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Singularities of Retractions of Chaotic Dynamical Eguchi-Hanson Space

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Abstract: This paper introduces types of chaotic retractions singularities in chaotic dynamical Eguchi-Hanson space. The retractions singularities of chaotic dynamical Eguchi-Hanson space by using Lagrangian equations are deduced. The effects of the retraction singularity on the variation of the dimension of the chaotic dynamical Eguchi-Hanson space by the time t are presented. The retractions singularities restriction imposed on the physical characteristics are also discussed. A new category of dynamical extension of chaotic dynamical Eguchi-Hanson space is presented. A new method for obtaining Eguchi-Hanson space's fractal retraction singularity dimension has been presented. Some applications are obtained.

Keywords: Chaotic dynamical Eguchi-Hanson space, Retraction, Singularities

1 Introduction and background

In recent years, mathematical investigations into chaotic dynamical systems have led to an improved understanding of real-world phenomena. The findings of chaos theory are not only applicable to natural sciences but also can be applied to philosophical and theological contexts. In this paper, we intend to provide an overview of some of the various applications of chaotic dynamical systems. We will focus on both Newton and his contemporaries who laid the foundations for scientific determinism as well as the theories that have undermined this foundation, namely quantum mechanics and chaos theory.

In mathematics, the dynamics of dynamical systems are examined, such as differential equations or maps that change over time. Typically, dynamical systems behave predictably towards a fixed or periodic attractor; however, with the assistance of a computer, a more random and unpredictable type of behavior can be demonstrated: chaotic behavior. Dynamical systems have undergone extensive modifications since the early 19th century, which has led to the emergence of the manifold. A chaotic manifold was developed by the end of the 19th century as a result of some alteration of the manifold. The study of any manifold is a special case of the geometry of a chaotic manifold. A major objective of this paper is to examine the chaos manifold which is affected by time (chaotic dynamical manifold) [1,10].

We advise the reader to possess a basic understanding of some topological concepts, such as differentiable manifolds, Hausdorff topological spaces, and differentiable structures on n-dimensional topological manifolds, which are described in detail in [1,9,11,12,13,18,19].

The following sections of the paper will discuss a particular type of retractions singularities of chaotic dynamical manifold, the Eguchi-Hanson space.

Definition 1. *[5] Let V be a set, $f : V \to V$ is said to be chaotic on V if*

(i) f has sensitive dependence on initial conditions.
(ii) f is topologically transitive.
(iii) periodic points are dense in V.

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Definition 2. *[9,19] A chaotic manifold refers to a manifold transformed by time into homeomorphic manifolds either with a fixed point $P_i, i = 1,2,\cdots,n$ or with a no-fixed point.*

Definition 3. *[7,8] A dynamical system in the space X is a function $q = f(p,t)$ such that for all $p \in X$ and $t \in R$, we find a point $q \in X$ such that:*

(i) $f(p,0) = p, \forall p \in X$.
(ii) $\lim\limits_{\substack{p \to p_o \\ t \to t_o}} f(p,t) = f(p_o,t_o)$.
(iii) $f(f(p,t_1),t_2) = f(p,t_1+t_2)$. *And* $f(p,t) = <\overline{p}>$ *is a direct change and not expected value of the position of* p, *which is the chaotic dynamical system.*

Definition 4. *[11,12,20] Let Z be a topological space, and $J \subset Z$. We call J a retract if there exists a continuous map $r: Z \to J$ such that $r(j) = j, \forall j \in J$.*

2 Main results

As a means of achieving our objectives, we introduce the following**:**

Definition 5. *Let Z be a topological space, and $J \subset Z$. We call J a* ***retraction singularity*** *if there exists a continuous map $r: Z \to J$ such that $r(j) = j$, $\forall j \in J$,* ***and at least one element of r(j)=0.***

2.1 *Eguchi - Hanson metric*

The Eguchi -Hanson metric is

$$dS^2 = \left(1 - \frac{\alpha^4}{R^4}\right)^{-1} dR^2 + \frac{1}{4}R^2\left(1 - \frac{\alpha^4}{R^4}\right)(d\Psi + \cos\theta d\Phi)^2 + \frac{1}{4}R^2\left(d\theta^2 + \sin^2\theta d\Phi^2\right), \quad \alpha = \text{constant [21,23]}.$$

