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Homotopy Analysis Fractional Sumudu Transform Method for Semi-analytic Solution of 2-D Time-Fractional Black–Scholes Equations

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Abstract: Through two assets of the European call option into the form of a time-fractional order Caputo derivative, we have addressed the time-fractional Black-Scholes equations in the current paper. The homotopy analysis fractional Sumudu transform technique has been used to acquire the analytical solution through the convergence of the series solution. The approach combines the Sumudu transform and homotopy analysis approaches. Several graphs at specific and varying parameter values display the solution analysis and variations.

Keywords: Fractional derivative, Homotopy Analysis, Fractional Sumudu Transform Method, Black–Scholes option pricing equation

1 Introduction

The Black-Scholes model (BSM) is a unique and essential notion in contemporary finance theory (BSM). It was introduced in 1973, and it is still frequently used today. The Black-Scholes mathematical model is known for pricing an options contract created by Black and Scholes [2]. Particularly, the Black-Scholes analysis estimates the variability of financial products throughout time. Because options are the most common and major derivatives in financial mathematics, the investigation into their evaluation is crucial for financial institutions and the global economy. To determine the theoretical values of financial options based on specific parameters such current stock prices, time till expiration, anticipated volatility, strike prices, etc., it refers to the BSM.

It is challenging to solve the BSM analytically or numerically. Finding a solution to BSM has recently received a lot of interest. The BSM solution has been obtained using a variety of analytical and numerical techniques that have been devised. For example, Homotopy perturbation method [4,3], Laplace Homotopy perturbation method [5,6] and Melin transform method [7] for analytical solutions, and finite difference method (FDM) [8,9,10,11,12], finite element method [13] and spline methods [14] for numerical solutions. Contrerasy et al. [15] have considered the solution of the multi-asset BSM with an important correlation parameter which has the solution of pricing model. Fasshauer et al. [16] have discussed the meshfree numerical technique for solving the multi-asset American option problems, and they have used a linearly implicit method for time discretization. A method for obtaining the 2D fractional BSE solution for pricing two-asset European options having financial funds fulfilling two different sets of constraints has been put forth by two Chinese academics [17]. For the purpose of resolving multidimensional BSEs, Jo and Kim [18] have created an operator splitting technique using the finite approximation. Luo and Sheng [19] have used optimal stopping theory, semilinear BS

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PDEs, to value multi-assets of American op,tions. They have obtained the unique viscosity solution of semilinear BS PDE from the viscosity solution PDE. The Laplace Homotopy perturbation method (LHPM), devised by Sawangtong et al.[20], finds the result to the BSM with two assets using the Liouville-Caputo fractional derivative. A brief overview of the finite difference method used to calculate the numerical results of one, two, and three-dimensional BSEs based on option pricing has been provided by Kim et al.[21]. Trachoo et al. [6] have obtained the explicit solution of two-dimensional BSE with the European call option using the LHPM.

Using the time-fractional (FT) Black-Scholes equation, Fadugba [22] has developed the Homotopy Analysis Method (HAM) for estimating a European call option (BSE). Arabas and Farhat [23] have studied multidimensional optimistic definite advection transport algorithm for the numerical solution of PDEs, which arise from stochastic models in quantitative finance. A method to use BSEs to predict the price of a company's stock across frontier markets, including the selling and buying costs of whatever stock, has been devised by Chowdhury et al. [24]. Roul [14] has discussed the numerical solution of TFBSE in European options price by quintic B-spline method, and he has defined the TF derivative in Caputo sense. Roul and Goura [25] constructed a higher-order condensed finite difference scheme for the numerical results of generalised BSE emerging in the financial market; they also demonstrated the method‘s convergence and studied the scheme‘s stability. To discover the numerical results of high-dimensional Black-Scholes PDEs, Ullah [25] has suggested a finite difference technique based on the radial basis function on the adaptive mesh. Those TFBSEs that have two assets from a European call option are:

$$\frac{\partial^\alpha \zeta}{\partial t^\alpha}+\frac{1}{2}\eta_1^2 y_1^2\frac{\partial^2\zeta}{\partial y_1^2}+\frac{1}{2}\eta_2^2 y_2^2\frac{\partial^2\zeta}{\partial y_2^2}+\rho\eta_1\eta_2 y_1 y_2\frac{\partial^2\zeta}{\partial y_1\partial y_2}+r\left[y_1\frac{\partial\zeta}{\partial y_1}+y_2\frac{\partial\zeta}{\partial y_2}\right]-r\zeta=0, \tag{1}$$

with the terminal condition:

$$\zeta\left(T,\ y_1,\ y_2\right)=\max\left(\gamma_1 y_1+\gamma_2 y_2-K,\ 0\right), for y_1,\ y_2\in[0,\ \infty),\ t\in[0,\ T], \tag{2}$$

