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Fractional Partial and Integral Differential Equations and Novel Conformable Double (Laplace -Sumudu) Transform

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Abstract: This article presents a novel methodology for dealing with fractional partial differential equations and fractional integral equations, subject to particular constraints, by combining the Laplace transform with the Sumudu transform. The conformable double Laplace-Sumudu transform (CDLST) method handles integrals and derivatives of fractional orders by using conformable derivatives. In this paper, we present a thorough examination of the fundamental traits and revolutionary developments related to the proposed shift. It is feasible to convert fractional partial differential equations and integral equations into algebraic equations by using the CDLST and its inherent properties. This modification makes finding solutions simpler, enabling quicker and more effective computations. The findings of our study highlight the potency and usefulness of this novel strategy in resolving numerous issues in the physics and engineering areas.

Keywords: Conformable single Laplace transform; Conformable single Sumudu transform; Conformable Double Laplace– Sumudu transform; Conformable partial Derivative; Fractional partial differential equations; Fractional integral differential equations.

1 Introduction

Scientific simulations in a variety of disciplines, such as physics, electrical circuits, fluid dynamics, optics, and mathematical biology, frequently make use of fractional partial differential equations [1,2,3,4,5,6,7]. Numerous definitions of fractional derivatives have been created throughout history, encompassing formulas credited to famous individuals like Rizez, Riemann-Liouville, Caputo, Hadamard, and others. It is noteworthy that the fractional derivatives of Riemann-Liouville and Caputo have gained popularity in the subject. However, their adherence to the accepted guidelines governing the chain, product, and quotient operations between functions has occasionally baffled researchers. Fractional derivatives' complex nature has created substantial obstacles for their integration into mathematical, physical, and engineering frameworks, leading to a variety of challenges [8,9,10, 11].

The conformal fractional derivative, a concept bridging complex analysis with fractional calculus, and is particularly valuable in modeling complex systems with intricate geometries, such as fluid dynamics and electromagnetism. It leverages conformal mapping to preserve angles between curves and analyze materials with complex structures. On the other hand, the Laplace-Sumudu transform method, combining Laplace and Sumudu transforms, is instrumental in solving fractional differential equations in various fields, offering a computer-friendly approach to studying viscoelasticity, diffusion, and more. In your research, these tools can enhance the analysis of complex systems involving fractional calculus, especially when geometry or intricate dynamics are paramount.

Fractional derivatives have been used in various fields including physics, engineering, and economics to model memory and hereditary properties of various materials and processes. They can also model anomalous diffusion

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processes, control theory, and viscoelastic materials. Numerically approximating fractional derivatives can be more challenging compared to classical derivatives. Various numerical schemes and software packages are available to handle fractional derivatives.

Khalil et al. [12,13,14,15,16] propose an alternative proposal in the form of the conformal fractional derivative, which demonstrates a notable alignment with the fundamental properties of derivatives. There has been a noticeable increase in scientific interest in the solving of conformable fractional partial differential equations. This interest derives from the recognition of the conformable fractional derivative's adaptable nature, which has opened up new avenues for investigation [17,18,19,20,21,22,23].

Fractional calculus, which encompasses fractional derivatives, is an extension of traditional calculus that allows for derivatives and integrals of non-integer order. It has interesting properties and applications, especially in fields like physics, engineering, and applied mathematics. The notion of a fractional derivative generalizes the concept of a derivative to non-integer orders, bridging the gap between differentiation and integration.

The application of the double Laplace-Sumudu transform method has emerged as a cutting-edge method for carrying out double integral transformations and has shown effective in the context of dealing with linear partial differential equations [24,25,26,27,28,29]. Similar to the restrictions experienced by other integral transform techniques, its application to nonlinear applications has nevertheless offered a substantial challenge. In order to address this specific problem [30,31,32,32], academics have worked to combine numerical techniques like variational iteration, decomposition, and perturbation methods with transformative paradigms. These initiatives have created new paths for obtaining complete solutions [13,14,15,16,17,18,19].

This information will be presented in the following sections: Section 2 clarifies the core framework by offering a thorough discussion of key terms and theorems important to conformable fractional derivatives. Section 3 begins with a full description of the CDLST, which contains essential definitions, significant traits, and overarching theorems. In Section 4, the research combines theory and validation to demonstrate transformation potential using seven instances, demonstrating dependability, convergence, and efficiency. The conclusive findings are described in Section 5.

2 Conformable Fractional Derivative

This section introduces the conformable fractional derivative as our main concept. The conformable fractional derivative provides a flexible framework for fractional differentiation qualities for single and multivariable functions. This article uses mathematical correlations to show the practicality of conformable fractional partial derivatives (CFPDs) in Proposition 1. This foundational knowledge prepares people to study the CDLST, a transformative technique that may solve complex mathematical physics problems.