2.2 *Geodesic in chaotic Eguchi-Hanson metric*

The chaotic Eguchi-Hanson metric is defined as:

$$\begin{aligned} dS_i^2 = {} & \left(1 - \frac{\alpha_i^4}{R^4(t)}\right)^{-1} dR^2(t) + \frac{1}{4}R^2(t)\left(1 - \frac{\alpha_i^4}{R^4(t)}\right)(d\Psi(t) + \cos\theta(t)\, d\Phi(t))^2 \\ & + \frac{1}{4}R^2(t)\left(d\theta^2(t) + \sin^2\theta(t) d\varnothing^2(t)\right), i = 1,2,\ldots\infty \end{aligned} \tag{1}$$

where $R(t), \Psi(t), \theta(t)$ and $\Phi(t)$ are time functions [6,10]. The coordinates of the chaotic Eguchi -- Hanson space E_i can then be determined as:

$$x_1 = \sqrt{C_1 + R^2(t) + a^2 \tanh^{-1}\frac{R^2(t)}{a^2}}$$
$$x_2 = \sqrt{C_2 + \frac{1}{4}R^2(t)\left(1 - \frac{\alpha^4}{R^4(t)}\right)\Psi^2(t)},$$
$$x_3 = \sqrt{C_3 + \frac{1}{4}R^2(t)\left[\left(1 + \frac{\alpha^4}{R^4(t)}\right)\cos^2\theta(t) + \sin^2\theta(t)\right]\varnothing^2(t)}$$
$$x_4 = \sqrt{C_4 + \frac{1}{4}R^2(t)\,\theta^2(t)}.$$

Where C_1, C_2, C_3 and C_4 are integration constants.

The Lagrangian equations

$$\frac{d}{dS}\left(\frac{\partial T}{\partial \in_i}\right)-\frac{\partial T}{\partial \in_i}=0, i=1,2,3,4,5.$$

are used to deduce a chaotic geodesic, a chaotic retraction of chaotic Eguchi -- Hanson space E_i.

Since $T=\frac{1}{2}dS^2$, therefore

$$T=\frac{1}{2}\left((1-\frac{\alpha^4}{R^4(t)})^{-1}R^2(t)+\frac{1}{4}R^2(t)(1-\frac{\alpha^4}{R^4(t)})(\Psi(t)+\cos\theta(t)\varnothing(t))^2+\frac{1}{4}R^2(t)(\theta'^2(t)+\sin^2\theta(t)\varnothing^2(t))\right) \quad (2)$$

Then, the Lagrangian equations are:

$$\frac{d}{dS}\left(1-\frac{\alpha^4}{R^4(t)}\right)R(t)-(\frac{2a^4}{R^5(t)}+\frac{a^4+R^4(t)}{4R^3(t)}(\Psi(t)+\cos\theta(t)\emptyset(t))^2$$
$$+\frac{R(t)}{4}(\theta^2(t)+\sin^2\theta(t)+\varnothing^2(t)) \qquad =0 \quad (3)$$

$$\frac{d}{dS}\left(\frac{R^2(t)}{4}\left(1-\frac{\alpha^4}{R^4(t)}\right)(\Psi(t)+\cos\theta(t)\emptyset(t))\right) \qquad =0 \quad (4)$$

$$\frac{d}{dS}\left(\frac{R^2(t)}{4}\left(1-\frac{\alpha^4}{R^4(t)}\right)\{\Psi(t)+\cos\theta(t)\varnothing(t)+\frac{R^2(t)}{4}(\sin^2\theta(t)\varnothing(t))\}\right) \qquad =0 \quad (5)$$

$$\frac{d}{dS}\left(\frac{R^2(t)}{4}\theta(t)\right)+\frac{R^2(t)}{4}\left(1-\frac{\alpha^4}{R^4(t)}\right)(\Psi(t)+\cos\theta(t)\emptyset(t))\sin\theta(t)\emptyset(t)$$
$$-\frac{R^2(t)}{8}\sin 2\theta(t)\emptyset^2(t) \qquad =0 \quad (6)$$

