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Application Of Fox-Wright Generalized Hypergeometric Functions to Multivalent Functions

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Abstract: In the present paper, we introduce a new class of multivalent analytic functions by using Fox-Wright generalized Hypergeometric functions and we obtain the coefficient bounds, extreme points, integral representations, distortion bounds, radii of starlikeness and convexity and neighbourhood.

Keywords: Multivalent function, Starlike and Convex function, Hypergeometric functions , Integral Operator. **AMS Subject Classification :** Secondary 30C45.

1 Introduction

Consider the subclass $A(n,p)$ of functions $f(z)\in\mathscr{A}$ of the form

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k, \quad (n,p\in\mathbb{N}) \tag{1}$$

analytic and multivalent functions in the unit disk $\mathscr{U}(1)=\{z:|z|<1\}$. The function $f(z)$ is said to be starlike of order $\delta(0\le\delta<p)$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\delta, \quad (z\in\mathscr{U}). \tag{2}$$

On the other hand $f(z)$ is said to be convex of order $\delta(0\le\delta<p)$ if and only if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\delta, \quad (z\in\mathscr{U}) \tag{3}$$

Definition : Let $\mathbb{H}$ denote a Hilbert space on the complex plane . Let $\mathbb{V}$ denote an operator on $\mathbb{H}$. For a complex-valued function f analytic on $\mathscr{U}(1)$, let $f(\mathbb{V})$ denote the operator on $\mathbb{H}$ defined by the Riesz-Dunford Integral [2]

$$f(\mathbb{V})=\frac{1}{2\pi i}\int_C(z\mathbb{I}-\mathbb{V})^{-1}f(z)dz \tag{4}$$

where $\mathbb{I}$ is the identity operator on $\mathbb{H}, C$ is positively oriented rectifiable Jordan cantour in $\mathscr{U}(1)$ and contain the spectrum of interior domain. The operator $f(\mathbb{V})$ has series representation $f(\mathbb{V})=\sum_{k=0}^{\infty}\frac{f^k(0)}{k!}\mathbb{V}^n$, which converges in the normed topology [4]. For complex parameters

$$\alpha_1,\cdots,\alpha_q\left(\frac{\alpha_j}{A_j}\ne 0,-1,-2,\cdots;j=1,\cdots,q\right)$$

and

$$\beta_1,\cdots,\beta_s\left(\frac{\beta_j}{B_j}\ne 0,-1,-2,\cdots;j=1,\cdots,s\right)$$

we define the Fox-Wright generalized hypergeometric function [5] (see also [7], [9], [13], [14]),

$$\begin{aligned} & {}_q\psi_s[(\alpha_1,A_1),\cdots,(\alpha_q,A_q);(\beta_1,B_1),\cdots,(\beta_s,B_s);z] \\ = & {}_q\psi_s[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s};z] \\ = & \sum_{k=0}\left\{\prod_{j=1}^{q}\Gamma(\alpha_j+A_jk)\right\}\left\{\prod_{j=1}^{s}\Gamma(\beta_j+B_jk)\right\}^{-1}\frac{z^k}{k!} \end{aligned} \tag{5}$$

$(A_j>0\ (j=1,\cdots,q);\ B_j>0\ (j=1,\cdots,s); 1+\sum_{j=1}^{s}B_j-\sum_{j=1}^{q}A_j\ge 0)$.

If $A_j=1\ (j=1,\cdots,q)$ and $B_j=1\ (j=1,\cdots,s)$, we have the relationship

$$w\,{}_q\psi_s[(\alpha_j,1)_{1,q};(\beta_j,1)_{1,s};z]={}_qF_s(\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s;z) \tag{6}$$

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where ${}_qF_s$ is generalized hypergeometric function and

$w = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_s)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_q)}, \ \ (q \leq s+1; q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \mathscr{U}(1)).$

