginals such as

$$H(x_1, \ldots, x_k) = C\big(F_1(x_1), \ldots, F_k(x_k)\big), \forall\, (x_1, \ldots, x_k) \in R^k \quad (1)$$

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Moreover, if $F_1, \dots, F_k$ are all continuous, then C is unique. Let $F_1^{-1}, \dots, F_k^{-1}$ be quasi-inverses of $F_1, \dots, F_k$ respectively. Then for any $u_1, \dots, u_k$ in $[0,1]^k$

$$C(u_1, \dots, u_k) = H(F_1^{-1}(u_1), \dots, F_k^{-1}(u_k)) \quad (2)$$

Thus, a copula density $c(u_1, \dots, u_k)$ can be calculated using the partial derivative with respect to $u_1, \dots, u_k$ that $\frac{\partial^k C(u_1,\dots,u_k)}{\partial u_1,\dots,\partial u_k}$.

A trivariate function $C(u_1, u_2, u_3)$ that maps $[0,1]^3$ to $[0,1]$ is a copula if it satisfies the conditions in the following definition.

Definition 2.1. A trivariate function (u_1, u_2, u_3), $C: [0,1]^3 \to [0,1]$, is a 3-dimensional copula if and only if C satisfies the following conditions:

1) *Boundary conditions*

$C(u_1, u_2, 0) = C(u_1, 0, u_3) = C(0, u_2, u_3) = 0$, $\forall u_1, u_2, u_3 \in [0,1]$;
$C(u_1, 1, 1) = u_1, C(1, u_2, 1) = u_2, C(1, 1, u_3) = u_3$

2) *Increasing property*

C-volume of any rectangle $R = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \subseteq [0,1]^3$ is positive.

$$V_C(R) = -C(x_1, y_1, z_1) + C(x_1, y_1, z_2) + C(x_1, y_2, z_1) - C(x_1, y_2, z_2) + C(x_2, y_1, z_1) - C(x_2, y_1, z_2) - C(x_2, y_2, z_1) + C(x_2, y_2, z_2) \geq 0.$$

Numerous copulas that impose various dependent relationships between the marginal distribution functions are found in the literature. Bivariate copulas and their characteristics are thoroughly covered by [9]. In this paper, several copulas that were used to construct a new trivariate copulas are presented.

- Product Copula:

$$C(u_1, u_2) = u_1 u_2 \quad (3)$$

The product Copula corresponds to independence.

- Ali-Mikhail-Haq Copula:

$$C(u_1, u_2) = \frac{u_1 u_2}{1 - \theta(1 - u_1)(1 - u_2)} \quad (4)$$

- Farlie-Gumbel-Morgenstern Copula:

$$C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2) \quad (5)$$

where, θ is the dependence parameter.

The description of three copulas, namely Product, Ali-Mikhail-Haq and Farlie-Gumbel-Morgenstern copula are shown in Table 1.

Table 1: Description of Copulas Used.

Copula Family	Copula $C(u_1, u_2)$	Copula Parameter	Kendall's tau
Product	$u_1 u_2$	-	0
Ali-Mikhail-Haq	$\frac{u_1 u_2}{1 - \theta(1 - u_1)(1 - u_2)}$	$\theta \in [-1,1]$	$\frac{3\theta - 2}{\theta} - \frac{2}{3}\left(1 - \frac{1}{\theta}\right)^2 \ln(1 - \theta)$
Farlie-Gumbel-Morgenstern	$u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2)$	$\theta \in [-1,1]$	$\frac{2\theta}{9}$

2.2 Multivariate Dependence Structure

Copula function give a full description to the dependence structure between a pair of random variables or more. To measure how strong the association is between random variables using association measures as Kendall's tau and Spearman's rho, see [2]. The partial and average conditional Kendall's tau and proposed a multivariate Kendall's tau and Spearman's rho for multivariate copula $C(u_1, u_2, \dots, u_k)$ are studied in [10] as follows,

$$\tau(C(u_1, u_2, \dots, u_p)) = \frac{1}{2^{k-1} - 1}\left\{2^k \int_{[0,1]^k} C(u_1, u_2, \dots, u_k)\, dC(u_1, u_2, \dots, u_k) - 1\right\} \quad (6)$$

$$\rho(C(u_1, u_2, \dots, u_p)) = \frac{k+1}{2^k - (k+1)}\left\{2^k \int_{[0,1]^k} C(u_1, u_2, \dots, u_k) du_1 du_2, \dots, du_k) - 1\right\} \quad (7)$$