$$\frac{2\alpha^4R(t)}{R^5(t)}R^2(t)+\left(1-\frac{\alpha^4}{R^4(t)}\right)R(t)R(\frac{((R^4(t)+\alpha^4)R(t))}{4R^3(t)}(\Psi(t)+\cos\theta(t)\emptyset(t))^2$$
$$+\frac{(R^4(t)-\alpha^4)}{4R^2(t)}(\Psi(t)+\cos\theta(t)\emptyset(t))(\Psi(t)\Psi(t)+(\cos\theta(t)\theta(t)\Psi(t)\Psi(t))$$
$$+\frac{R(t)R(t)}{4}(\theta^2(t)+\sin^2\theta(t)\varnothing^2(t))+\frac{R^2(t)}{4}(\theta(t)\theta''(t)+\sin\theta(t)\cos\theta(t)\theta(t)\varnothing^2(t)$$
$$+\sin^2\theta(t)\varnothing'(t)\varnothing''(t)) \qquad =0 \quad (7)$$

Hence, we have $\frac{R^2(t)}{4}\left(1-\frac{\alpha^4}{R^4(t)}\right)(\Psi(t)+\cos\theta(t)\emptyset(t))=$ constant, say η, if $\eta=0$, we get $R(t)=0$. Then the chaotic sphere S^3_{il} in the chaotic Eguchi -- Hanson metric is we acquired, which represented by:

$$x_1=C_1$$

$$x_2=C_2$$

$$x_3=C_3$$

$$x_4=C_4$$

Also, $S^3_{il}: x_1^2+x_2^2+x_3^2+x_4^2=K^2$, is a geodesic and retraction in chaotic Eguchi-- Hanson space E_i.

Now, if $\left(1-\frac{\alpha^4}{R^4(t)}\right)=0$, the following coordinates are obtained as a result:

$$x_1=\sqrt{C_1+R^2(t)+a^2\tanh^{-1}\frac{R(t)}{\alpha^2}}$$
$$x_2=C_2$$
$$x_3=\sqrt{C_3+\frac{1}{4}R^2(t)\sin^2\theta(t)\varnothing^2(t)}$$
$$x_4=\sqrt{C_4+\frac{1}{4}R^2(t)\,\theta^2(t)}$$

Which is the chaotic hyper affine subspace E_{il} in chaotic Eguchi -- Hanson space E_i. This geodesic is a retraction in E_i.

If $(\Psi(t)+\cos\theta(t)\emptyset(t))=0$. Then we have the following case $\Psi(t),\emptyset(t)=0$, then the chaotic affine subspace of Eguchi -- Hanson space E_{i2} is represented by the following coordinate:

$$x_1=\sqrt{C_1+R^2(t)+a^2\tanh^{-1}\frac{R(t)}{\alpha^2}}$$
$$x_2=C_2$$
$$x_3=C_3$$
$$x_4=\sqrt{C_4+\frac{1}{4}R^2(t)\,\theta^2(t)}$$

Which is a geodesic in Eguchi-Hanson space E_i. Also, this geodesic is a retraction on E_i.

Now, if $(\Psi(t)+\cos\theta(t)\emptyset(t))=0$, then we have $\Psi(t)=\cos\theta(t)=0$ and we get the following coordinates:

$$x_1=\sqrt{C_1+R^2(t)+a^2\tanh^{-1}\frac{R(t)}{\alpha^2}}$$
$$x_2=C_2$$
$$x_3=C_3$$
$$x_4=C_4$$

which is an affine subspace in chaotic Eguchi-Hanson space E_i which is a retraction, also this retraction is a geodesic. The connection between the deformation retract of the original Eguchi-Hanson space E and the deformation retract of its retractions singularities is given by the following commutative diagram, see Figure 1.

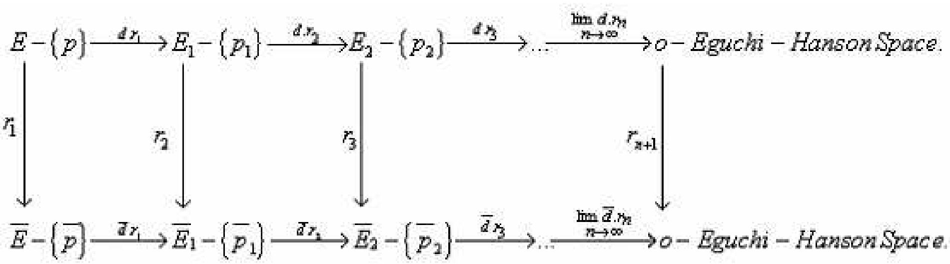

Fig. 1: The connection between the deformation retract of the original Eguchi-Hanson space E and the deformation retract of its retractions singularities

The induced connection between the deformation retract of the chaotic Eguchi - Hanson space E_i and the deformation retract of these retractions singularities is summarized in the following two commutative diagrams, see Figure 2.