and conditions for boundary values are:

$$\zeta\left(T,\ y_1,\ y_2\right)=0\ as\ y_1,\ y_2\to(0,0)\ , \tag{3}$$

and

$$\zeta\left(T,\ y_1,\ y_2\right)=\max\left(\gamma_1 y_1+\gamma_2 y_2-K,\ 0\right), for y_1,\ y_2\in[0,\ \infty),\ t\in[0,\ T], \tag{4}$$

where
ζ is the call option depending on the underlying asset prices y_1, y_2 at the time t.
y_1, y_2 are the asset price variables,
$\eta_1,\ \eta_2$ are the volatility function of underlying assets,
$\gamma_1,\ \gamma_2$ are coefficients so that all risky asset prices are at the same level,
ρ is the volatility of y_1 and y_2,
r is the risk-free interest rate,
T is the expiration date,
K is the strike price of the underlying asset.

Fractional calculus (FC) has been widely applied for real-world problems over the last two decades in many areas of engineering and science [27,28,29,30,31,32,33,34,35]. The fractional derivatives of Caputo and Atangana-Baleanu (AB) on a regression analysis have been provided by Can et al. [36], who also performed error analysis for conformable derivatives and regression models using analytical and numerical approaches. It is observed that traditional regression models provide more accuracy to conformable derivatives. The bibliometric research on AB operators in FC has been done by Templeton [37]. The rheological models based on FC with just one component that can actually reflect the viscoelastic behaviour of two dissimilar materials, a polymer and a low strength concrete, have been discussed by Ribeiro et al. [38]. Tan et al. [39] have merged the three numerical models to explore the relationship between process-microstructure-property of the manufactured Al/Al2O3 compound.. Fernandez and Ustaoglu [40] have investigated the analytical properties of tempered FC based on integral and derivative operators and also introduced numerous new properties. The associations with the Riemann–Liouville (RL) model of FC have been demonstrated to obtain special functions like hypergeometric and Appell's functions. The stability analysis and numerical simulations are investigated via FC in numerical models about tumor dormancy [41]. Song [42] explores the quantitative analysis of economic management and economic theory after studying gigantic financial models in FC.. Sun et al. [43] have focused utilization of FC in the real world. Carlos et al. [44] have utilized tumor growth models for clinical proof.

An efficient analysis approach for getting convergent series solutions to complex boundary value problems is the Homotopy analysis method (HAM) [45,46,47,48,49,50,51,52,53,54,55]. Renuka et al. [56] have investigated the effects of nanofluid flow through two stretchable rotating discs, joule heating, and viscous dissipation on the formation of entropy. They have constructed the energy equation through heat generation/absorption and radiation effects and obtained mathematical equations that HAM has solved. Jia et al. [57] have provided the ideal HAM for solving linear optimum control problems. In order to determine the solution of fractional nonlinear differential equations with modified Riemann-Liouville derivative, Saratha et al. [58] developed hybrid fractional generalised HAM. To solve the problem of significant bending of the variable-thickness plate resting on a nonlinear triparametric elastic foundation, Yu [59] has looked into a unified bran-new wavelet technique. The HAFSTM has been explored by Pandey and Mishra [60,61] for solving the time-space fractional telegraph equation. Additionally, they have discussed the HAM and Sumudu transform (ST) methods for obtaining equations of the heat solution wave type. In order to obtain the numerical results of the fractional reaction-diffusion equation, Yadav et al. [62] employed the q-HASTM. The double Laplace Sumudu transform method for partial differential equations(PDEs) has been researched by Ahmed et al. [63]. The semi-analytic solution of time-fractional-order Zakharov–Kuznetsov equations has been computed by Mishra and Pandey [64] using HAFSTM. The Homotopy Sumudu transform approach has been put forth by Alomari et al. [65] to solve the delay fractional Bagley-Torvik problem.

The 2-D time-fractional Black-Scholes equations were solved semi-analytically in this publication using the HAFSTM (TFBSE). The remainder of the text is structured as follows: Section 2 provides preliminary information. The Caputo derivative is used to explain HAFSTM in section 3. In section 4, a semi-analytic 2-D TFBSE solution is presented. Section 5 provides the obtained results and a 2-D TFBSE analysis, while Section 6 provides a summary of the work's conclusions.

2 Preliminaries

This section presents some elementary properties of the FC and Sumudu transforms, which is crucial in examining the proposed work. Furthermore, some fractional derivatives viz. Caputo, Riemann-Liouville and Sumudu transform definitions are used.