Using Eq.(6) and Eq.(7), the Kendall's tau and Spearman's rho for trivariate copula $C(u_1, u_2, u_3)$ are given by

$$\tau\big(C(u_1, u_2, u_3)\big) = \frac{1}{3}\left\{\int_0^1 \int_0^1 \int_0^1 8\, C(u_1, u_2, u_3)\, dC(u_1, u_2, u_3) - 1\right\} \quad (8)$$

$$\rho\big(C(u_1, u_2, u_3)\big) = 8 \int_0^1 \int_0^1 \int_0^1 C(u_1, u_2, u_3)\, du_1\, du_2\, du_3 - 1 \quad (9)$$

3 Trivariate Nested Hierarchical Copula

The main idea for trivariate copulas is examining the correlation between three different variables. There are several methods for constructing new trivariate copulas. Building trivariate copula from bivariate copulas as a fully nested hierarchical copula is one way to account for the correlations more effectively between variables two by two; for more information see [2]. The fully nested hierarchical trivariate copula is defined as follows,

$$C(u_1,u_2,u_3)=C_1(C_2(u_1,u_2),u_3) \quad (10)$$

When testing a fully nested hierarchical copula, [11] only employ one bivariate copula and do not differentiate between C_1 and C_2. Using Eq. (10) to construct a trivariate copula with any bivariate copulas C_1 and C_2, we must check that the new function is a copula and satisfies the properties in Definition 2.1.

3.1 Farlie-Gumbel-Morgenstern Product Nested Copula (FGPN)

Construction of a trivariate nested copula will be made by taking C_1 is a Farlie-Gumbel-Morgenstern copula given by Eq. (5) and C_2 is a product copula given by Eq. (3) and substituting in Eq.(10), we get

$$C(u_1,u_2,u_3)=u_1u_2u_3[1+\theta(1-u_1u_2)(1-u_3)],$$

$$0\le u_1,u_2,u_3\le 1 \quad (11)$$

where $\theta\in[-1,1]$. By some computation we get Eq. (11) satisfies all conditions in Definition 2.1, so Eq. (11) called a trivariate copula. Therefore $C(u_1,u_2,u_3)$ in Eq. (11) called Farlie-Gumbel-Morgenstern Product nested *(FGPN)* trivariate copula.

3.2 Ali-Mikhail-Haq Product Nested Copula (AMPN)

A trivariate nested hierarchical copula is presented using Ali-Mikhail-Haq and product bivariate copulas in Eq. (10), let C_1 be Ali-Mikhail-Haq copula in Eq. (4) and C_2 is a product copula in Eq. (3). Substituting by Eqs. (3) and (4) in Eq. (10) we get

$$C(u_1,u_2,u_3)=\frac{u_1u_2u_3}{1-\theta(1-u_1u_2)(1-u_3)},$$

$$0\le u_1,u_2,u_3\le 1 \quad (12)$$

where $\theta\in[-1,1]$. Eq. (12) satisfies all properties in Definition 2.1, so this function is a trivariate copula. Hence, $C(u_1,u_2,u_3)$ in Eq. (12) is called Ali-Mikhail-Haq Product nested *(AMPN)* trivariate copula.

4 New Trivariate Mixed Copulas

In the case of building trivariate functions from bivariate copulas, we will introduce a new method to construct a trivariate copula with more flexibility. This way is the simplest for creating trivariate copula based on mixed of two bivariate copulas. The new trivariate mixed copula is presented as follows,

$$C(u_1,u_2,u_3)=3u_1u_2u_3-u_3C_1(u_1,u_2)-u_1C_2(u_2,u_3) \quad (13)$$

We must check that the function in Eq. (13) is a copula and satisfies the properties in Definition 2.1. We found the best fitting trivariate copulas using bivariate copulas Product and Farlie-Gumbel-Morgenstern copula and also Product and Ali-Mikhail-Haq copulas which has been successfully applied for water quality measurements data.