Definition 1. *[4] Let $m < \vartheta_2 \leq m+1$, $m \in \mathbb{N}$, and $\xi : (0,\infty) \to \mathbb{R}$, then the ${\vartheta_2}^{th}$ order conformable fractional derivative of ξ is defined by*

$$D_y^{\vartheta_2}\xi\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right) = \lim_{\delta\to 0}\frac{\xi^{([\vartheta_2]-1)}\left(\frac{y^{\vartheta_2}}{\vartheta_2}+\delta y^{([\vartheta_2]-\vartheta_2)}\right)-\xi^{([\vartheta_2]-1)}\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)}{\delta}, \quad \frac{y^{\vartheta_2}}{\vartheta_2}>0, \quad \vartheta_2\in(m,m+1]. \tag{1}$$

As a special case, if $0 < \vartheta_2, \leq 1$, then we have:

$$D_y^{\vartheta_2}\xi\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right) = \lim_{\delta\to 0}\frac{\xi\left(\frac{y^{\vartheta_2}}{\vartheta_2}+\delta y^{(1-\vartheta_2)}\right)-\xi\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)}{\delta}, \quad \frac{y^{\vartheta_2}}{\vartheta_2}>0, \quad \vartheta_2\in(0,1]. \tag{2}$$

Definition 2. *[20] The CFPDs of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right) : \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ order ϑ_1 and ϑ_2 of the function $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ is defined by*

$$D_z^{\vartheta_1}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right) = \lim_{\delta\to 0}\frac{\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}+\delta z^{(1-\vartheta_1)},\frac{y^{\vartheta_2}}{\vartheta_2}\right)-\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)}{\rho}, \tag{3}$$

$$D_y^{\vartheta_2}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right) = \lim_{\delta\to 0}\frac{\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}+\delta y^{(1-\vartheta_2)}\right)-\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)}{\delta}, \tag{4}$$

where $\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}>0$, $0<\vartheta_1,\vartheta_2\leq 1$.

The CFPDs of several functions are mentioned in the following proposition.

Proposition 1. Suppose $0<\vartheta_1,\vartheta_2\leq 1$, and c_1, c_2, m_1, m_2, γ, and $\eta\in\mathbb{R}$; then

- $D_z^{\vartheta_1}\left(c_1\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)+c_2\psi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right) = c_1\left(D_z^{\vartheta_1}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right)+c_2\left(D_z^{\vartheta_1}\psi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right).$
- $D_y^{\vartheta_2}\left(e^{\gamma\frac{z^{\vartheta_1}}{\vartheta_1}+\eta\frac{y^{\vartheta_2}}{\vartheta_2}}\right) = \eta e^{\gamma\frac{z^{\vartheta_1}}{\vartheta_1}+\eta\frac{y^{\vartheta_2}}{\vartheta_2}}.$

- $D_z^{\vartheta_1}\left(e^{\gamma\frac{z^{\vartheta_1}}{\vartheta_1}+\eta\frac{y^{\vartheta_2}}{\vartheta_2}}\right)=\gamma e^{\gamma\frac{z^{\vartheta_1}}{\vartheta_1}+\eta\frac{y^{\vartheta_2}}{\vartheta_2}}$.
- $D_z^{\vartheta_1}\left(\sin\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right)=\cos\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)$.
- $D_y^{\vartheta_2}\left(\sin\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right)=\sin\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\cos\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)$.

3 Conformable double Laplace - Sumudu transform

In this section, we analyze the CDLST, a novel and strong integral transformation technique that underpins our mathematical study. A synergistic combination of the conformable Laplace transform (CLT) and conformable Sumudu transform (CST) is proposed. With appropriate conditions, this method solves fractional partial differential and integral equations reliably. Essential definitions and complicated theorems explain the approach's uniqueness and analytical efficacy.

Definition 3. *[19] The CLT of real valued $\xi:(0,\infty)\to\mathbb{R}$ order ϑ_1 is defined as*

$$L_z^{\vartheta_1}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\right]=\Xi(u)=\int_0^\infty e^{-u\frac{z^{\vartheta_1}}{\vartheta_1}}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)z^{\vartheta_1-1}dz,\quad u\in\mathbb{C}. \tag{5}$$

Definition 4. *[19] The CST of real valued $\xi:(0,\infty)\to\mathbb{R}$ order ϑ_2 is defined as*

$$S_y^{\vartheta_2}\left[\xi\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right]=\Xi(v)=\frac{1}{v}\int_0^\infty e^{-\frac{y^{\vartheta_2}}{v\vartheta_2}}\xi\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)y^{\vartheta_2-1}dy,\quad v\in\mathbb{C}. \tag{6}$$

Definition 5. *[19] The CDLST of a piecewise continuous $\xi:(0,\infty)\times(0,\infty)\to\mathbb{R}$ order ϑ_1 and ϑ_2 is defined as*

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right]=\Xi(u,v)$$
$$=\frac{1}{v}\int_0^\infty\int_0^\infty e^{-\left(u\frac{z^{\vartheta_1}}{\vartheta_1}+\frac{y^{\vartheta_2}}{v\vartheta_2}\right)}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)z^{\vartheta_1-1}y^{\vartheta_2-1}dzdy,$$
$$u,v\in\mathbb{C},\quad \vartheta_1,\vartheta_2\in(0,1]. \tag{7}$$

The relationship between the usual and the CDLST is provided in the following theorem.