Definition 2.1: The left-sided Riemann-Liouville fractional order integral operator of a function $f(y) \in C_\lambda,\ \lambda \geq -1$ for an order $\alpha \in \mathbf{R}^+$ is defined as follows [66,67]:

$$I_+^\alpha f(y) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^y (y-\zeta)^{\alpha-1} f(\zeta)\, d\zeta, & \alpha > 0,\ y > 0 \\ f(y), & \alpha = 0 \end{cases} \tag{5}$$

Definition 2.2: The function $f(y),\ y > 0$,be in the space $C_\lambda,\ \lambda \in \mathbf{R}$ if there exists $q \in \mathbf{R}$ and$q(>\lambda)$ such that $f(y) = y^q f_1(y)$, where $f_1(y) \in C[0,\ \infty)$, it is in the space C_λ^m as$f^m \in C_\lambda,\ m \in N$ [30].

Definition 2.3: The Riemann–Liouville fractional differential operator of an order $\alpha \geq 0$, [68].

$$D_{*+}^\alpha f(y) = \frac{d}{dy} I^{n-\alpha} f(y),\ n-1 \leq \alpha \leq n,\ n \in N. \tag{6}$$

Definition 2.4: The left-sided Caputo of $f(y)$derivative is defined as [30]:

$$D_y^\alpha f(y) = \begin{cases} J^{n-\alpha} D^n f(y) \\ \frac{1}{\Gamma(n-\alpha)} \int_0^y (y-\theta)^{n-\alpha-1} f^n(\theta)\, d\theta \end{cases}, \tag{7}$$

where$n-1 < \alpha \leq n,\ n \in N,\ y > 0$.

Definition 2.5: The Sumudu transform is defined over the set of functions [69]

$$A = \left\{ f(y) |\ \exists M,\ \tau_1,\ \tau_2 > 0,\ |f(y)| < Me^{\frac{|y|}{\tau_i}},\ if\, y \in (-1)^i \times [0, \infty) \right\}, \tag{8}$$

through the subsequent formula

$$\bar{f}(u) = S[f(y)] = \int_0^\infty f(uy) e^{-y} dy,\ u \in (-\tau_1,\ \tau_2). \tag{9}$$

Definition 2.6: The Sumudu transform is defined as [70]

$$S[y^\alpha] = \int_0^\infty e^{-y} y^\alpha dy = \Gamma(\alpha+1) u^\alpha, \quad R(\alpha) > 0. \tag{10}$$

Definition 2.7: The Sumudu transform $S[f(y)]$of the Caputo fractional derivative is defined as [70]

$$S\left[D_y^\alpha f(y)\right] = u^{-\alpha} S[f(y)] - \sum_{k=0}^{n-1} u^{-\alpha+k} f^k\left(0^+\right),\ \text{where}\ \ n-1 \leq \alpha \leq n. \tag{11}$$

3 Transformation of 2-D Black Scholes equations as a diffusion equation

Applying the transformation on Eq. 1, so that the Eq. 1 changes to time-fractional diffusion type problem [71]

$$x_1 = \ln(y_1) - \left(r - \frac{1}{2}\eta_1^2\right)\tau,\, x_2 = \ln(y_2) - \left(r - \frac{1}{2}\eta_2^2\right)\tau. \tag{12}$$

Then the Eq. 1 becomes

$$\frac{\partial^\alpha \zeta}{\partial \tau^\alpha} + \frac{1}{2}\eta_1^2 \frac{\partial^2 \zeta}{\partial x_1^2} + \frac{1}{2}\eta_2^2 \frac{\partial^2 \zeta}{\partial x_2^2} + \rho\eta_1\eta_2 \frac{\partial^2 \zeta}{\partial x_1 \partial x_2} - r\zeta = 0,\ \ (x_1, x_2, \tau) \in [0,\ \infty) \times [0,\ \infty) \times [0, T] \tag{13}$$

Now, the terminal and boundary conditions are also transformed according to Eqs. 12 and 13
and BCs:

$$\zeta(x_1,\ x_2,\ \tau) = 0\ as\ x_1,\ x_2 \to -\infty, \tag{14}$$

and

$$\zeta(T,\ y_1,\ y_2) = \gamma_1 e^{x_1 + \left(r - \frac{1}{2}\eta_1^2\right)\tau} + \gamma_2 e^{x_2 + \left(r - \frac{1}{2}\eta_2^2\right)\tau} - Ke^{-r(T-\tau)}, for x_1 \text{ or } x_2 \to \infty \tag{15}$$