Now let $q,s \in \mathbb{N}$ and suppose that $\alpha_1,\cdots,\alpha_q$ and $\beta_1,\cdots,\beta_s$ are also positive real numbers. Then, we define the function

$${}_q\phi_s[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s};z] = wz_q\psi_s[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s};z]$$

and consider the linear operator [12] $L[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}] : \mathscr{A} \to \mathscr{A}$ defined by

$$L[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}]f(z) = {}_q\phi_s[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s};z] * f(z). \tag{7}$$

For simplicity, we write

$$L[\alpha_1]f(z) = L[(\alpha_1,A_1),\cdots,(\alpha_q,A_q);(\beta_1,B_1),\cdots,(\beta_s,B_s)]f(z). \tag{8}$$

We note that special cases of this operator were investigated by Dziok and Srivastava [3], by letting $A_j = 1\ (j=1,\cdots,q)$ and $B_j = 1\ (j=1,\cdots,s)$ in 7, and includes the Noor Integral operator [8].

Now we define the class of functions $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$ consisting of functions f defined by 1 which satisfy the condition

$$Re\left\{\frac{L[\alpha_1]f(\mathbb{V})}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})}\right\} > \alpha\left|\frac{L[\alpha_1]f(\mathbb{V}))}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} - p\right| + \beta, \tag{9}$$

for $\alpha \geq 0, 0 \leq < p$ and for all operator $\mathbb{V}$ such that $\mathbb{V} \neq \mathbb{O}$ and $\|\mathbb{V}\| < 1, \mathbb{O}$ being the null operator on $\mathbb{H}$.

We need the following,

Let $f(z) = z^p - \sum\limits_{k=n+p}^{\infty} a_k z^k$ and $g(z) = z^p - \sum\limits_{k=n+p}^{\infty} b_k z^k$ then the Hadamard product $f * g$ is defined as

$$(f*g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k. \tag{10}$$

2 Coefficient bounds

At first, we prove necessary and sufficient condition for the function $f(z)$ as defined by 1 to belong to the class $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Theorem 2.1 : Let a function $f(z) \in T(n,p)$. Then $f(z) \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$ if and only if

$$\sum_{k=n+p}^{\infty} \sigma_k a_k \left(\frac{(1+\alpha) - k(\alpha p+\beta)}{(1+\alpha) - p(\alpha p+\beta)}\right) < 1, \tag{11}$$

$0 \leq < p, \alpha \geq 0$ and where

$$\sigma_k = \frac{\Gamma(\alpha_1 + A_1(k-n)]\cdots\Gamma(\alpha_q + A_q(k-n)]}{\Gamma(_1 + B_1(k-n)]\cdots\Gamma(_s + B_s(k-n)](k-n)!}, \ \ k \in \mathbb{N}. \tag{12}$$

Proof : Let $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$. Using the fact for real γ

$$Re(w) > \alpha|w-p| + \Leftrightarrow Re[w(1+\alpha e^{i\gamma}) - p\alpha e^{i\gamma}] > \beta$$

and letting $w = \frac{L[\alpha_1]f(\mathbb{V}))}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})}$ in 4, we obtain

$$Re\left(\frac{L[\alpha_1]f(\mathbb{V})}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})}(1+\alpha e^{i\gamma}) - p\alpha e^{i\gamma}\right) > \beta$$

or

$$Re\left(\frac{(\mathbb{V})^p - \sum\limits_{k=n+p}^{\infty} a_k\sigma_k(\mathbb{V})^k}{(\mathbb{V})(p(\mathbb{V})^{p-1} - \sum\limits_{k=n+p}^{\infty} ka_k\sigma_k}(k)(\mathbb{V})^{k-1}\left(1+\alpha e^{i\gamma}\right) - p\alpha e^{i\gamma} - \beta\right) > 0$$

Setting $\mathbb{V} = r\mathbb{I}\ (0 < r < 1)$ and letting $r \to 1^-$, yields

$$Re\left(\frac{(1-\beta p) + \left(1-p^2\right)\alpha e^{i\gamma} - \sum\limits_{k=n+p}^{\infty}\left(1+\alpha e^{i\gamma} - kp\alpha e^{i\gamma} - \beta k\right)a_k\sigma_k}{p - \sum\limits_{k=n+p}^{\infty} ka_k\sigma_k}\right) > 0$$