Theorem 1. *[10] Suppose that $c_1,c_2\in\mathbb{R}$ and $0<\vartheta_1,\vartheta_2\leq 1$, then the followings hold*

1. $L_z^{\vartheta_1}S_y^{\vartheta_2}[c_1]=L_zS_y[c_1]=\frac{c_1}{u},\quad u>0,$
2. $L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)^{m_1}\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)^{m_2}\right]=L_zS_y\left[(z)^{m_1}\left((y)^{m_2}\right)\right]$
$=\frac{(m_1!)(m_2!)v^{m_2}}{u^{m_1+1}},\quad m_1,m_2\in\mathbb{Z}^+,$
3. $L_z^{\vartheta_1}S_y^{\vartheta_2}\left[e^{c_1\frac{z^{\vartheta_1}}{\vartheta_1}+c_2\frac{y^{\vartheta_2}}{\vartheta_2}}\right]=L_zS_y\left[e^{c_1z+c_2y}\right]$
$=\frac{1}{(u-c_1)(1-c_2v)},$
4. $L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\sin\left(c_1\frac{z^{\vartheta_1}}{\vartheta_1}\right)\sin\left(c_2\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right]$
$=L_zS_y\left[\sin(c_1z)\sin(c_2z)\right]=\frac{c}{(\upsilon^2+c^2)}\frac{d\omega}{1+d^2\omega^2},$
5. $L_z^{\vartheta_1}S_y^{\vartheta_2}\left[e^{c_1\frac{z^{\vartheta_1}}{\vartheta_1}+c_2\frac{y^{\vartheta_2}}{\vartheta_2}}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right]$
$=\frac{1}{1-c_2v}\Xi\left(u-c_1,\frac{v}{1-c_2v}\right).$

Theorem 2. *[10] Suppose that $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ is a function of exponential order a and b defined on the interval $(0,Z)$ and $(0,Y)$, then CDLST of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ well-defined for all u and $\frac{1}{v}$ provided that $Re[u]>a$ and $Re\left[\frac{1}{v}\right]>b$.*

Theorem 3. *[19]. Let $L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right]=\Xi(u,v)$, where $0<\vartheta_1,\vartheta_2\leq 1$. Then the CDLST of the conformable partial derivatives of order ϑ_1 and ϑ_2 is given by*

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\frac{\partial^{\vartheta_1}\xi}{\partial z^{\vartheta_1}}\right]=u\Xi(u,v)-\Xi(0,v), \tag{8}$$

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\frac{\partial^{\vartheta_2}\xi}{\partial y^{\vartheta_2}}\right]=v^{-1}\Xi(u,v)-v^{-1}\Xi(u,0), \tag{9}$$

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\frac{\partial^{2\vartheta_1}\xi}{\partial z^{2\vartheta_1}}\right]=u^2\Xi(u,v)-u\Xi(0,v)-\Xi_z(0,v), \tag{10}$$

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\frac{\partial^{2\vartheta_2}\xi}{\partial y^{2\vartheta_2}}\right]=v^{-2}\Xi(u,v)-v^{-2}\Xi(u,0)-v^{-1}\Xi_y(u,0). \tag{11}$$

They can be generalized as

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\frac{\partial^{m\vartheta_1}\xi}{\partial z^{m\vartheta_1}}\right]=u^m\Xi(u,v)-\sum_{k=0}^{m-1}u^{m-1-k}S_y^{\vartheta_2}\left[\frac{\partial^{k\vartheta_1}}{\partial z^{k\vartheta_1}}\left(\xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)\right)\right], \tag{12}$$

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\frac{\partial^{n\vartheta_2}\xi}{\partial y^{n\vartheta_2}}\right]=v^{-n}\Xi(u,v)-\sum_{j=0}^{n-1}v^{-n+j}L_z^{\vartheta_1}\left[\frac{\partial^{j\vartheta_2}}{\partial z^{j\vartheta_2}}\left(\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)\right)\right]. \tag{13}$$

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Theorem 4. *[21] (CDL-CDLS duality) If the CDLST of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ exist, then*