Now, assuming the change in the dependent variable for the elimination last term in Eq. 13, we define ξ as

$$\xi(x_1, x_2,\ \tau) = e^{-r(T-\tau)}\zeta(x_1, x_2,\ \tau), \tag{16}$$

Using Eqs. 13 and 16, we have

$$\frac{\partial^\alpha \xi}{\partial \tau^\alpha} + \frac{1}{2}\eta_1^2 \frac{\partial^2 \xi}{\partial x_1^2} + \frac{1}{2}\eta_2^2 \frac{\partial^2 \xi}{\partial x_2^2} + \rho\eta_1\eta_2 \frac{\partial^2 \xi}{\partial x_1 \partial x_2} = 0,\ \ (x_1, x_2, \tau) \in R \times R \times [0, T] \tag{17}$$

Again, the terminal and BCs are transformed according to Eqs. 16 and 17

$$\xi(x_1,\ x_2, T) = \max\left(\gamma_1 e^{x_1 + \left(r - \frac{1}{2}\eta_1^2\right)T} + \gamma_2 e^{x_2 + \left(r - \frac{1}{2}\eta_2^2\right)T} - K,\ 0\right), for x_1,\ x_2 \in [0,\ \infty),\ t \in [0,\ T], \tag{18}$$

and BCs:

$$\xi(x_1,\ x_2,\ \tau) = 0\ as\ x_1,\ x_2 \to -\infty, \tag{19}$$

and

$$\xi(T,\ y_1,\ y_2) = \gamma_1 e^{x_1 + \left(r - \frac{1}{2}\eta_1^2\right)\tau} + \gamma_2 e^{x_2 + \left(r - \frac{1}{2}\eta_2^2\right)\tau} - Ke^{-r(T-\tau)}, for x_1 \text{ or } x_2 \to \infty \tag{20}$$

Consider the forward time coordinate change as $t = T - \tau$ in Eq. 17, we have

$$\frac{\partial^\alpha \xi}{\partial t^\alpha} = \frac{1}{2}\eta_1^2 \frac{\partial^2 \xi}{\partial x_1^2} + \frac{1}{2}\eta_2^2 \frac{\partial^2 \xi}{\partial x_2^2} + \rho\eta_1\eta_2 \frac{\partial^2 \xi}{\partial x_1 \partial x_2},\ \ (x_1, x_2, t) \in R \times R \times [0, T] \tag{21}$$

with the ICs

$$\xi(x_1,\ x_2, T) = \max\left(B_1 e^{x_1} + B_2 e^{x_2} - K,\ 0\right), \tag{22}$$

and BCs

$$\xi(x_1,\ x_2, t) = 0\ as\ x_1,\ x_2 \to -\infty, \tag{23}$$

and

$$\xi(T,\ x_1,\ x_2) = B_1 e^{x_1 + \frac{1}{2}\eta_1^2 t} + B_2 e^{x_2 + \frac{1}{2}\eta_2^2 t} - K, for x_1 \text{ or } x_2 \to \infty \tag{24}$$

where $B_1 = \gamma_1 e^{\left(r - \frac{1}{2}\eta_1^2\right)T}$ and $B_2 = \gamma_2 e^{\left(r - \frac{1}{2}\eta_2^2\right)T}$.

4 Application of HAFSTM for the solution 2-D time-fractional Black Scholes equation

We take into account the 2-D time-fractional BSE that has been translated and apply the Sumudu transform to both sides of Eq. 21

$$\frac{S[\xi(x_1,x_2,t)]}{u^{\alpha}}-\sum_{k=0}^{n-1}\frac{\xi^{(k)}(0)}{u^{(\alpha-k)}}=S\left[\frac{1}{2}\eta_1^2\frac{\partial^2\xi(x_1,x_2,t)}{\partial x_1^2}+\frac{1}{2}\eta_2^2\frac{\partial^2\xi(x_1,x_2,t)}{\partial x_2^2}+\rho\eta_1\eta_2\frac{\partial^2\xi(x_1,x_2,t)}{\partial x_1\partial x_2}\right] \tag{25}$$

The following definition applies to the 2-D TFBSE nonlinear operator:

$$\begin{aligned} N[\varphi(x_1,x_2,t;q)] = S[\varphi(x_1,x_2,t;q)]-\textstyle\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(0)}{u^{(-k)}} \\ +u^{\alpha}S\left[\frac{1}{2}\eta_1^2\frac{\partial^2\varphi(x_1,x_2,t;q)}{\partial x_1^2}+\frac{1}{2}\eta_2^2\frac{\partial^2\varphi(x_1,x_2,t;q)}{\partial x_2^2}+\rho\eta_1\eta_2\frac{\partial^2\varphi(x_1,x_2,t;q)}{\partial x_1\partial x_2}\right], \end{aligned} \tag{26}$$

where $\varphi(x_1,x_2,t;q)$ is a real function of x_1,x_2,t,q and $0\leq q\leq 1$ is an embedding parameter. We create a homotopy as shown below:

$$(1-\mathrm{q})\,S[\varphi(x_1,x_2,t;q)-\xi_0(x_1,x_2,t)]=\hbar\mathrm{qH}(x_1,x_2,t)N[\varphi(x_1,x_2,t;q)] \tag{27}$$

where $\mathrm{H}(x_1,x_2,t)\neq 0$. and$\hbar$ is an auxiliary parameter that is not zero. An additional purpose The function $\varphi(x_1,x_2,t;q)$ is an unidentified function, and $\xi_0(x_1,x_2,t)$ is a first guess at $\xi(x_1,x_2,t)$. It‘s essential to have a lot of discretion when choosing auxiliary settings in HAFSTM. It true when $q=0$ and $q=1$.

$$\varphi(x_1,x_2,t;0)=\xi_0(x_1,x_2,t),\qquad \varphi(x_1,x_2,t;1)=\xi(x_1,x_2,t) \tag{28}$$

When a result, the solution changes from the initial guess, $\xi_0(x_1,x_2,t)$, to the solution, $\xi(x_1,x_2,t)$., as q rises from 0 to 1. Now, extending on Taylor‘s series about how we get

$$\varphi(x_1,x_2,t;q)=\xi_0(x_1,x_2,t)+\sum_{m=1}^{\infty}q^m\xi_m(x_1,x_2,t) \tag{29}$$

where

$$\xi_m(x_1,x_2,t)=\frac{1}{\Gamma(m-1)}\left.\frac{\partial^m\varphi(x_1,x_2,t;q)}{\partial q^m}\right|_{q=0} \tag{30}$$

The series solution 29‘s convergence is controlled by $\hbar$. The series 29 converges at if the initial estimate, the auxiliary linear operator, the control parameter$\hbar$, and the auxiliary function are properly chosen. As a result, we are

$$\xi(x_1,x_2,t)=\xi_0(x_1,x_2,t)+\sum_{m=1}^{\infty}\xi_m(x_1,x_2,t) \tag{31}$$

Eq. 31 is indeed a solution of Eq. 21. The initial guess $\xi_0(x_1,x_2,t)$ and the integer order exact solution $\xi(x_1,x_2,t)$ are related each other by 31 by the iterative terms $\xi_m(x_1,x_2,t)$ $(m=1,2,3,\ldots)$, which further steps will determine.
Define the vectors

$$\overrightarrow{\xi}=\{\xi_0(x_1,x_2,t),\xi_1(x_1,x_2,t),\xi_2(x_1,x_2,t),\ldots,\xi_m(x_1,x_2,t)\}. \tag{32}$$

We obtain the m^{th} order deformation equation by differentiating Eq. 27 m times with respect to the embedding parameter q then parameter and dividing with $m!$ then

$$S[\xi_m(x_1,x_2,t)-\chi_m\xi_{m-1}(x_1,x_2,t)]=\hbar H(x_1,x_2,t)R_m\left(\overrightarrow{\xi}_{m-1},x_1,x_2,t\right). \tag{33}$$

When both sides of Eq. 33 are transformed using the inverse Sumudu method, we get

$$\xi_m(x_1,x_2,t)=\chi_m\xi_{m-1}(x_1,x_2,t)+\hbar S^{-1}\left[H(x_1,x_2,t)R_m\left(\overrightarrow{\xi}_{m-1},x_1,x_2,t\right)\right], \tag{34}$$

where

$$R_m\left(\overrightarrow{\xi}_{m-1},x_1,x_2,t\right)=\frac{1}{\Gamma(m)}\left.\frac{\partial^{m-1}\varphi(x_1,x_2,t;q)}{\partial q^{m-1}}\right|_{q=0} \tag{35}$$

and

$$\chi_m = \begin{cases} 0, \ m \leq 1, \\ 1 \ \ m > 1. \end{cases} \tag{36}$$

In the 2-D TFBSE, the formulation for the iterative solution of Eq. 21 is as follows:

$$\begin{aligned} R_m\left(\overrightarrow{\xi}_{m-1}, x_1, x_2, t\right) = \frac{S[\xi(x_1,x_2,t)]}{u^{\alpha}} - \sum_{k=0}^{n-1} \frac{\xi^{(k)}(0)}{u^{(-k)}} - u^{\alpha} S\left[\frac{1}{2}\eta_1^2 \frac{\partial^2 \xi(x_1,x_2,t)}{\partial x_1^2}\right. \\ \left. + \frac{1}{2}\eta_2^2 \frac{\partial^2 \xi(x_1,x_2,t)}{\partial x_2^2} + \rho \eta_1 \eta_2 \frac{\partial^2 \xi(x_1,x_2,t)}{\partial x_1 \partial x_2}\right], \ \ 0 < \alpha \leq 1, \end{aligned} \tag{37}$$