By mean value theorem, we have

$$Re\left((1-\beta p) + (1-p^2)\alpha e^{i\gamma} - \sum_{k=n+p}^{\infty}(1+\alpha e^{i\gamma} - kp\alpha e^{i\gamma} - \beta k)a_k\sigma_k\right) > 0$$

Therefore,

$$\sum_{k=n+p}^{\infty}((1+\alpha) - k(p\alpha+\beta))a_k\sigma_k < (1+\alpha) - p(\beta+\alpha p).$$

Conversely, for $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$, it is enough to show that

$$\left|\frac{L[\alpha_1]f(\mathbb{V}))}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} - \left(p + \alpha\left|\frac{L[\alpha_1]f(\mathbb{V}))}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} - p\right| + \beta\right)\right|$$
$$< \left|\frac{L[\alpha_1]f(\mathbb{V}))}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} + \left(p - \alpha\left|\frac{L[\alpha_1]f(\mathbb{V}))}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} - p\right| - \beta\right)\right|$$

by using the fact $Re(w) > \alpha \Leftrightarrow |w-(p+\alpha)| < |w+(p-\beta)|$.

Now let $M = \frac{L[\alpha_1]f(\mathbb{V}))}{|\mathbb{W}L'[\alpha_1]f(\mathbb{V})|}$, then

$$\begin{aligned}
T &= \left|\frac{L[\alpha_1]f(\mathbb{V})}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} + p - \beta - \alpha\left|\frac{L[\alpha_1]f(\mathbb{V})}{\mathbb{W}L'[\alpha_1]f(\mathbb{V})} - p\right|\right| \\
&= \frac{1}{|\mathbb{W}L'[\alpha_1]f(\mathbb{V})|} \\
&\quad \left|L[\alpha_1]f(\mathbb{V}) + (p-\beta)(\mathbb{W}L'[\alpha_1]f(\mathbb{V}) - \alpha M\left|L[\alpha_1]f(\mathbb{V}) - p\mathbb{W}(\mathbb{W}L'[\alpha_1]f(\mathbb{V})\right|\right| \\
&= \frac{1}{|\mathbb{W}L'[\alpha_1]f(\mathbb{V}|}\Bigg|\mathbb{V}^p - \sum_{k=n+p}^{\infty} a_k\sigma_k\mathbb{V}^k + (p-\beta)\times \\
&\quad \left(p\mathbb{W}^p - \sum_{k=n+p}^{\infty} ka_k\sigma_k\mathbb{V}^k\right) - \alpha M\left|\mathbb{V}^p - \sum_{k=n+p}^{\infty} a_k\sigma_k\mathbb{V}^k - p(p\mathbb{W}^p - \sum_{k=n+p}^{\infty} k\sigma_k a_k\mathbb{V}^k)\right|\Bigg| \\
&= \frac{1}{|\mathbb{W}L'[\alpha_1]f(\mathbb{V})|}|\mathbb{V}^p + (p-\beta)p(\mathbb{V})^p - \alpha\mathbb{V}^p + \alpha p(p\mathbb{W}^p) \\
&\quad - \sum_{k=n+p}^{\infty}(1+(p-\beta)k + \alpha - \alpha pk)\sigma_k a_k\mathbb{V}^k| \\
&> \frac{|\mathbb{V}|^p}{|\mathbb{W}L'[\alpha_1]f(\mathbb{V})|}\left|((1-\alpha) + p(p-\beta+\alpha p) - \sum_{k=n+p}^{\infty}((1+\alpha) - k(\beta+\alpha p) + kp)a_k\sigma_k a_k\right|
\end{aligned}$$