4.1 Product Farlie-Gumbel-Morgenstern Mixed Copula (PFGM)

Let we have a product and a Farlie-Gumbel-Morgenstern copulas given by Eqs. (3) and (5), respectively, as C_1 and C_2 in Eq. (13). Thus, we obtained a construction of new trivariate mixed copula, namely *PFGM* as follows,

$$C(u_1,u_2,u_3)=u_1u_2u_3[1-\theta(1-u_2)(1-u_3)] \quad (14)$$

where $\theta\in[-1,1]$. This new trivariate copula satisfies all boundary and increasing properties in Definition 2.1. Therefore $C(u_1,u_2,u_3)$ in Eq. (14) is called Product Farlie-Gumbel-Morgenstern mixed *(PFGM)* trivariate copula.
If U_1,U_2,U_3 are pairwise independent, then the conditional distribution of $(U_2,U_3)'$ given $U_1=u_1$ is given by

$$C_{2,3|1}(u_2,u_3|u_1)=\frac{\partial}{\partial u_1}C(u_1,u_2,u_3)=u_2u_3[1-\theta(1-u_2)(1-u_3)]$$

The probability density function of the trivariate *PFGM* copula, which is given in Eq. (14) is

$$c(u_1,u_2,u_3)=\frac{\partial^3C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}=1-\theta(1-2u_2)(1-2u_3) \quad (15)$$

In Proposition 1 and Proposition 2, an explicit expression of Kendall’s tau and Spearman’s rho for Product Farlie-Gumbel-Morgenstern mixed (*PFGM*) trivariate copula are given.

Proposition 1. Given $\theta\in[-1,1]$ is the copula parameter, the Kendall's tau for *PFGM* trivariate copula is given by

$$\tau_{PFGM}(C(u_1,u_2,u_3))=-\frac{2\theta}{27} \quad (16)$$

Proof. To compute Kendall's tau for *PFGM* trivariate copula we use Eq. (8) and substituting by Eq. (14) and the third partial derivatives of the new copula Eq. (15). Hence,

$$\tau_{PFGM}(C(u_1,u_2,u_3)) = \frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\,C(u_1,u_2,u_3)\,dC(u_1,u_2,u_3)-1\right\}$$

$$=\frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\,C(u_1,u_2,u_3)\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}du_1\,du_2\,du_3-1\right\}$$

$$=\frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\,(u_1u_2u_3[1-\theta(1-u_2)(1-u_3)])(1-\theta(1-2u_2)(1-2u_3)\ )\,du_1\,du_2\,du_3-1\right\}$$

$$=\frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\,u_1u_2u_3(1+(-1+\theta u_2(2-4u_3)+2u_3))(1+\theta(-1+u_2+u_3-u_2u_3))\,du_1\,du_2\,du_3-1\right\}$$

$$=\frac{1}{3}\left\{8\left(\frac{1}{8}-\frac{\theta}{36}\right)-1\right\}=-\frac{2\theta}{27}$$

Proposition 2. Given $\theta\in[-1,1]$ is the copula parameter, the Spearman's rho *PFGM* trivariate copula is given by

$$\rho_{PFGM}\big(C(u_1,u_2,u_3)\big)=-\frac{\theta}{9} \tag{17}$$

Proof. To compute Spearman's rho for *PFGM* trivariate copula we substitute by Eq.(14) in Eq. (9), thus we get

$$\rho_{PFGM}\big(C(u_1,u_2,u_3)\big)= 8\int_0^1\int_0^1\int_0^1 C(u_1,u_2,u_3)\,du_1\,du_2\,du_3-1$$

$$=8\int_0^1\int_0^1\int_0^1 u_1u_2u_3[1-\theta(1-u_2)(1-u_3)]\,du_1\,du_2\,du_3-1$$

$$=4\int_0^1\int_0^1 u_2u_3[1-\theta(1-u_2)(1-u_3)]\,du_1\,du_2\,du_3-1$$

$$=8\left(\frac{9-\theta}{72}\right)-1=-\frac{\theta}{9}$$

4.2 Product Ali-Mikhail-Haq Mixed Copula (PAMM)

Now we combine two other bivariate copulas, Product with Ali-Mikhail-Haq copulas, using new mixed construction form in Eq. (13) to obtain new trivariate mixed copula, named Product Ali-Mikhail-Haq mixed (*PAMM*) copula. By substituting of Eqs. (3) and (4) in Eq. (13) a new copula is created as

$$C(u_1,u_2,u_3)=u_1u_2u_3\left[2-\frac{1}{1-\theta(1-u_2)(1-u_3)}\right] \tag{18}$$

This new trivariate copulas fulfills all boundary and increasing properties in Definition 2.1. Therefore $C(u_1,u_2,u_3)$ in Eq. (18) is called Product Ali-Mikhail-Haq mixed *(PAMM)* trivariate copula.