Through this technique, it is clear to obtain $\xi_m(x_1,x_2,t)$ for $m \geq 1$, at N^{th} order

$$\xi(x_1,x_2,t) = \sum_{m=0}^{N} \xi_m(x_1,x_2,t), \tag{38}$$

where $N \to \infty$, we have got a precise approximation of the Eq. 21.
On solving the Eq. 34 by using Eqs. 22 and 37 from $m = 1, 2, 3, \ldots$,

$$\xi_0(x_1,x_2,t) = \max\left[B_1 \mathrm{e}^{x_1} + B_2 \mathrm{e}^{x_2} - K, 0\right],$$

$$\begin{aligned} \xi_1(x_1,x_2,t) = -\frac{\hbar t^{\alpha}}{2\Gamma(\alpha+1)} \Bigg(\eta_1^2 \Bigg(\begin{cases} e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \\ +\eta_2^2 \Bigg(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg)\Bigg) \end{aligned}$$

$$\begin{aligned} \xi_2(x_1,x_2,t) = -\frac{\hbar t^{\alpha}}{2\Gamma(\alpha+1)} \Bigg(\eta_1^2 \Bigg(\begin{cases} e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \\ +\eta_2^2 \Bigg(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_1}\gamma_2 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg)\Bigg) \\ +\frac{\eta_1^4\hbar^2 t^{2\alpha}}{4\Gamma(2\alpha+1)} \Bigg(\begin{cases} e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \\ -\frac{\eta_2^2\hbar t^{\alpha}}{2\Gamma(\alpha+1)} \Bigg(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \\ -\frac{\eta_2^2\hbar^2 t^{\alpha}}{2\Gamma(\alpha+1)} \Bigg(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \\ -\frac{\eta_2^2\hbar^2 t^{\alpha}}{2\Gamma(\alpha+1)} \Bigg(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \\ +\frac{\eta_2^4\hbar^2 t^{2\alpha}}{4\Gamma(2\alpha+1)} \Bigg(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2 & -K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2>0 \\ 0 & \text{True} \end{cases} \Bigg) \end{aligned}$$

$$
\begin{aligned}
\xi_3(x_1,x_2,t) = &-\frac{\hbar t^{\alpha}\eta_1^2}{2\Gamma(\alpha+1)}\left(\left(\begin{cases} e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right)\right. \\
&\left.-\frac{\hbar^2 t^{\alpha}\eta_1^2}{\Gamma(\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_1}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right)\right) \\
&-\frac{\eta_1^2\hbar^3 t^{\alpha}}{2\Gamma(\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&+\frac{\eta_1^4\hbar^2 t^{2\alpha}}{2\Gamma(2\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_1}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&+\frac{\eta_1^4\hbar^3 t^{2\alpha}}{2\Gamma(2\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&-\frac{\eta_1^6\hbar^3 t^{3\alpha}}{8\Gamma(3\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&-\frac{\eta_2^2\hbar t^{\alpha}}{2\Gamma(\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&-\frac{\eta_2^2\hbar^2 t^{\alpha}}{\Gamma(\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&-\frac{\eta_2^4\hbar^3 t^{2\alpha}}{2\Gamma(\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&+\frac{\eta_2^4\hbar^2 t^{2\alpha}}{2\Gamma(2\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&-\frac{\eta_2^4\hbar^3 t^{2\alpha}}{2\Gamma(2\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right) \\
&-\frac{\eta_2^6\hbar^3 t^{3\alpha}}{8\Gamma(3\alpha+1)}\left(\begin{cases} e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2}\gamma_2-K+e^{\left(r-\frac{\eta_1^2}{2}\right)T+x_1}\gamma_1+e^{\left(r-\frac{\eta_2^2}{2}\right)T+x_2} & \gamma_2>0 \\ 0 & \text{True} \end{cases}\right)
\end{aligned}
$$

$$\vdots$$

Etc.

In the series solution $m \geq 4$, the other terms of Eq. 34 can be inferred. Eq. 38 provides the series solution to Eq. 21. It has been found that the careful selection of the auxiliary parameter $\hbar$. determines the precision and convergence of the HAFSTM series solution. The convergent form for the R^{+} approaches the following form at $\hbar = -1$.

$$
\begin{aligned}
\xi(x_1,x_2,t) = \max\left(B_1 e^{x_1} + B_2 e^{x_2} - K, 0\right) + \left(\frac{\eta_1^2 t^{\alpha}}{2}\right)\max\left(B_1 e^{x_1}, 0\right) E_{\alpha,\alpha+1}\left(\frac{\eta_1^2 t^{\alpha}}{2}\right) \\
+ \left(\frac{\eta_2^2 t^{\alpha}}{2}\right)\max\left(B_2 e^{x_2}, 0\right) E_{\alpha,\alpha+1}\left(\frac{\eta_2^2 t^{\alpha}}{2}\right)
\end{aligned} \tag{39}
$$

If taking the $\alpha = 1$ then approximate solutions of Eq. 21 converges to Eq. 24.