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):(u,v)\right] = \frac{1}{v}L_z^{\vartheta_1}L_y^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):\left(u,\frac{1}{v}\right)\right], \tag{14}$$

where

$$L_u^{\vartheta_1}L_v^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):(u,v)\right]=\Xi(u,v) = \int_0^\infty\int_0^\infty e^{-u\frac{z^{\vartheta_1}}{\vartheta_1}-v\frac{y^{\vartheta_2}}{\vartheta_2}}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)d_{\vartheta_1}zd_{\vartheta_2}y.$$

Theorem 5. *(Convolution Theorem) Assume that $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ and $\varphi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ are two functions with the CDLST, then,*

$$L_z^{\vartheta_1}S_y^{\vartheta_2}\left[(\xi**\varphi)\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):(u,v)\right]=v\Xi(u,v)\Psi(u,v). \tag{15}$$

Proof. Using Theorems 4 and 2.2 in [22], we obtain,

$$\begin{aligned}
&L_z^{\vartheta_1}S_y^{\vartheta_2}\left[(\xi**\varphi)\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):(u,v)\right]\\
&=\frac{1}{v}L_z^{\vartheta_1}L_y^{\vartheta_2}\left[(\xi**\varphi)\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):\left(u,\frac{1}{v}\right)\right]\\
&=\frac{1}{v}L_z^{\vartheta_1}L_y^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):\left(u,\frac{1}{v}\right)\right]\\
&\qquad L_z^{\vartheta_1}L_y^{\vartheta_2}\left[\varphi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):\left(u,\frac{1}{v}\right)\right]\\
&=vL_z^{\vartheta_1}S_y^{\vartheta_2}\left[\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):(u,v)\right]\\
&\qquad L_z^{\vartheta_1}S_y^{\vartheta_2}\left[\varphi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right):(u,v)\right]\\
&=v\Xi(u,v)\Psi(u,v).
\end{aligned}$$

4 Applications of CDLST to Partial and Integral Differential Equations

This section provides a variety of examples that demonstrate the application of the CDLST method in solving linear conformable fractional partial and integral differential equations. Every example is accompanied with its corresponding equation, initial and boundary conditions, steps for solving, and the resulting solution. The evaluation of the CDLST approach's accuracy involves a comparison between the solutions obtained using this method and well-established analytical solutions. In addition, the CDLST approach utilizes visual representations in the form of figures to illustrate the approximate answers obtained.

Example 1. Consider the linear Euler -Bernoulli equation of CFPD

$$\frac{\partial^{2\vartheta_2}\xi}{\partial y^{2\vartheta_2}}+\frac{\partial^{4\vartheta_1}\xi}{\partial z^{4\vartheta_1}}=\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)+\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)^2, \quad \vartheta_1,\vartheta_2\in(0,1], \tag{16}$$

subject to the initial conditions (ICs) and the boundary conditions (BCs),

$$\begin{aligned}
\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=0,\\
\xi_y\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=\frac{1}{120}\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)^5,\\
\xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=\frac{1}{12}\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)^4,\\
\frac{\partial^{k\vartheta_1}}{\partial z^{k\vartheta_1}}\xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=0,\quad k=1,2,3.
\end{aligned} \tag{17}$$

Solution. Employing the CDLST on Eq. (16) and single conformable Laplace transform (CLT) and conformable Sumudu transform (CST) on Eq. (17), we get

$$\begin{aligned}
&v^{-2}\Xi(u,v)-v^{-2}\Xi(u,0)-v^{-1}\Xi_y(u,0)+u^4\Xi(u,v)\\
&\quad-u^3\Xi(0,v)-u^2\Xi_z(0,v)-u\Xi_{zz}(0,v)\\
&\quad-\Xi_{zzz}(0,v)=\frac{v}{u^2}+\frac{2v^2}{u},
\end{aligned} \tag{18}$$

substituting

$$\begin{aligned}
&\Xi(u,0)=0,\\
&\Xi_y(u,0)=\frac{1}{u^6},\\
&\Xi(0,v)=2v^4,\\
&\Xi_z(0,v)=\Xi_{zz}(0,v)=\Xi_{zzz}(0,v)=0.
\end{aligned}$$

in Eq. (18), we get

$$\left(v^{-2}+u^4\right)\Xi(u,v)=\frac{v}{u^2}+\frac{2v^2}{u}+\frac{v^{-1}}{u^6}+2u^3v^4.$$

Simplifying

$$\begin{aligned}
&\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\\
&=\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{2v^4}{u}+\frac{v}{u^6}\right]\\
&=\frac{1}{12}\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)^4+\frac{1}{5!}\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)^5,
\end{aligned} \tag{19}$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\xi(z,y)=\frac{2y^4}{4!}+\frac{yz^5}{5!}, \tag{20}$$

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which is consistent with the solution found in [23].