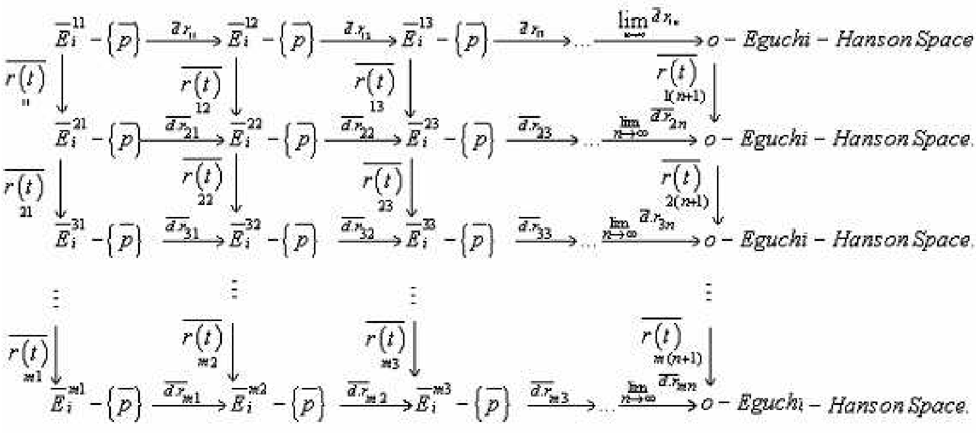

Fig. 2: The connection between the deformation retract of the chaotic Eguchi - Hanson space E_i and the deformation retract of the retractions singularities

2.3 *Retractions of chaotic dynamical Eguchi - Hanson space* E_i^n

In what follows, we discuss the variation of the chaotic dynamical Eguchi -- Hanson space E_i^n by the time t. Now, let $X_i(t) = \{x_1(t),\ x_2(t), \cdots,\ x_n(t),\ x_{n+1}(t)\}$, then after times $t_1,\ t_2, \cdots, t_n,\ t_{n+1}$ we have the following sequence

$$\begin{aligned} &\{x_1(t), x_2(t), \cdots, x_n(t),\ x_{n+1}(t)\} \\ &\xrightarrow{t_1}\{x_1(t), x_2(t), \cdots, x_n(t), 0\} \\ &\xrightarrow{t_2}\{x_1(t),\ x_2(t), \cdots,\ x_{n-1}(t), 0, 0\}, \cdots, \\ &\xrightarrow{t_n}\{x_1(t), \cdots, 0, 0\}\overset{t_{n+1}}{\rightarrow}\{0, 0, \cdots, 0, 0\} = \text{ zero chaotic dynamical Eguchi-Hanson space } E_i^o. \end{aligned}$$

Then the variation of the time t induce minimal chaotic retractions singularities in chaotic dynamical Eguchi-Hanson space E_i^n. Also, the end of the limit of chaotic retractions singularities of zero-dimensional chaotic Eguchi -- Hanson space E_i^o is 0--dimensional geometry set. This means that, if r_i are restricted on the pure chaotic retractions singularities on E_i^o, then the pure chaotic retractions singularities will reduce the degree of chaotic and we obtain the following chains

$$\mu^n \xrightarrow{r_1^1} \mu_1^n \xrightarrow{r_2^1} \mu_2^n, \cdots, \xrightarrow{\lim\limits_{n\to\infty} r_n^1} \mu^{n-1},$$

$$\mu^{n-1} \xrightarrow{r_1^2} \mu_1^{n-1} \xrightarrow{r_2^2} \mu_2^{n-1}, \cdots, \xrightarrow{\lim\limits_{n\to\infty} r_n^2} \mu^{n-2},$$

$$\mu^1 \xrightarrow{r_1^n} \mu_1^1 \xrightarrow{r_2^n} \mu_2^1, \cdots, \xrightarrow{\lim\limits_{n\to\infty} r_n^n} \mu^o \equiv E^o$$

Following that, the limits ending of retractions singularities is 0--dimensional geometric Eguchi -- Hanson space E^o.