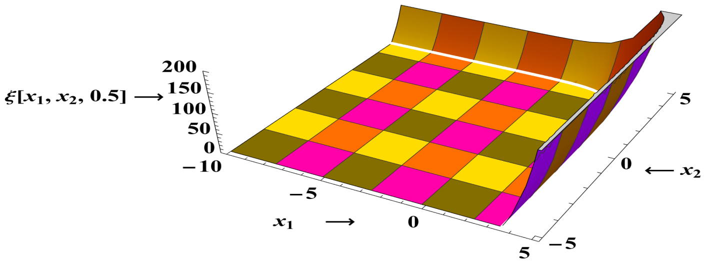

Fig. 1: Solution plot of $\xi(x_1,x_2,t)$ for x_1, x_2 on fixed values $t=0.5$, $\alpha=0.5$, $K=70$, $r=0.05$, $T=1$, $\eta_1=0.1$, $\eta_2=0.2$, $\rho=0.5$, $\hbar=-1$.

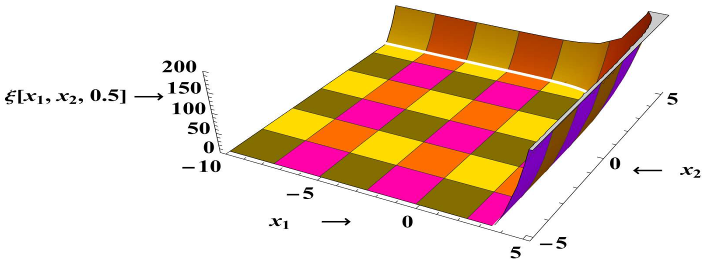

Fig. 2: Solution plot of $\xi(x_1,x_2,t)$ for x_1, x_2 on fixed values $t=0.5$, $\alpha=0.6$, $K=70$, $r=0.05$, $T=1$, $\eta_1=0.1$, $\eta_2=0.2$, $\rho=0.5$, $\hbar=-1$.

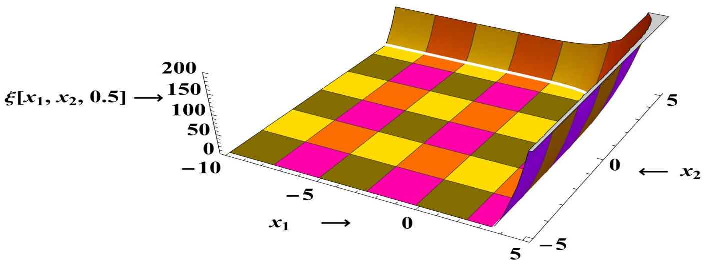

Fig. 3: Solution plot of $\xi(x_1,x_2,t)$ for x_1, x_2 on fixed values $t=0.5$, $\alpha=0.7$, $K=70$, $r=0.05$, $T=1$, $\eta_1=0.1$, $\eta_2=0.2$, $\rho=0.5$, $\hbar=-1$.

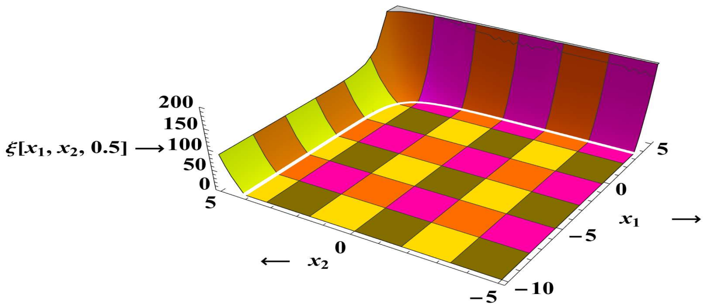

Fig. 4: Solution plot of $\xi(x_1,x_2,t)$ for x_1, x_2 on fixed values $t=0.5$, $\alpha=0.8$, $K=70$, $r=0.05$, $T=1$, $\eta_1=0.1$, $\eta_2=0.2$, $\rho=0.5$, $\hbar=-1$.