If U_1,U_2,U_3 are pairwise independent, then the conditional distribution of (U_2,U_3) given $U_1=u_1$ is given by

$$C_{2,3|1}(u_2,u_3|u_1)=\frac{\partial}{\partial u_1}C(u_1,u_2,u_3)$$
$$=u_2u_3\left[2-\frac{1}{1-\theta(1-u_2)(1-u_3)}\right]$$

The probability density function of the *PAMM* copula, is given by,

$$c(u_1,u_2,u_3)=\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}$$

$$=2+\frac{1+\theta(-2+u_2+u_3+u_2u_3+\theta(u_2-1)(u_3-1))}{(-1+\theta(u_2-1)(u_3-1))^3} \tag{19}$$

In Proposition 3 and Proposition 4, an explicit expression of Kendall's tau and Spearman's rho for Product Ali-Mikhail-Haq mixed (*PAMM*) trivariate copula are obtained.

Proposition 3. Given $\theta\in[-1,1]$ is the copula parameter, the Kendall's tau for *PAMM* trivariate copula is given by,

$$\tau_{PAMM}\big(C(u_1,u_2,u_3)\big)=\frac{\theta(15\theta+142)-2(\theta^2+46\theta-47)ln\,(\theta-1)-48(\theta+1)Li_2(\theta)}{9\theta^2} \tag{20}$$

where $Li_2(\theta)$ is the polylogarithm function.

Proof. To compute Kendall's tau for *PAMM* trivariate copula we use Eq. (7) and substituting by Eq. (18) and the third partial derivatives of the new copula Eq. (19). Hence,

$$\tau_{PAMM}\big(C(u_1,u_2,u_3)\big)=$$

$$\frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\,C(u_1,u_2,u_3)\,dC(u_1,u_2,u_3)-1\right\}$$

$$=\frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\,C(u_1,u_2,u_3)\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}du_1\,du_2\,du_3-1\right\}$$

$$=\frac{1}{3}\left\{\int_0^1\int_0^1\int_0^1 8\left(u_1u_2u_3\left[2-\frac{1}{1-\theta(1-u_2)(1-u_3)}\right]\right)\left(2+\frac{1+\theta(-2+u_2+u_3+u_2u_3+\theta(u_2-1)(u_3-1))}{(-1+\theta(u_2-1)(u_3-1))^3}\right)du_1\,du_2\,du_3-1\right\}$$

$$=\frac{1}{3}\left\{\frac{8\theta(9\theta+71)-(\theta-1)(\theta+47)ln\,(\theta-1)-24(\theta+1)Li_2(\theta)}{12\theta^2}-1\right\}$$

$$=\frac{\theta(15\theta+142)-2(\theta^2+46\theta-47)ln\,(\theta-1)-48(\theta+1)Li_2(\theta)}{9\theta^2}$$

Proposition 4. Given $\theta\in[-1,1]$ is the copula parameter, the Spearman's rho for Product Ali-Mikhail-Haq mixed

(*PAMM*) trivariate copula is given by

$$\rho_{PAMM}\left(C(u_1,u_2,u_3)\right)=\frac{\theta(\theta+12)-8(\theta-1)ln(\theta-1)-4(\theta+1)Li_2(\theta)}{\theta^2} \quad (21)$$

where $Li_2(\theta)$ is the polylogarithm function.

Proof. To compute Spearman's rho for *PAMM* trivariate copula we substitute by Eq. (18) in Eq. (7), thus we get

$$\rho_{PAMM}(C(u_1,u_2,u_3))=8\int_0^1\int_0^1\int_0^1 C(u_1,u_2,u_3)\,du_1\,du_2\,du_3-1$$

$$=8\int_0^1\int_0^1\int_0^1 u_1u_2u_3\left[2-\frac{1}{1-\theta(1-u_2)(1-u_3)}\right]du_1\,du_2\,du_3-1$$

$$=8\left(\frac{\theta(\theta+6)-4(\theta-1)\ln(\theta-1)-2(\theta+1)Li_2(\theta)}{4\theta^2}\right)-1$$

$$=\frac{\theta(\theta+12)-8(\theta-1)ln(\theta-1)-4(\theta+1)Li_2(\theta)}{\theta^2}$$

5 Application: Water Quality Analysis

In addition to the alteration of the natural water cycle brought on by human activity, problems with the quality of the water (including but not limited to bacteria, temperature, dissolved oxygen, and phosphorus) have also been identified. According to the availability of the water quality dataset published by USGS, temperature (Tem), dissolved oxygen (DO), and phosphorus (Phs) are selected for the Chattahoochee River.