Now, if $X_i(t) = \{x_1(t), x_2(t), \cdots, x_n(t), x_{n+1}(t)\}$, then after times $t_1, t_2, \cdots, t_n, t_{n+1}$ we obtain the following sequence

$$\begin{aligned} &\{x_1(t), x_2(t), \cdots, x_n(t), x_{n+1}(t)\} \\ &\xrightarrow{t_1}\{x_1(t), x_2(t), \cdots, x_n(t), x_{n+1}(t), x_{n+2}(t)\} \\ &\xrightarrow{t_2}\{x_1(t), x_2(t), \cdots, x_{n+1}(t), x_{n+2}(t), x_{n+3}(t)\}, \cdots, \\ &\xrightarrow{t_n}\{x_1(t), x_2(t), \cdots, x_n(t), x_{n+1}(t), x_{n+2}(t), \cdots, x_{n+s}(t)\} \end{aligned}$$

Then, the variation of the chaotic dynamical n-dimensional Eguchi -- Hanson space E_i^n by the time t induce a variation in $(n+s)$- dimensional Eguchi -- Hanson space E_i^{n+s}. Also, the increase in dimension of the chaotic dynamical Eguchi -- Hanson space E_i^{n+s} is a new type of dynamical extension of the chaotic dynamical Eguchi-Hanson space E_i^{n+s}.

Moreover, the extension and the limits of extension of the chaotic dynamical Eguchi-Hanson space E_i^n is wild Eguchi-Hanson space $\bar{E_i^n}$. This means that if $exr_1 : E_i^n \to \bar{E_i^n}$ be an extension of chaotic dynamical Eguchi-Hanson space E_i^n onto $\bar{E_i^n}$ then $exr_1(E_i^n) = \bar{E_i^n}$, the maximum radius $R(t)$ of $\bar{E_i^n}$ maximum radius $R(t)$ of E_i^n and $E_i^n \subset \bar{E_i^n}$,

$$\begin{aligned}
&exr_2 : exr_1(E_i^n) \to exr_1(\bar{E_i^n}),\\
&exr_3 : exr_2(exr_1(E_i^n)) \to exr_2(exr_1(\bar{E_i^n}), \cdots,\\
&exr_n : exr_{n-1}(exr_{n-2} \cdots exr_2(exr_1(E_i^n))\ldots) \to exr_{n-1}(exr_{n-2} \cdots exr_2(exr_1(\bar{E_i^n}))\ldots),\\
&\lim_{n\to\infty} exr_n(exr_{n-1}(exr_{n-2}\ldots exr_2(exr_1(E_i^n)))\cdots) = \text{ wild Eguchi-Hanson space } \tilde{E}_i^n.
\end{aligned}$$

Now, we discuss the variation of chaotic dynamical Eguchi -- Hanson space E_{ih}^n by the physical character μ_i, and density $d_i, i = 1, 2, \ldots, \infty$.

Let $(X_{ih}(t), \mu_i) = ((x_1(t), \mu_1), (x_2(t), \mu_2), \ldots, (x_n(t), \mu_n))$, then we have a sequence of chaotic retractions given by

$$\begin{aligned}
&((x_1(t), \mu_1), (x_2(t), \mu_2), \ldots, (x_n(t), \mu_n))\\
&\xrightarrow{r_1}((x_1(t), \mu_{11}), (x_2(t), \mu_{21}), \ldots, (x_n(t), \mu_{n1}))\\
&\xrightarrow{r_2}((x_1(t), \mu_{12}), (x_2(t), \mu_{22}), \ldots, (x_n(t), \mu_{n2})), \cdots,\\
&\xrightarrow{r_n}((x_1(t), \mu_{1n}), (x_2(t), \mu_{2n}), \ldots, (x_n(t), \mu_{nn})) \cdots\\
&\xrightarrow{r_\infty}((x_1(t), \mu_{1\infty}), (x_2(t), \mu_{2\infty}), \cdots, (x_n(t), \mu_{n\infty}))\\
\text{where } &(\mu_1, \mu_2 \cdots, \mu_n) \supset (\mu_{11}, \mu_{21} \cdots, \mu_{n1}) \supset (\mu_{12}, \mu_{22}, \cdots, \mu_{n2})\\
&\cdots \supset (\mu_{1n}, \mu_{2n} \cdots, \mu_{nn}) \supset \cdots \supset (\mu_{1\infty}, \mu_{2\infty} \cdots, \mu_{n\infty})
\end{aligned}$$