$$R = \frac{1}{|WL'[\alpha_1]f(\mathbb{V})|}\left|L[\alpha_1]f(\mathbb{V}) - (p+\beta)WL'[\alpha_1]f(\mathbb{V})\right| - \alpha M\left|L[\alpha_1]f(\mathbb{V}) - pWL'[\alpha_1]f(\mathbb{V})\right|$$
$$< \frac{|\mathbb{V}|^p}{|WL'[\alpha_1]f(\mathbb{V})|}\left|1 - p(p+\beta) - \alpha + \alpha p + \sum_{k=n+p}^{\infty}(-kp + (1+\alpha) - k(\beta+\alpha p))a_k\sigma_k\right|$$

It is easy to verify that $T - R > 0$ if (11) holds, and the proof is complete.

3 Extreme Points and Distortion Bounds

Now we obtain the extreme points for the class $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$

Theorem 3.1 : Let $f_1(z) = z^p$ and

$$f_k(z) = z^p - \frac{(1+\alpha) - p(\alpha p+\beta)}{((1+\alpha) - k(\alpha p+\beta))\sigma_k}z^k$$

where, $k \geq n+p, n, p \in \mathbb{N}$, then $f \in \mathscr{M}_p^{(\mathbb{W})}(\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty}\lambda_k f_k(z) \qquad (13)$$

where

$$\lambda_1 + \sum_{k=n+p}^{\infty}\lambda_k = 1 \qquad (\lambda_1 \geq 0, \lambda_k \geq 0). \qquad (14)$$

Proof : Let f can be expressed by the form 13, then

$$f(z) = \lambda_1 z^p + \sum_{k=n+p}^{\infty}\left[\lambda_k z^p - \frac{(1+\alpha) - p(\alpha p+)\lambda_k z^k}{\{(1+\alpha) - k(\alpha p+)\}\sigma_k}\right]$$

$$= z^p(\lambda_1 + \sum_{k=n+p}^{\infty}\lambda_k) - \sum_{k=n+p}^{\infty}t_k z^k = z^p - \sum_{k=n+p}^{\infty}t_k z^k \qquad (15)$$

where

$$t_k = \frac{((1+\alpha) - p(\alpha p+\beta))\lambda_k}{((1+\alpha) - k(\alpha p+\beta))\sigma_k}$$

Since

$$\sum_{k=n+p}^{\infty}\frac{[(1+\alpha) - k(\alpha p+)]\sigma_k}{(1+\alpha) - p(\alpha p+)}t_k = \sum_{k=n+p}^{\infty}\lambda_k = 1 - \lambda_1 < 1$$

then we conclude that $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Conversely,let $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$, then by 11

$$a_k < \frac{(1+\alpha) - p(\alpha p+\beta)}{[(1+\alpha) - k(\alpha p+\beta)]\sigma_k}, \quad k \geq n+p, n \in \mathbb{N},$$

so, if we set

$$\lambda_k = \frac{[(1+\alpha) - k(\alpha p+\beta)]\sigma_k}{(1+\alpha) - p(\alpha p+\beta)}a_k < 1,$$

and $\lambda_1 = 1 - \sum_{k=n+p}\lambda_k$. Then

$$f(z) = z^p - \sum_{k=n+p}^{\infty}a_k z^k$$
$$= z^p - \sum_{k=n+p}^{\infty}\frac{((1+\alpha) - p(\alpha p+\beta))\lambda_k}{[(1+\alpha) - k(\alpha p+\beta)]\sigma_k}z^k$$
$$= z^p - \sum_{k=n+p}^{\infty}\lambda_k(z^p - f_k(z))$$
$$= \left(1 - \sum_{k=n+p}^{\infty}\lambda_k\right)z^p + \sum_{k=n+p}^{\infty}\lambda_k f_k(z).$$

Next, we derive the distribution bound for $L[\alpha_1]f(\mathbb{V})$.