Table2: Monthly Water Quality Measurements for the Chattahoochee River Watershed.

Time	Tem (°C)	DO (mg/L)	Phs (mg/L)	*Time*	Tem (°C)	DO (mg/L)	Phs (mg/L)
Sep-06	26.5	5.8	0.127	Dec-08	12.5	8.8	0.256
Oct-06	23.4	8.3	0.056	Jan-09	10.8	10.4	0.215
Nov-06	18.4	8.2	0.082	Feb-09	10.3	10.1	0.149
Dec-06	13.1	9.2	0.05	Mar-09	7.8	12.3	0.103
Jan-07	12.6	8.6	0.163	Apr-09	15.5	7.8	0.074
Feb-07	11.5	10.8	0.065	May-09	22.4	7.1	0.092
Mar-07	21.9	8.4	0.041	Jun-09	25.6	6.5	0.095
Apr-07	17.4	7.8	0.081	Jul-09	24.6	7.2	0.074
May-07	23	9.1	0.065	Aug-09	27.4	7.2	0.096
Jun-07	25.9	6.3	0.104	Sep-09	21.6	6.2	0.542
Jul-07	27.2	7	0.083	Oct-09	15.1	8.4	0.103
Aug-07	29.3	6.3	0.055	Nov-09	13.8	8.8	0.03
Sep-07	20.6	7.8	0.079	Dec-09	11.6	9.2	0.035
Oct-07	20.9	7.9	0.07	Jan-10	8	10.9	0.078
Nov-07	15.4	8.7	0.064	Feb-10	6.6	11.8	0.051
Dec-07	16.7	9	0.038	Mar-10	9.6	10.8	0.054
Jan-08	7.5	10.6	0.072	Apr-10	14.6	9	0.078
Feb-08	12.8	9.2	0.069	May-10	16.1	9.3	0.068
Mar-08	15.1	8.5	0.124	Jun-10	25	7.3	0.092
Apr-08	15.3	8.5	0.059	Jul-10	28.3	7.4	0.065
May-08	22.6	6.7	0.073	Aug-10	25.4	7.6	0.056
Jun-08	26	6.9	0.067	Sep-10	24.4	7.6	0.025
Jul-08	26.5	6.4	0.112	Oct-10	18.3	9.2	0.054
Aug-08	27.6	6.5	0.061	Nov-10	10.7	10.3	0.036
Sep-08	20.3	7.9	0.079	Dec-10	5.6	12.2	0.032
Oct-08	20.4	7.6	0.08	Jan-11	7	11.5	0.034
Nov-08	9.8	11.1	0.086				

River watershed. To examine the proposed copula, the period with continuous measurements is selected, that is, September 2006–January 2011. Water quality parameters listed in Table 2.

The main descriptive statistics for the given data are summarized in Table 3. The selection of the most suitable probability distribution and associated parameter estimation procedure are the fundamental step for data analysis. [12] showed that most water quality variables were best fit to different types of probability distribution functions, including the gamma, Weibull, lognormal, exponential, and Logistic distributions. In Table 4, we use the Kolmogorov-Smirnov test were used to select the marginal probability distribution which best fitted for Chattahoochee River watershed data. We notice that the common fit model for the three variables (Tem, DO, Phs) is the Weibull distribution.

Via goodness-of-fit criteria, we make a comparison between the proposed new trivariate copulas against the trivariate nested hierarchical copulas. The popular criteria for model selection are used which are the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC) and Hannan–Quinn Information Criterion (HQIC) which are defined respectively as

$$\mathrm{AIC} = -2l + 2q$$
$$\mathrm{BIC} = -2l + q\log(n)$$
$$\mathrm{HQIC} = -2l + 2q\log(\log(n))$$
$$\mathrm{CAIC} = -2l + \frac{2qn}{n-q-1}$$

where l denotes the log-likelihood function evaluated at the maximum likelihood estimates for parameters, q is the number of parameters and *n* is the sample size.

