Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170607

Symmetric Colorings of $\mathbb{Z}_p^n$

Yuliya Zelenyuk

School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa

Received: 9 Apr. 2023, Revised: 1 Oct. 2023, Accepted: 13 Oct. 2023
Published online: 1 Nov. 2023

Abstract: Symmetries on a group G are the mappings $G \ni x \mapsto gx^{-1}g \in G$, where $g \in G$. A coloring $\chi : G \to \{1,\dots,r\}$ of G is symmetric if it is invariant under some symmetry. We count the number $S_r(\mathbb{Z}_p^n)$ of symmetric r-colorings of $\mathbb{Z}_p^n$, the direct product of n copies of the cyclic group of prime order p. As a consequence we obtain that $S_r(\mathbb{Z}_p^n) = p^n r^{\frac{p^n+1}{2}} + S_r(\mathbb{Z}_p^{n-1})$.

Keywords: Symmetric coloring, equivalent colorings, elementary abelian p-group, Gaussian coefficient.

Let G be a finite group and let $r \in \mathbb{N}$. An *r-coloring* of G is any mapping $\chi : G \to \{1,\dots,r\}$. Let r^G denote the set of all r-colorings of G. The group G naturally acts on r^G. For any $\chi \in r^G$ and $g \in G$, $\chi g \in r^G$ is defined by $\chi g(x) = \chi(xg^{-1})$. Colorings χ and ψ are *equivalent* if there exists $g \in G$ such that $\chi(xg^{-1}) = \psi(x)$ for all $x \in G$ (that is, χ and ψ belong to the same orbit). Let $c_r(G)$ denote the number of equivalence classes of r-colorings of G (= the number of orbits of r^G). Applying Burnside's Lemma [1, 1.7] shows that

$$c_r(G) = \frac{1}{|G|}\sum_{g\in G} r^{|G:\langle g\rangle|},$$

where $\langle g\rangle$ is the subgroup generated by g. For $\mathbb{Z}_n$, the cyclic group of order n, this formula simplifies to

$$c_r(\mathbb{Z}_n) = \frac{1}{n}\sum_{d|n}\varphi(d)r^{\frac{n}{d}},$$

where φ is the Euler function [2].

For every $g \in G$, the *symmetry* on G with respect to g is the mapping

$$G \ni x \mapsto gx^{-1}g \in G.$$

This is an old notion, which can be found in the book [3]. For $\mathbb{Z}_n$, identifying it with the vertices of a regular n-gon, the symmetries are the reflections of the polygon in an axis through one of the vertices (if n is odd, the symmetries are all the reflections). A coloring $\chi \in r^G$ is *symmetric* if it is invariant under some symmetry (that is, there exists $g \in G$ such that $\chi(gx^{-1}g) = \chi(x)$ for all $x \in G$). A coloring equivalent to a symmetric one is also symmetric [4, Lemma 2.1]. Let $S_r(G)$ denote the number of symmetric r-colorings of G and $s_r(G)$ the number of equivalence classes of symmetric r-colorings of G (= the number of symmetric orbits of r^G). If G is abelian, then

$$S_r(G) = \sum_{X\le G}\sum_{Y\le X}\frac{\mu(Y,X)|G/Y|}{|(G/Y)[2]|}r^{\frac{|G/X|+|(G/X)[2]|}{2}},$$

$$s_r(G) = \sum_{X\le G}\sum_{Y\le X}\frac{\mu(Y,X)}{|(G/Y)[2]|}r^{\frac{|G/X|+|(G/X)[2]|}{2}},$$

where X runs over subgroups of G, Y over subgroups of X, $\mu(Y,X)$ is the Möbius function on the lattice of subgroups of G, and $H[2] = \{x \in H : x^2 = 1\}$ [5]. Similar but more complicated formulas hold also in the non-abelian case [4].

For $\mathbb{Z}_n$, the general formulas simplify to

$$S_r(\mathbb{Z}_n) = \begin{cases} \sum_{d|n} d\prod_{p|\frac{n}{d}}(1-p)r^{\frac{d+1}{2}} & \text{if } n \text{ is odd}\\ \sum_{d|\frac{n}{2}} d\prod_{p|\frac{n}{2d}}(1-p)r^{d+1} & \text{if } n \text{ is even,}\end{cases}$$

$$s_r(\mathbb{Z}_n) = \begin{cases} r^{\frac{n+1}{2}} & \text{if } n \text{ is odd}\\ \frac{1}{2}(r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}) & \text{if } n \text{ is even,}\end{cases}$$

where p is a variable of prime value and m is the greatest odd divisor of n [5]. For the dihedral group D_n, the semidirect product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$, the numbers $S_r(D_n)$ and $s_r(D_n)$ were counted in [6].