Theorem 3.2 : Let $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$. Then

$$r^p - r^{p+n}\frac{(1+\alpha) - p(\alpha p+\beta)}{(1+\alpha) - (n+p)(\alpha p+\beta)} < ||L[\alpha_1]f(\mathbb{V})|| < r^p + r^{p+n}\frac{(1+\alpha) - p(\alpha p+\beta)}{(1+\alpha) - (n+p)(\alpha p+\beta)}.$$

Proof : Since $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$, then it follows from 13 that

$$\sum_{k=n+p}^{\infty}a_k\sigma_k < \frac{(1+\alpha) - p(\alpha p+\beta)}{(1+\alpha) - (n+p)(\alpha p+\beta)}.$$

Therefore,

$$||L[\alpha_1]f(\mathbb{V})|| = |r^p - \sum_{k=n+p}^{\infty}a_k\sigma_k r^k|$$
$$\leq r^p + r^{p+n}\sum_{k=n+p}^{\infty}a_k\sigma_k$$
$$< r^p + \frac{(1+\alpha) - p(\alpha p+\beta)}{(1+\alpha) - (n+p)(\alpha p+\beta)}r^{p+n}$$

and

$$||L[\alpha_1]f(\mathbb{V})|| \geq r^p - \sum_{k=n+p}^{\infty}a_k\sigma_k r^k$$
$$\geq r^p - r^{p+n}\sum_{k=n+p}^{\infty}a_k\sigma_k$$
$$> r^p - \frac{(1+\alpha) - p(p\alpha+\beta)}{(1+\alpha) - (n+p)(\alpha p+\beta)}r^{p+n}.$$

4 Radii of Starlikeness and Convexity

Now we obtain the radius of starlikeness and convexity for the class $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Theorem 4.1 : The radius of starlikeness for the class $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$ is given by,

$$r_1(\alpha,\beta,\lambda,\mu,p,\gamma) = \inf_k \left[\frac{(p-\gamma)[(1+\alpha)-k(\alpha p+\beta)]}{[(1+\alpha)-p(\alpha p+\beta)](k-\gamma)}\sigma_k\right]^{\frac{1}{k-p}}.$$

Proof : For $0 \le \gamma < p$, we want to show that

$$\left|\left|\frac{zf'}{f}-p\right|\right| < p-\gamma$$

or equivalently,

$$\frac{\sum_{k=n+p}^{\infty}(k-p)a_k r^{k-p}}{1-\sum_{k=n+p}^{\infty}a_k r^{k-p}} < p-\gamma \Rightarrow \sum_{k=n+p}^{\infty}\frac{k-\gamma}{p-\gamma}a_k r^{k-p} < 1.$$

By 11 it is easy to see that the above inequality holds if

$$r^{k-p} < \left[\frac{(p-\gamma)\{(1+\alpha)-k(\alpha p+\beta)\}}{\{(1+\alpha)-p(\beta+\alpha p)\}(k-\gamma)}\right]\sigma_k.$$

Now, since f is convex if and only if zf' is starlike, then we have:

Theorem 4.2 : The radius of convexity for the class $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$ is given by,

$$r_2(\alpha,\beta,\lambda,\mu,p,\gamma) = \inf_k \left[\frac{[(1+\alpha)-k(\alpha p+\beta)](p-\gamma)\sigma_k}{k(k-\gamma)[(1+\alpha)-p(\alpha p+\beta)]}\right]^{\frac{1}{k-p}}.$$

5 Neighbourhoods

Now we extend the concept of neighbourhoods of analytic function for the class $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$. Goodman [6] introduced this concept and, then generalized by Ruscheweyh [10].

Let $\alpha \ge 0, 0 \le< p, \lambda > -1, \delta \ge 0$, we define the δ-neighbourhoods of a function $f(z) = z^p - \sum_{k=n+p}^{\infty}a_k z^k$ and denote by $N_{\delta,p}^{\lambda,\mu}(f)(z)$ consisting of all functions

$$g(z) = z^p - \sum_{k=n+p} b_k z^k \text{ satisfying}$$

$$\sum_{k=n+p}^{\infty}\frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)}\sigma_k|a_k-b_k| \le \delta.$$

Theorem 5.1 : Let $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$, then $N_{\delta,p}^{\lambda,\mu}(f) \subset \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Before proving this theorem we need the following two lemmas can be found the proofs in [11].