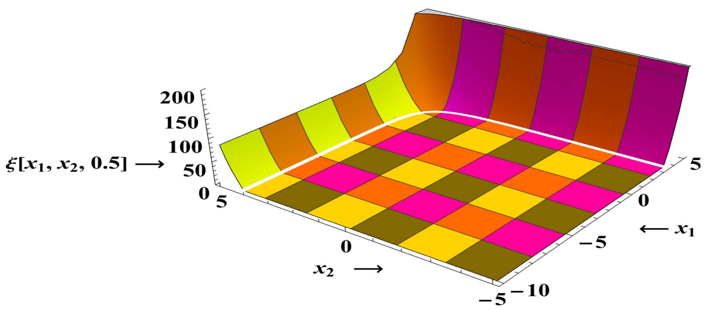

Fig. 5: Solution plot of $\xi(x_1,x_2,t)$ for x_1, x_2 on fixed values $t=0.5$, $\alpha=0.9$, $K=70$, $r=0.05$, $T=1$, $\eta_1=0.1$, $\eta_2=0.2$, $\rho=0.5$, $\hbar=-1$.

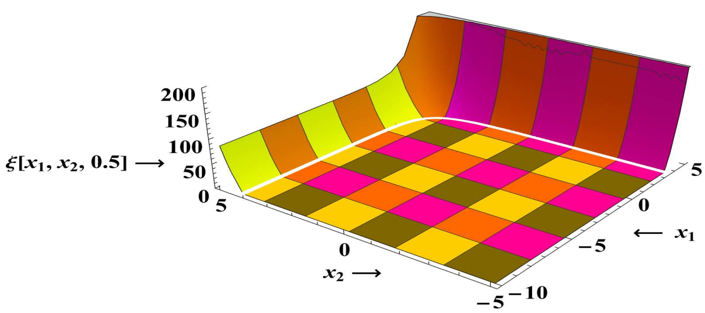

Fig. 6: Solution plot of $\xi(x_1,x_2,t)$ for x_1, x_2 on fixed values $t=0.5$, $\alpha=1$, $K=70$, $r=0.05$, $T=1$, $\eta_1=0.1$, $\eta_2=0.2$, $\rho=0.5$, $\hbar=-1$.

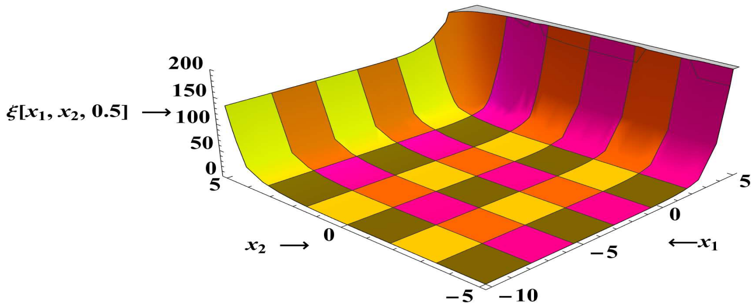

Fig. 7: Exact Solution plot of $\xi(x_1, x_2, t)$ for x_1, x_2 on fixed value$t = 0.5$, $K = 70$, $r = 0.05$, $T = 1$, $\eta_1 = 0.1$, $\eta_2 = 0.2$, $\rho = 0.5$.

5 Result and Analysis

In this part, we use Mathematica to show how to solve the TFBSE Eq. 38 analytically using fractional derivatives of the Caputo type for double assets of the European call option. $B_1 = 2, B_2 = 1$ and $K = 70$. are the strike prices in condition Eq. 36.

$$\xi(x_1,\ x_2,T) = \max(2e^{x_1} + e^{x_2} - 70,\ 0),$$

The volatilities of transformed variable assets x_1 and x_2 are $\eta_1 = 0.1$, $\eta_2 = 0.2$ respectively.
The risk-free interest rate each year is5%, hence, $r = 0.05$; and the maturity term is $T = 1$, expressed in years. The volatility of the underlying assets, y_1 and y_2, is $y_1 = 5\%$ and $y_2 = 10\%$ respectively.
Figs. 1, 2, 3, 4, 5, 6, are represents the approximate solutions at $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ respectively. Fig. 7 is plot of exact solution at $t = 0.5$, $K = 70$, $r = 0.05$, $T = 1$, $\rho = 0.5$, $\hbar = -1$. The deformation of three–dimensional graphs shows the agreement with integer and non-integer order relationships.

6 Conclusions

In this work, an iterative procedure, i.e., HAFSTM, is successfully applied for obtaining the exact solution of TFBSE through high precision. The convergence analysis is also discussed to authenticate the effectiveness and powerfulness of the so-called procedure. The perfect agreement of the exact solution of convergent iterations is displayed in the TFBSE solution plots, demonstrating the method's strong convergence to analytical conclusions. The solutions in particular cases achieved from the proposed method agree with the other techniques described in [2,28,29,30,31,32,33,34,35,71] which shows that the process is effective, suitable, and gives a closed-form solution in the series form.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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