* Corresponding author e-mail: yuliya.zelenyuk@wits.ac.za

In this note we consider elementary abelian p-group $\mathbb{Z}_p^n$, the direct product of n copies of $\mathbb{Z}_p$, where p is prime. If $p=2$, then every coloring is symmetric, so

$$S_r(\mathbb{Z}_2^n)=r^{2^n},$$

$$s_r(\mathbb{Z}_2^n)=c_r(\mathbb{Z}_2^n)=\frac{1}{2^n}(r^{2^n}+(2^n-1)r^{2^{n-1}}).$$

And if $p>2$, then

$$s_r(\mathbb{Z}_p^n)=r^{\frac{p^n+1}{2}},$$

which is a partial case of a more general fact (we prove it in the end of the note). In [7], $S_r(\mathbb{Z}_p^n)$ was counted for $n=2,3$. Notice that a symmetry of $\prod_{i=1}^n G_i$ is a mapping $\prod_{i=1}^n \sigma_i$, where σ_i is a symmetry of G_i, so the symmetries of $\mathbb{Z}_p^n$ $(p>2)$ are the coordinate-wise reflections.

The aim of this note is to count the number $S_r(\mathbb{Z}_p^n)$ for all n. We show that

Theorem 1. *For all $r,n\in\mathbb{N}$ and prime $p>2$,*

$$S_r(\mathbb{Z}_p^n)=p^n r^{\frac{p^n+1}{2}}+p^{n-1}(1-p^n)r^{\frac{p^{n-1}+1}{2}}+ +p^{n-2}(1-p^{n-1})(1-p^n)r^{\frac{p^{n-2}+1}{2}}+\dots +p(1-p^2)(1-p^3)\dots(1-p^n)r^{\frac{p+1}{2}}+ +(1-p)(1-p^2)\dots(1-p^n)r.$$

Proof. The number of subgroups of $\mathbb{Z}_p^n$ of order p^k is

$$\binom{n}{k}_p=\frac{(p^n-1)(p^{n-1}-1)\dots(p^{n-k+1}-1)}{(p^k-1)(p^{k-1}-1)\dots(p-1)},$$

the Gaussian coefficient [1, 3.11], and if $Y\le X\le\mathbb{Z}_p^n$ and $|Y|=p^k$ and $|X|=p^m$, then

$$\mu(Y,X)=(-1)^{m-k}p^{\binom{m-k}{2}}$$

[1, 4.20]. Here,

$$\binom{n}{k}=\frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot\dots\cdot k},$$

and if $k>n$, then $\binom{n}{k}=0$ and $\binom{n}{k}_p=0$. Thus, the general formula for counting $S_r(G)$ gives us that

$$S_r(\mathbb{Z}_p^n)=\sum_{m=0}^{n}\binom{n}{m}_p\sum_{k=0}^{m}\binom{m}{k}_p(-1)^{m-k}p^{\binom{m-k}{2}+n-k}r^{\frac{p^{n-m}+1}{2}}.$$

Comparing, we conclude that in order to prove the theorem, it suffices to show that

$$\binom{n}{m}_p\sum_{k=0}^{m}\binom{m}{k}_p(-1)^{m-k}p^{\binom{m-k}{2}+n-k}= =p^{n-m}(1-p^{n-m+1})\dots(1-p^n).$$

If $m=0$, both sides are equal to p^n, so let $m\ge 1$.

The left-hand side of the equality is equal to

$$\frac{(p^n-1)\dots(p^{n-m+1}-1)}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}\frac{(p^m-1)\dots(p^{m-k+1}-1)}{(p^k-1)\dots(p-1)}(-1)^{m-k}p^{\binom{m-k}{2}+n-k}$$
$$=(p^n-1)\dots(p^{n-m+1}-1)\sum_{k=0}^{m}\frac{(-1)^{m-k}p^{\binom{m-k}{2}+n-k}}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)}$$
$$=\frac{(p^n-1)\dots(p^{n-m+1}-1)}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}\frac{(p^m-1)\dots(p-1)\cdot(-1)^{m-k}p^{\binom{m-k}{2}+n-k}}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)}$$
$$=\frac{(1-p^n)\dots(1-p^{n-m+1})}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}\frac{(p^m-1)\dots(p-1)\cdot(-1)^k p^{\binom{m-k}{2}+n-k}}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)},$$

and since

$$\frac{(p^m-1)\dots(p-1)}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)}=\binom{m}{k}_p$$

and $n-k=(n-m)+(m-k)$ and

$$\binom{m-k}{2}+m-k=1+2+\dots+(m-k-1)+(m-k)= =\binom{m-k+1}{2},$$

it is equal to

$$\frac{p^{n-m}(1-p^{n-m+1})\dots(1-p^n)}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}(-1)^k\binom{m}{k}_p p^{\binom{m-k+1}{2}}.$$

Then the next lemma finishes the proof.

Lemma 1.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}_p p^{\binom{m-k+1}{2}}=(p^m-1)(p^{m-1}-1)\dots(p-1).$$

Proof. If $m=1$, the left-hand side is

$$(-1)^0\binom{1}{0}_p p^{\binom{2}{2}}+(-1)^1\binom{1}{1}_p p^{\binom{1}{2}}=p-1,$$

so the equality holds.