Lemma 5.1 : If for every complex number ξ with $|\xi| < \delta(0 \le \delta)$ and $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$ then $\frac{f(z)+\xi z^p}{1+\xi} \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Lemma 5.2 : $f \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta) \Leftrightarrow \frac{(f*\psi)(z)}{z^p} \ne 0, z \in \mathscr{U}-\{0\}$ where $\psi(z) = z^p - \sum_{k=n+p} b_k z^k$ and

$$|b_k| \le \frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)}\sigma_k.$$

Proof of Theorem 5.1 : Since $f \in \mathscr{M}_p^{(\mathbb{W})}(\alpha,\beta)$, then by Lemma 5.1, we have

$$\frac{f(z)+\xi z^p}{1+\xi} \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta),$$

therefore,

$$\left(\frac{f(z)+\xi z^p}{1+\xi} * \psi(z)\right) \ne 0$$

then

$$z^{-p}\left(\frac{f(z)+\xi z^p}{1+\xi} * \psi(z)\right) \ne 0,$$

and so

$$\frac{(f*\psi)(z)}{(1+\xi)z^p} + \frac{\xi}{1+\xi} \ne 0.$$

Let $\left|\left|\frac{(f*\psi)(z)}{z^p}\right|\right| < \delta$, then we must have

$$\left|\left|\frac{(f*\psi)(z)}{z^p(1+\xi)} + \frac{\xi}{1+\xi}\right|\right| \ge \frac{|\xi|}{|1+\xi|} - \frac{1}{|1+\xi|}\left|\left|\frac{(f*\psi)(z)}{z^p}\right|\right| > \frac{|\xi|-\delta}{1+|\xi|} \ge 0,$$

which is a contradiction with $|\xi| < \delta$, however we have $\left|\frac{(f*\psi)(z)}{z^p}\right| > \delta$. Let

$$h(z) = z^p - \sum_{k=n+p}^{\infty} e_k z^k \in N_{\delta,p}^{\lambda,\mu}(f),$$

then

$$\delta - \left|\left|\frac{(h*\psi)(z)}{z^p}\right|\right| \le \left|\left|\frac{((f-h)*\psi)(z)}{z^p}\right|\right| \le \sum_{k=n+p}|a_k-e_k||b_k||z^k|$$

$$< \frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)}|a_k-e_k|\sigma_k \le \delta.$$

So we obtain $\frac{(h*\psi)(z)}{z^p} \ne 0$, and by Lemma 5.2 we have $h \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

6 Some Properties of Class $\mathscr{M}(\mathbb{V})_p(\alpha,\beta)$

Theorem 6.1 : Let $f_i(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,i} z^k$ belongs to $\mathscr{M}_p^{(\mathbb{W})}(\alpha,\beta)$ and $0 < \lambda_i < 1$ such that $\sum_{i=1}^{m}\lambda_i = 1$, then the function $G(z) = \sum_{i=1}^{m}\lambda_i f_i(z)$ is also in $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Proof : Since $f_i(z) \in \mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$, then by 11 we have

$$\sum_{k=n+p}^{\infty}\frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(kp+\beta)}a_{k,i}\sigma_k < 1 \quad (i = 1,\cdots,m)$$

$$G(z)=\sum_{i=1}^{m}\lambda_i f_i(z)=\sum_{i=1}^{m}\lambda_i\left(z^p-\sum_{k=n+p}^{\infty}a_{k,i}z^k\right)$$
$$=z^p\sum_{i=1}^{m}\lambda_i-\sum_{k=n+p}^{\infty}\left(\sum_{i=1}^{m}\lambda_i a_{k,i}\right)z^k$$
$$=z^p-\sum_{k=n+p}^{\infty}\left(\sum_{i=1}^{m}\lambda_i a_{k,i}\right)z^k.$$