In the following figure, Figure 1 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (19) at $\vartheta_1=\vartheta_2=1$.

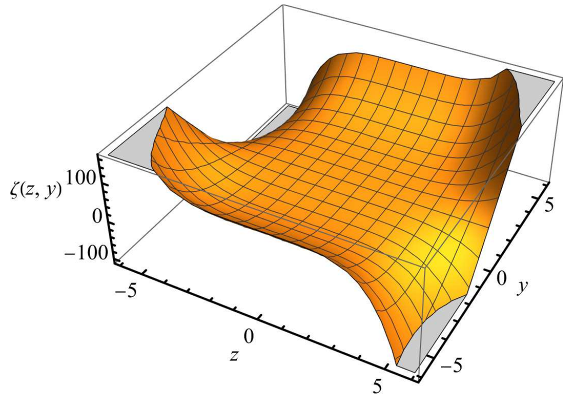

Fig. 1: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (19) at $\vartheta_1=\vartheta_2=1$.

Example 2. Consider the Klein-Gordon equation of CFPD

$$\frac{\partial^{2\vartheta_2}\xi}{\partial y^{2\vartheta_2}}-\frac{\partial^{2\vartheta_1}\xi}{\partial z^{2\vartheta_1}}-2\xi=-2\sin\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\sin\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right),\quad \vartheta_1,\vartheta_2\in(0,1], \tag{21}$$

subject to the ICs and the boundary conditions BCs,

$$\begin{aligned}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=0,\\ \xi_y\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=\sin\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right),\\ \xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=0,\\ \xi_z\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=\sin\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right).\end{aligned} \tag{22}$$

Solution. As described previously in Example 1, the answer to problem (21) can be expressed as

$$\begin{aligned}&\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\\ &=\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{1}{(v^{-2}-u^2-2)}\right.\\ &\left.\left(\frac{v^{-1}}{1+u^2}-\frac{v}{1+u^2}-\frac{2v}{(1+u^2)(1+v^2)}\right)\right].\end{aligned} \tag{23}$$

Simplifying

$$\begin{aligned}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{v}{(1+u^2)(1+v^2)}\right]\\ &=\sin\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)\sin\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right),\end{aligned} \tag{24}$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\xi(z,y)=\sin(y)\sin(z), \tag{25}$$

which is consistent with the solution found in [23].

In the following figure, Figure 2 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (24) at $\vartheta_1=\vartheta_2=1$.

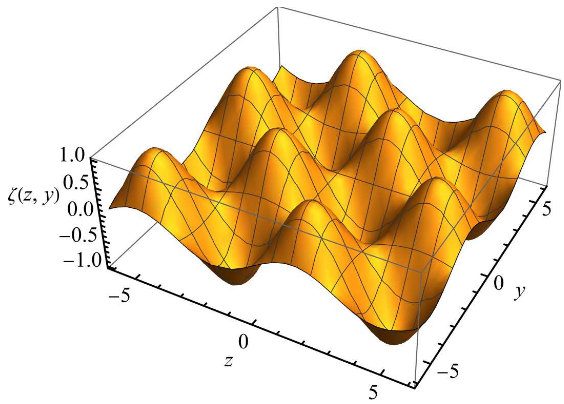

Fig. 2: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (24) at $\vartheta_1=\vartheta_2=1$.

Example 3. Consider the advection - diffusion problem of CFPD

$$\frac{\partial^{\vartheta_2}\xi}{\partial y^{\vartheta_2}}=\frac{\partial^{2\vartheta_1}\xi}{\partial z^{2\vartheta_1}}-\frac{\partial^{\vartheta_1}\xi}{\partial z^{\vartheta_1}},\quad \vartheta_1,\vartheta_2\in(0,1], \tag{26}$$

subject to the ICs and the boundary conditions BCs,

$$\begin{aligned}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=e^{\frac{z^{\vartheta_1}}{\vartheta_1}}-\frac{z^{\vartheta_1}}{\vartheta_1},\\ \xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=\frac{y^{\vartheta_2}}{\vartheta_2}+1,\\ \xi_z\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=0.\end{aligned} \tag{27}$$

Solution. Employing the CDLST on Eq. (26) and single CLT and CST on Eq. (27), we get

$$\begin{aligned}v^{-1}\Xi(u,v)-v^{-1}\Xi(u,0)&=u^2\Xi(u,v)-u\Xi(0,v)\\ &\quad-\Xi_z(0,v)-(u\Xi(u,v)-\Xi(0,v)).\end{aligned} \tag{28}$$