Now fix $m>1$ and suppose that the equality holds for $m-1$. Since

$$\binom{m}{k}_p=\binom{m-1}{k-1}+p^k\binom{m-1}{k}$$

[1, 3.34], we obtain that

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}_p p^{\binom{m-k+1}{2}}=\sum_{k=0}^{m}(-1)^k\binom{m-1}{k-1}_p p^{\binom{m-k+1}{2}}+ +\sum_{k=0}^{m}(-1)^k\binom{m-1}{k}_p p^{\binom{m-k+1}{2}+k}.$$

The first sum is equal to

$$-\sum_{k=1}^{m}(-1)^{k-1}\binom{m-1}{k-1}_p p^{\binom{(m-1)-(k-1)+1}{2}} = -\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}_p p^{\binom{(m-1)-k+1}{2}},$$

and the second one to

$$\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}_p p^{\binom{m-k+1}{2}+k} = p^m\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}_p p^{\binom{(m-1)-k+1}{2}}$$

because

$$\binom{m-k+1}{2}+k = 1+2+\ldots+(m-k-1)+(m-k)+k =$$
$$=\binom{m-k}{2}+m.$$

Consequently, by the inductive hypothesis,

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}_p p^{\binom{m-k+1}{2}} = -(p^{m-1}-1)\ldots(p-1)+$$
$$+p^m(p^{m-1}-1)\ldots(p-1)$$
$$=(p^m-1)(p^{m-1}-1)\ldots(p-1).$$

As a consequence we obtain from Theorem 1 that

Corollary 1. *For all $r \geq 1$, $n \geq 2$, and prime $p > 2$,*

$$S_r(\mathbb{Z}_p^n) = p^n r^{\frac{p^n+1}{2}} + S_r(\mathbb{Z}_p^{n-1}).$$

We conclude this note by counting the number $s_r(G)$ for every finite abelian group G of odd order.

Let G be a finite group and let $r \in \mathbb{N}$. For every $\chi \in r^G$, let

$$[\chi] = \{\chi g : g \in G\} \text{ and } St(\chi) = \{g \in G : \chi g = \chi\},$$

and let

$$Z(\chi) = \{g \in G : \chi \text{ is symmetric with respect to } g\}.$$

For every symmetric $\chi \in r^G$ and for every $h \in G$, $Z(\chi h) = Z(\chi)h$ [4, Lemma 2.1], so there are colorings in $[\chi]$ symmetric with respect to 1, and their number is $\frac{|Z(\chi)|}{|St(\chi)|}$ [4, Lemma 2.5]. Furthermore, if χ is symmetric with respect to 1 and G is abelian, then $Z(\chi) = \{g \in G : g^2 \in St(\chi)\}$ [4, Corollary 2.9], and consequently, if the order of G is odd, then $Z(\chi) = St(\chi)$.

Lemma 2. *If G is a finite abelian group of odd order, then*

$$s_r(G) = r^{\frac{|G|+1}{2}}.$$

Proof. Every symmetric orbit of r^G has a coloring symmetric with respect to 1, and since G is abelian of odd order, there is only one such coloring. Consequently, $s_r(G)$ is equal to the number of r-colorings of G symmetric with respect to 1, which is equal to the number of r-colorings of the set $\{1\} \cup \{\{x, x^{-1}\} : x \in G \setminus \{1\}\}$ whose cardinality is $1 + \frac{|G|-1}{2} = \frac{|G|+1}{2}$.

Acknowledgement

This work is supported by NRF grant SRUG200318509932. The part of this work was done during the visit to Instituto de Ciencias Matemáticas, ICMAT, Madrid. The author is grateful to Instituto de Ciencias Matemáticas, ICMAT, Madrid and Fundación Mujeres Por África for their hospitality and support.

References

[1] M. Aigner, *Combinatorial Theory*. Springer-Verlag, Berlin-Heidelberg-New York, (1979).

[2] E. Bender and J. Goldman, On the applications of Möbius inversion in combinatorial analysis, *Amer. Math. Monthly*, **82**, 789-803, (1975).

[3] O. Loos, *Symmetric Spaces*. Benjamin: New York, NY, USA, (1969).

[4] Y. Zelenyuk, *Symmetric colorings of finite groups*, in Proceedings of Groups St Andrews 2009, Bath, UK, LMS Lecture Note Series, **388**, 580-590, (2011).

[5] Y. Gryshko (Zelenyuk), Symmetric colorings of regular polygons, *Ars Combinatoria*, **78**, 277-281, (2006).

[6] J. Phakathi, W. Toko, Y. Zelenyuk, and Yu. Zelenyuk, Symmetric colorings of the dihedral group, *Communications in Algebra*, **46**, 1554-1559, (2018).

[7] Yu. Zelenyuk, Computing the number of symmetric colorings of elementary Abelian groups, *Alexandria Engineering Journal*, **60**, 2075-2081, (2021).

Yuliya Zelenyuk is a Professor of Mathematics at the University of the Witwatersrand, South Africa. She received a PhD degree in Mathematics at Kyiv University, Ukraine in 2003. She works in the areas of Algebra, Topology and Combinatorics.