This means that the retraction of the physical characteristics will be decreased.

$$\begin{aligned}
\text{If } &(\mu_1, \mu_2 \cdots, \mu_n) \subset (\mu_{11}, \mu_{21} \cdots, \mu_{n1}) \subset (\mu_{12}, \mu_{22}, \cdots, \mu_{n2})\\
&\cdots \subset (\mu_{1n}, \mu_{2n} \cdots, \mu_{nn}) \subset \cdots \subset (\mu_{1\infty}, \mu_{2\infty} \cdots, \mu_{n\infty})
\end{aligned}$$

Then, the retraction of the physical character is a type of extension of chaotic dynamical Eguchi -- Hanson space E_{ih}^n.

In the following we will discusses the retraction in the volume of the physical character μ. Let

$$(X_{ih}(t), \mu) = ((x_1(t), \mu_1), (x_2(t), \mu_2), \ldots, (x_n(t), \mu_n)),$$

if $\mu = (\mu_1, \mu_2 \ldots, \mu_n) < \bar{\mu} = (\bar{\mu_1}, \bar{\mu_2} \ldots, \bar{\mu}_n)$, then the volume of the physical character μ will increase, also if $\mu < \bar{\mu}_1 < \bar{\mu}_2 < \cdots < \bar{\mu}_n$, then $\lim_{n\to\infty} \bar{\mu}_n$ is a wild volume of the physical character μ in chaotic dynamical Eguchi -- Hanson space E_{ih}^n.

If $\mu = (\mu_1, \mu_2 \ldots, \mu_n) > \bar{\mu} = (\bar{\mu_1}, \bar{\mu_2} \ldots, \bar{\mu}_n)$, then the volume of the physical character μ will be decreased. Also, if $\mu < \mu_1 < \mu_2 < \cdots < \mu_n$, then $\lim_{n\to\infty} \mu_n$ is a dynamical Eguchi-Hanson space E^n.

Moreover if $\mu = (\mu_1, \mu_2 \ldots, \mu_n) = \bar{\mu} = (\bar{\mu_1}, \bar{\mu_2} \ldots, \bar{\mu}_n)$ then the volume of the physical character is constant.

Now we will discuss the fractal cases of variation of chaotic dynamical Eguchi-Hanson space by time t. And the limit of this variation. Let $X_i(t) = \{x_1(t), x_2(t), \ldots, x_n(t), x_{n+1}(t)\}$. Then

$$\xrightarrow{t_1} \{\frac{x_1(t)}{\eta_1}, \frac{x_2(t)}{\eta_2}, \ldots, \frac{x_n(t)}{\eta_n}, \frac{x_{n+1}(t)}{\eta_{n+1}}\}, \eta_i > 1, i = 1, 2, \ldots, n+1.$$

Also,

$$\begin{aligned}
&\{\frac{x_1(t)}{\eta_1}, \frac{x_2(t)}{\eta_2}, \ldots, \frac{x_n(t)}{\eta_n}, \frac{x_{n+1}(t)}{\eta_{n+1}}\} \xrightarrow{t_2} \{\frac{x_i(t)}{m_1}, \frac{x_2(t)}{m_2}, \cdots, \frac{x_n(t)}{m_n}, \frac{x_{n+1}(t)}{m_{n+1}}\}\\
&\xrightarrow{t_3} \{\frac{x_1(t)}{S_1}, \frac{x_2(t)}{S_2}, \cdots, \frac{x_n(t)}{S_n}, \frac{x_{n+1}(t)}{S_{n+1}}\}, S_i > m_i > \eta_i > 1, \cdots\\
&\xrightarrow{t_n} \{\frac{x_1(t)}{h_1}, \frac{x_2(t)}{h_2}, \ldots, \frac{x_n(t)}{h_n}, \frac{x_{n+1}(t)}{h_{n+1}}\}, h_i > \ldots > S_i > m_i > \eta_i > 1.
\end{aligned}$$

Thus, $\lim_{h_i\to\infty}\{\frac{x_1(t)}{h_1},\frac{x_2(t)}{h_2},...,\frac{x_n(t)}{h_n},\frac{x_{n+1}(t)}{h_{n+1}}\}=\{0,0,...,0,0\}=$ 0-dimentional Eguchi-Hanson space which is the limit of variation of Eguchi-Hanson space by the time and retractions singularities.