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Substituting

$$\Xi(u,0) = \frac{1}{u-1} - \frac{1}{u^2},$$
$$\Xi(0,v) = v+1, \Xi_z(0,v) = 0,$$

in Eq. (28), we get

$$\left(v^{-1} - u^2 - u\right)\Xi(u,v) = v^{-1}\left(\frac{1}{u-1} - \frac{1}{u^2}\right) - u(v+1). \tag{29}$$

Simplifying

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right) = \left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{1}{u-1} - \frac{1}{u^2} + \frac{v}{u}\right] = e^{\frac{z^{\vartheta_1}}{\vartheta_1}} - \frac{z^{\vartheta_1}}{\vartheta_1} + \frac{y^{\vartheta_2}}{\vartheta_2}, \tag{30}$$

the exact solution if $\vartheta_1 = \vartheta_2 = 1$ is

$$\xi(z,y) = e^z - z + y, \tag{31}$$

which is consistent with the solution found in [24].

In the following figure, Figure 3 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (30) at $\vartheta_1 = \vartheta_2 = 1$.

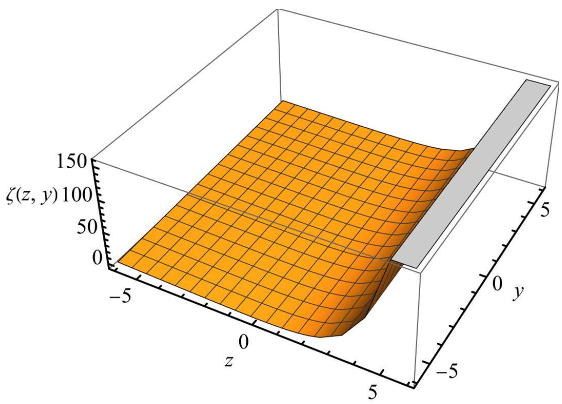

Fig. 3: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (30) at $\vartheta_1 = \vartheta_2 = 1$.

Example 4. Consider the telegraph Equation of CFPD

$$\frac{\partial^{2\vartheta_1}\xi}{\partial z^{2\vartheta_1}} - \frac{\partial^{2\vartheta_2}\xi}{\partial y^{2\vartheta_2}} - \frac{\partial^{\vartheta_2}\xi}{\partial y^{\vartheta_2}} - \xi = 1 - \left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)^2 - \frac{y^{\vartheta_2}}{\vartheta_2}, \quad \vartheta_1, \vartheta_2 \in (0,1], \tag{32}$$

subject to the ICs and the boundary conditions BCs,

$$\begin{aligned} \xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, 0\right) &= \left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)^2, \\ \xi_y\left(\frac{z^{\vartheta_1}}{\vartheta_1}, 0\right) &= 1, \\ \xi\left(0, \frac{y^{\vartheta_2}}{\vartheta_2}\right) &= \frac{y^{\vartheta_2}}{\vartheta_2}, \\ \xi_z\left(0, \frac{y^{\vartheta_2}}{\vartheta_2}\right) &= 0. \end{aligned} \tag{33}$$

Solution. As described previously in the Example 3, the answer to problem (32) can be expressed as

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right) = \left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{1}{(u^2 - v^{-2} - v^{-1} - 1)}\left(\frac{1}{u} - \frac{2}{u^3} - v + \frac{v}{1+v^2} - \frac{v^{-1}}{1+u^2}\right)\right]. \tag{34}$$

Simplifying

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right) = \left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{2}{u^3} + \frac{v}{u}\right] = \left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)^2 + \frac{y^{\vartheta_2}}{\vartheta_2}, \tag{35}$$

the exact solution if $\vartheta_1 = \vartheta_2 = 1$ is

$$\xi(z,y) = z^2 + y, \tag{36}$$

which is consistent with the solution found in [25].

In the following figure, Figure 4 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (35) at $\vartheta_1 = \vartheta_2 = 1$.

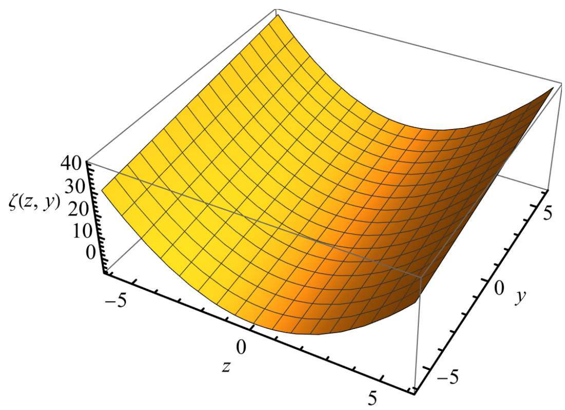

Fig. 4: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1}, \frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (35) at $\vartheta_1 = \vartheta_2 = 1$.