The model with minimum AIC (or BIC, CAIC and HQIC) value is chosen as the best model to fit the data. From Table 5, we note that the new Product Farlie-Gumbel-Morgenstern mixed (*PFGM*) copula Eq. (14) is fitting better than Farlie-Gumbel-Morgenstern Product nested (*FGPN)* copula Eq. (11). Also, the obtained results in Table 6 indicate the new Product Ali-Mikhail-Haq mixed (*PAMM)* copula Eq. (16) is fitting better than Ali-Mikhail-Haq Product nested (*AMPN)* copula Eq. (12).

Table 3: Descriptive Statistics of Water Quality Measurements.

Variable	***n***	***Min***	***Max***	***Mean***	***Variance***	***SD***	***Skewness***	***Kurtosis***
***Tem* (˚C)**	53	5.6	29.3	17.85	47.13	6.86	-0.058	-1.26
DO (mg/L)	53	5.8	12.3	8.57	2.77	1.66	0.51	-0.46
Phs (mg/L)	53	0.025	0.542	0.089	0.0058	0.076	4.45	24.53

Table 4: K-S and the p-values for of Five Distributions.

Variable	***Distribution***	***K-S***	***p-value***
Tem	Weibull Gamma Lognormal Logistic Exponential (2p)	0.10329 0.13305 0.14119 0.11346 0.18525	0.5877 0.27961 0.2195 0.46846 0.04588
DO	Weibull Gamma Lognormal Logistic Exponential (2p)	0.10679 0.08595 0.07305 0.10795 0.19533	0.54556 0.79682 0.9202 0.53181 0.03019
Phs	Weibull Gamma Lognormal Logistic Exponential (2p)	0.13351 0.22949 0.11461 0.25811 0.17782	0.27592 0.00617 0.45582 0.00133 0.06157

Table 5: The MLEs and Goodness of Fit Criteria for New *PFGM* and Nested *FGPN* Copulas.

Copula	***MLEs***	***Estimated dependence parameter***	***Log(l)***	***AIC***	***BIC***	***CAIC***	***HQIC***
PFGM	$\hat{\beta}_1 = 0.0101$ $\hat{\beta}_2 = 0.0075$ $\hat{\beta}_3 = 2.0581$ $\hat{\gamma}_1 = 1.5665$ $\hat{\gamma}_2 = 2.1301$ $\hat{\gamma}_3 = 0.6416$	$\hat{\theta} = -0.9496$	-279.066	572.133	585.925	574.621	577.436
FGPN	$\hat{\beta}_1 = 0.7435$ $\hat{\beta}_2 = 0.0058$ $\hat{\beta}_3 = 1.2233$ $\hat{\gamma}_1 = 0.2863$ $\hat{\gamma}_2 = 2.1149$ $\hat{\gamma}_3 = 0.9531$	$\hat{\theta} = 0.9648$	-480.086	974.172	987.964	976.661	979.476

Table 6: The MLEs and Goodness of Fit Criteria for New *PAMM* and Nested *AMPN*.

Copula	*MLEs*	*Estimated dependence parameter*	*Log(l)*	*AIC*	*BIC*	*CAIC*	*HQIC*
PAMM	$\hat{\beta}_1 = 0.0075$ $\hat{\beta}_2 = 0.0104$ $\hat{\beta}_3 = 2.4717$ $\hat{\gamma}_1 = 1.6652$ $\hat{\gamma}_2 = 2.0064$ $\hat{\gamma}_3 = 0.6902$	$\hat{\theta} = -0.8349$	-273.466	560.93	574.725	563.421	566.236
AMPN	$\hat{\beta}_1 = 2.2592$ $\hat{\beta}_2 = 0.0110$ $\hat{\beta}_3 = 10.247$ $\hat{\gamma}_1 = 0.1008$ $\hat{\gamma}_2 = 2.7454$ $\hat{\gamma}_3 = 1.3078$	$\hat{\theta} = 0.7881$	-318.641	651.28	665.074	653.770	656.585

6 Conclusion

In this paper, we introduced a new method for constructing original trivariate copulas called *PFGM* and *PAMM* copulas using mixed of different bivariate copulas. These copulas fulfill all boundary and increasing properties. Moreover, Kendall's tau and Spearman's rho calculated for these new copulas. Finally, we pointed out the applicability of these new trivariate copulas for the real data set of monthly water quality measurements for the Chattahoochee River watershed. We also compared our results for the *PFGM* and *PAMM* copulas with the trivariate nested hierarchical copulas *FGPN* and *AMPN* via goodness-of-fit criteria, finding that *PFGM* and *PAMM* copulas are the best for fitting the data set.

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