Also, we have $\{\zeta_1 x_1(t),\zeta_2 x_2(t),...,\ \zeta_n x_n(t),\zeta_{n+1}x_{n+1}(t)\}$ the smallest chaotic dynamical Eguchi-Hanson space before the limit.

Furthermore, let

$$\begin{aligned}
X_i(t) =&\{x_1(t),x_2(t),\cdots,x_j(t),\cdots,x_{n+1}(t)\}\\
&\xrightarrow{t_1^1}\{x_1(t),x_2(t),\cdots,\frac{x_j(t)}{\eta_1},\cdots,x_n(t),x_{n+1}(t)\}\\
&\xrightarrow{t_2^1}\{x_1(t),x_2(t),\cdots,\frac{x_j(t)}{\eta_2},...,x_n(t),x_{n+1}(t)\}\\
&\xrightarrow{t_3^1}\{x_1(t),x_2(t),\cdots,\frac{x_j(t)}{\eta_3},...,x_n(t),x_{n+1}(t)\},\cdots\\
&\xrightarrow{t_n^1}\{x_1(t),x_2(t),...,\frac{x_j(t)}{\eta_n},...,x_n(t),x_{n+1}(t)\},\quad \eta_n>...>\eta_3>\eta_2>\eta_1>1\\
&\lim_{n\to\infty} t_n^1\{x_1(t),x_2(t),...,\frac{x_j(t)}{\eta_n},...,x_n(t),x_{n+1}(t)\}=\{x_1(t),x_2(t),\cdots,0,\cdots,x_n(t),x_{n+1}(t)\}
\end{aligned}$$

which is a type of a retraction singularity of chaotic Eguchi-Hanson space. Additionally, before the limit we will obtain:

$$\{x_1(t),x_2(t),\cdots,\frac{x_j(t)}{\eta_n},...,x_n(t),x_{n+1}(t)\}=\{x_1(t),x_2(t),\cdots,\zeta_j x_j(t),\cdots,x_n(t),x_{n+1}(t)\}$$

which is a fractal retraction of chaotic dynamical Eguchi-Hanson space with dimension $E_i^{n-\frac{p}{q}}$.

Moreover, let

$$\begin{aligned}
X_i(t) =&\{x_1(t),x_2(t),...,0,x_{j+1}(t),...,x_{n+1}(t)\}\\
&\xrightarrow{t_1^2}\{x_1(t),x_2(t),\cdots,\frac{x_{j+1}(t)}{m_1},\cdots,x_{n+1}(t)\}\\
&\xrightarrow{t_2^2}\{x_1(t),x_2(t),\cdots,\frac{x_{j+1}(t)}{m_2},\cdots,x_{n+1}(t)\}\\
&\xrightarrow{t_3^2}\{x_1(t),x_2(t),\cdots,\frac{x_{j+1}(t)}{m_3},\cdots,x_{n+1}(t)\},\cdots,\\
&\xrightarrow{t_n^2}\{x_1(t),x_2(t),\cdots,\frac{x_{j+1}(t)}{m_n},\cdots,x_{n+1}(t)\},\quad m_n>m_{n-1}>\cdots>m_2>m_1>1,\\
&\lim_{n\to\infty} t_n^2\{x_1(t),x_2(t),...,\frac{x_j(t)}{m_n},...,x_{n+1}(t)\}=\{x_1(t),x_2(t),...,0,0,...,x_{n+1}(t)\}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\{x_1(t),x_2(t),0,0,\cdots,x_{n+1}(t)\}&\xrightarrow{t_1^3}\{x_1(t),x_2(t),\cdots,0,0,\cdots,\frac{x_{n+1}(t)}{\upsilon_1}\}\\
&\xrightarrow{t_2^3}\{x_1(t),x_2(t),cdots,0,0,\cdots,\frac{x_{n+1}(t)}{\upsilon_2}\},\cdots,\\
&\xrightarrow{t_n^3}\{x_1(t),x_2(t),\cdots,0,0,\cdots,\frac{x_{n+1}(t)}{\upsilon_n}\},\\
&\lim_{n\to\infty} t_n^3\{x_1(t),x_2(t),0,0,\cdots,\frac{x_{n+1}(t)}{\upsilon_n}\}=\{x_1(t),x_2(t),\cdots,0,0,\cdots,0\}.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} t_n^n\{\frac{x_1(t)}{\alpha_n},0,...,0,0,...,0\} = \{0,0,...,0,0,...,0\} \tag{8}$$