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Example 5. Consider the following Volterra – Integral Equation of conformable fractional derivative

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right) = a-\lambda\int_0^z\int_0^y \xi\left(\frac{(z-\eta)^{\vartheta_1}}{\vartheta_1},\frac{(y-\gamma)^{\vartheta_2}}{\vartheta_2}\right)d\eta d\gamma, \quad \vartheta_1,\vartheta_2\in(0,1], \tag{37}$$

Solution. Applying the CDLST on Eq. (37) and using Theorem 5, we get

$$\Xi(u,v)=\frac{a}{u}-\frac{\lambda v}{u}\Xi(u,v). \tag{38}$$

Consequently,

$$\Xi(u,v)=\frac{a}{u+\lambda v}. \tag{39}$$

Taking $\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}$ for Eq. (39), we obtain the solution $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ of Eq. (37).

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)=\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{a}{u+\lambda v}\right] = aJ_0\left(2\sqrt{\lambda\frac{z^{\vartheta_1}}{\vartheta_1}\frac{y^{\vartheta_2}}{\vartheta_2}}\right), \tag{40}$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\xi(z,y)=a\ J_0\left(2\sqrt{\lambda yz}\right), \tag{41}$$

which is consistent with the solution found in [11].

In the following figure, Figure 5 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (40) at $\vartheta_1=\vartheta_2=1$ and $\lambda=1$.

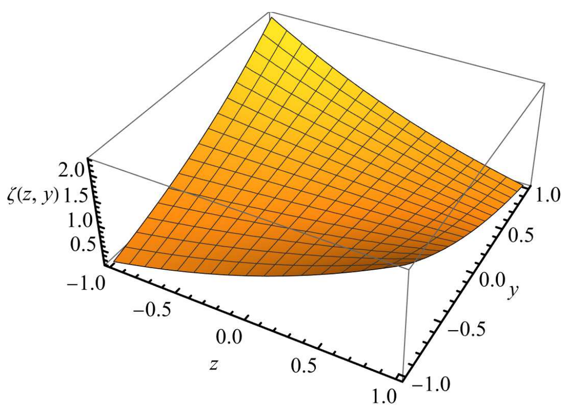

Fig. 5: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (40) at $\vartheta_1=\vartheta_2=1$ for $a=1$, and $\lambda=1$.

Example 6. Consider the following Volterra Integro – Partial Differential Equation of conformable fractional derivative

$$\frac{\partial^{\vartheta_1}\xi}{\partial z^{\vartheta_1}}+\frac{\partial^{\vartheta_2}\xi}{\partial y^{\vartheta_2}}=-1+e^{\frac{z^{\vartheta_1}}{\vartheta_1}}+e^{\frac{y^{\vartheta_2}}{\vartheta_2}}+e^{\frac{z^{\vartheta_1}}{\vartheta_1}+\frac{y^{\vartheta_2}}{\vartheta_2}} +\int_0^z\int_0^y\xi\left(\frac{(z-\eta)^{\vartheta_1}}{\vartheta_1},\frac{(y-\gamma)^{\vartheta_2}}{\vartheta_2}\right)d\eta d\gamma, \quad \frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}>0,\quad \vartheta_1,\vartheta_2\in(0,1], \tag{42}$$

with conditions

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)=e^{\frac{z^{\vartheta_1}}{\vartheta_1}},\quad \xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)=e^{\frac{y^{\vartheta_2}}{\vartheta_2}}. \tag{43}$$

Solution. Operating the CDLST on Eq. (42), and single CLT, and CST, on Eq. (43), and simplifying we get,

$$\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)=\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{1}{(u-1)(1-v)}\right] = e^{\frac{z^{\vartheta_1}}{\vartheta_1}+\frac{y^{\vartheta_2}}{\vartheta_2}}, \tag{44}$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\xi(z,y)=e^{z+y}, \tag{45}$$

which is consistent with the solution found in [11].

In the following figure, Figure 6 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (44) at $\vartheta_1=\vartheta_2=1$.

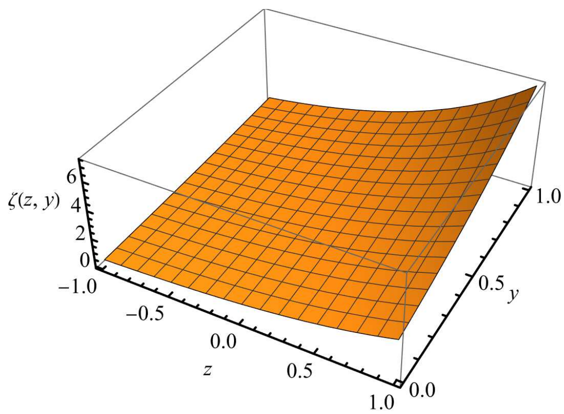

Fig. 6: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (44) at $\vartheta_1=\vartheta_2=1$.