Now

$$\sum_{k=n+p}^{\infty}\frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)}\left(\sum_{i=1}^{m}\lambda_i a_{k,i}\right)\sigma_k$$
$$=\sum_{i=1}^{m}\lambda_i\left[\sum_{k=n+p}^{\infty}\frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+)}a_{k,i}\sigma_k\right]<\sum_{i=1}^{m}\lambda_i=1,$$

then $G(z)\in\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Here we introduce an integral operator due to Bernardi [1]

$$L_e[f]=\frac{p+e}{z^e}\int_0^z f(t)t^{e-1}dt\quad(e>-p).$$

Theorem 6.2 : If $f\in\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$. Then $L_e[f]$ also belongs to $\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.

Proof : Let $f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k$, then

$$L_e[f]=\frac{p+e}{z^e}\int_0^z\left(t^p-\sum_{k=n+p}^{\infty}a_kt^k\right)t^{e-1}dt$$
$$=\frac{p+e}{z^e}\left[\left(\frac{1}{p+e}t^{p+e}-\sum_{k=n+p}^{\infty}\frac{1}{k+e}a_kt^{k+e}\right)\Big|_0^z\right]$$
$$=z^p-\sum_{k=n+p}^{\infty}\frac{p+e}{k+e}a_kz^k.$$

Since $e>-p,k\geq n+p>p$, then $\frac{p+e}{k+e}\leq 1$. So we have

$$\sum_{k=n+p}^{\infty}\left[\frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+)}\right]\left[\frac{p+e}{k+e}\right]\sigma_ka_k$$
$$\leq\sum_{k=n+p}^{\infty}\frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)}\sigma_ka_k<1.$$

By assumption $f\in\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$.Thus $L_e[f]\in\mathscr{M}_p^{(\mathbb{V})}(\alpha,\beta)$

References

[1] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969), 429-446.
[2] N. Dunford and J. T. Schwartz; Linear operators. Part I : General Theory (Reprinted from the (1958) original). A Wiley-Interscience Publication, John Wiley and Sons, New York, 1988.
[3] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput., 103 (1999), 1-13.
[4] K. Fan, Julia's lemma for operators, Math. Ann., 239 (1979), 241-245.
[5] C. Fox, The asymptotic expansion of generalized hypergeometric functions, Proc. Landon Math. Soc. (Ser. 2), 27 (1928), 389-400.
[6] A. W. Goodman, Univalent functions with analytic curves, Proc. Amer. Math. Soc., 8 (1957), 598-601.
[7] G. Murugusundaramoorthy and T. Rosy ,Subclasses of analytic functions associated with Fox-Wright's generalized hypergeometric functions based on Hilbert space operator, Stud. Univ. Babe-Bolyai, Math., 56(3), (2011),61-72.
[8] K. I. Noore and M. A. Noore, On integral operators, J. Math. Anal. Appl., 238 (1999), 341-352.
[9] J. Ram, and S. Chandak, Unified fractional derivative formulas for the Fox-Wright generalized hypergeometric function. Proc. Natl, Acad. Sci. India, Sect. A, Phys. Sci. 79(1), (2009),51-57 .
[10] S. Ruscheweyh, Neighbourhood of univalent functions, Proc. Amer. Math. Soc., 81 (1981), 521-527.
[11] S. Shams and S. R. Kulkarni, Certain properties of the class of univalent functions defined by Ruscheweyh derivative, Bull. Cal. Math. Soc. 97 (3) (2005), 223-234.
[12] H. M. Srivastava, Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators, Appl. Anal. and Discrete Math., 1 (2007), 56-71.
[13] C. Wei ,D. Gong, and H. Hao, A family of summation formulas on the Fox-Wright function, J. Math. Anal. Appl., 393(1), (2012),316-327.
[14] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, J. London Math. Soc., 10 (1935), 286-293.

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