which is the 0-dimensional space, it is retractions singularities. Also, before the end of limits we will obtain $\{\zeta_n x_1(t),0,\cdots,0\}$ which is the pure Eguchi-Hanson space with fractal dimension $E^{\frac{p}{q}}$. Hence, the end of limit of variation of time of any space is the null space. Now, consider the coordination of the chaotic Eguchi-Hanson space cc_i given by:

$$x_1 = \sqrt{C_1 + R^2(t) + a^2 \tanh^{-1}\frac{R^2(t)}{\alpha^2}}$$
$$x_2 = \sqrt{C_2 + \frac{1}{4}R^2(t)\left(1-\frac{\alpha^4}{R^4(t)}\right)\Psi^2(t)}$$
$$x_3 = \sqrt{C_3 + \frac{1}{4}R^2(t)\left[\left(1+\frac{\alpha^4}{R^4(t)}\right)\cos^2\theta(t) + \sin^2\theta(t)\right]\varnothing^2(t)}$$
$$x_4 = \sqrt{C_4 + \frac{1}{4}R^2(t)\,\theta^2(t)}$$

Let $X_4 \approx 0$, this means that $C_4 + \frac{1}{4}R^2(t)\,\theta^2(t) \simeq 0$ and $R^2(t) \simeq -\frac{4C_4}{\theta^2(t)}$, then the coordinate of chaotic Eguchi-Hanson space E_i are

$$x_1 = \sqrt{C_1 - \frac{4C_4}{\theta^2(t)} + a^2\tanh^{-1}\left(\frac{-4C_4}{\alpha^2\theta^2(t)}\right)}$$
$$x_2 = \sqrt{C_2 - \frac{C_4}{\theta^2(t)}\left(1-\frac{\alpha^4\theta^4(t)}{16C_4^2}\right)\Psi^2(t)}$$
$$x_3 = \sqrt{C_3 - \frac{C_4}{\theta^2(t)}\left[\left(1+\frac{\alpha^4\theta^4(t)}{16C_4^2}\right)\cos^2\theta(t) + \sin^2\theta(t)\right]\varnothing^2(t)}$$

This method is a type of fractal retractions singularities of Eguchi-Hanson space with dimension $E^{3+\frac{p}{q}}$.

3 Applications

1. The density functions have a wide range of applications. It has been observed in the medical field, for instance, that a person suffering from cancer will also suffer from many other diseases that affect the blood, kidney, heart, prostate, skin, and the majority of organs. Also, the victim of the Hepatitis virus C suffers from a number of symptoms as a consequence of the disease. The mathematical expression of these factors and symptoms is a density function, where $(C(\lambda_1,\lambda_2,\ldots,\lambda_n),\mu_1,\mu_2,\ldots,\mu_n),\mu_1\in[0,1],\mu_2,\ldots,\mu_n\in[-1,1]$, the effective functions.

2. It is known that a real phenomenon can be represented by a geometric metric function, if it is a Riemannian metric. If it is a pseudo-Riemannian metric, it represents phenomena that are not real, such as a complex potential function and complex radiation.

4 Conclusion

A variation of the chaotic dynamical Eguchi-Hanson space by the time t has been described, which induces variation in its dimension by decreasing and increasing. Also, the increase in the dimension of chaotic dynamical Eguchi -- Hanson space has been acquired, which is a new type of extension. The classification of the retractions singularities and extension of chaotic dynamical Eguchi-Hanson space are deduced. The relationship between the retractions singularities of geometry and pure chaotic dynamical Eguchi-Hanson space are discussed. The connection between physical character, density, and the retractions singularities of chaotic dynamical Eguchi -- Hanson space are obtained. The technic for obtaining the fractal retractions singularities of Eguchi -- Hanson space has been achieved. The limit of this type of fractal retractions singularities is presented. Some applications are discussed.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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