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Example 7. Consider the following Integro – Partial Differential Equation of conformable fractional derivative

$$\begin{aligned}\frac{\partial^{2\vartheta_2}\xi}{\partial y^{2\vartheta_2}}-\frac{\partial^{2\vartheta_1}\xi}{\partial z^{2\vartheta_1}}+\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)\\+\int_0^z\int_0^y e^{\frac{(z-\eta)^{\vartheta_1}}{\vartheta_1}+\frac{(y-\gamma)^{\vartheta_2}}{\vartheta_2}}\xi(\eta,\gamma)d\eta d\gamma\\=e^{\frac{z^{\vartheta_1}}{\vartheta_1}+\frac{y^{\vartheta_2}}{\vartheta_2}}+\left(\frac{z^{\vartheta_1}}{\vartheta_1}\right)\left(\frac{y^{\vartheta_2}}{\vartheta_2}\right)e^{\frac{z^{\vartheta_1}}{\vartheta_1}+\frac{y^{\vartheta_2}}{\vartheta_2}},\\\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}>0,\qquad \vartheta_1,\vartheta_2\in(0,1],\end{aligned}\tag{46}$$

with conditions

$$\begin{aligned}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=e^{\frac{z^{\vartheta_1}}{\vartheta_1}},\\\xi_y\left(\frac{z^{\vartheta_1}}{\vartheta_1},0\right)&=e^{\frac{z^{\vartheta_1}}{\vartheta_1}},\\\xi\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=e^{\frac{y^{\vartheta_2}}{\vartheta_2}},\\\xi_z\left(0,\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=e^{\frac{y^{\vartheta_2}}{\vartheta_2}}.\end{aligned}\tag{47}$$

Solution. Operating the CDLST on Eq. (46), and single CLT, and CST, on Eq. (47), and simplifying we get,

$$\begin{aligned}\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)&=\left(L_z^{\vartheta_1}\right)^{-1}\left(S_y^{\vartheta_2}\right)^{-1}\left[\frac{1}{(u-1)(1-v)}\right]\\&=e^{\frac{z^{\vartheta_1}}{\vartheta_1}+\frac{y^{\vartheta_2}}{\vartheta_2}},\end{aligned}\tag{48}$$

the exact solution if $\vartheta_1=\vartheta_2=1$ is

$$\xi(z,y)=e^{z+y},\tag{49}$$

which is consistent with the solution found in [11].

In the following figure, Figure 7 we sketch the approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (48) at $\vartheta_1=0.9$, and $\vartheta_2=0.7$.

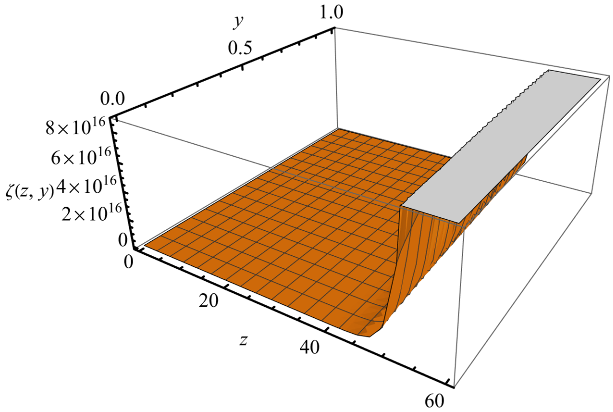

Fig. 7: The approximate solution of $\xi\left(\frac{z^{\vartheta_1}}{\vartheta_1},\frac{y^{\vartheta_2}}{\vartheta_2}\right)$ for Eq. (48) at $\vartheta_1=0.9$, and $\vartheta_2=0.7$.

5 Conclusion

In this research, we used the conformable double Laplace-Sumudu transform (CDLST) to solve a wide class of linear partial and integral differential equations involving conformable fractional derivatives from various real-life sciences. Based on the results, the suggested technique is efficient, appropriate, reliable, and sufficient to solve linear partial and integral differential equations with beginning and boundary conditions. Compared to other approaches, CDLST computations are small [11,23, 24,25]. Thus, these methods work for many linear fractional partial differential equation systems. Future applications of the CDLST approach include solving increasingly complex differential equations utilizing conformable fractional derivatives. Its efficiency and versatility make it ideal for complex scientific challenges. Integrating additional methods may improve its capabilities, possibly extending to nonlinear problems. Collaboration may lead to specialized solutions and wider adoption, advancing science and problem-solving in various disciplines. Based on our study, we conclude that the Laplace Sumudu transform is efficient for solving conformable fractional problems, whenever it can be applied. Thus, unfortunately the method for solving integral equations can be hold only if it is of convolution type.

In the future, we intend to solve models of fractional differential equations and system of fractional integro differential equations in the conformable sense.

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