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A Variant of Accelerated Ramadan Group Adomian Decomposition Method for Numerical Solution of Fractional Riccati Differential Equations

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Abstract: Due to the vast range of applications in many scientific domains, researchers have recently become interested in quadratic Riccati differential equations of fractional order and their solutions. In this research, we propose a new method for solving particular classes of quadratic Riccati fractional differential equations that combines the Ramadan group transform (RGT) and a variant of the accelerated Adomian decomposition method (AADM). It is worth noting that RGT is a generalization for both Laplace and Sumudu transforms. El-kalla proposed the AADM, where the main advantages of AADM are that the polynomials generated are recursive and do not have derivative terms, so the formula is easy to programme and saves much time on the same processor as the traditional Adomian polynomials formula, and thus the solution obtained using this proposed hybrid method, accelerated Ramadan group Adomian decomposition method (RGAADM), converges faster than the traditional Adomian decomposition method According to the findings of this work, the solutions obtained by solving a class of quadratic Riccati differential equations of fractional order are extremely compatible with those found via exact solutions. We obtained good performance in all applied cases, which may lead to a promising strategy for many applications.

Keywords: Ramadan group transform, Adomian decomposition method, Accelerated Adomian decomposition method, quadratic Riccati differential equations of fractional order, Fractional calculus, accuracy.

2020 Mathematics Subject Classification. Primary 34A08 ; Secondary 26A33

1 Introduction

Fractional differential equations (FDEs) have become increasingly common in various research areas over the last few decades, and they can provide a better description of many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, cosmology, and materials science. As a result, the solution of the fractional differential equations [1] has received a lot of interest. Fractional differential equations (FDEs) are generalizations of integer-order classical differential equations. Because exact solutions cannot be achieved for most nonlinear FDEs, approximation techniques are used to solve these equations. For obtaining series solutions for FDEs, this method is used. These methods include the Adomian decomposition method [[2], [3], [4]], the homotopy perturbation method [[5], [6]], the variational iterative method [7], the homtopy analysis method [8] and the Chebyshev wavelet method [9].

The fractional Riccati differential equation (FRDE) is an essential FDE with several applications in engineering and applied research. Random processes, optimum control, and diffusion issues [10], stochastic realization theory, optimal control, resilient stabilization, network synthesis, and financial mathematics [11] are some of the applications.

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The Adomian decomposition method is a powerful tool for solving fractional differential equations that emerge in the modelling of real-world physical situations. George Adomian (1970-1990s) [[12], [13]], the chairman of applied mathematics at the University of Georgia, devised this method. The "Adomian polynomial," which gives the convergence of series solutions without any linearization or discretization of the problem's nonlinear elements, is a key part of the method. These polynomials derive a Maclaurin's series about an arbitrary parameter, the Adomian polynomial formula is given by [13]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[[N(\sum_{i=0}^{n}\lambda^i u_i)]]_{\lambda=0}, n = 0,1,2,3,.... \tag{1}$$

El-kalla et al. developed a new modified Adomian decomposition method in [14], this new method uses an accelerated formula of Adomian Polynomial called El-kalla polynomial, which takes the form (See [14]).

$$\bar{A_n} = N(S_n) - \sum_{n=0}^{n-1}\bar{A_i}, n = 0,1,2,..., \tag{2}$$

where the partial sum $S_n = \sum_{i=0}^{n} y_n(t)$. For example, for the nonlinear term $Ny = y^2$
$\bar{A_0} = N(S_0) = N(y_0) = y_0^2,$
$\bar{A_1} = N(S_1) - \bar{A_0} = N(y_0 + y_1) - \bar{A_0} = (y_0 + y_1)^2 - y_0^2 = 2y_0y_1 + y_1^2,$
$\bar{A_2} = N(S_2) - \bar{A_0} - \bar{A_1} = N(y_0 + y_1 + y_2) - \bar{A_0} - \bar{A_1} = (y_0 + y_1 + y_2)^2 - y_0^2 - 2y_0y_1 - y_1^2 = 2y_0y_2 + 2y_1y_2 + y_2^2,$
and similarly, we can compute $\bar{A_3}, \bar{A_4}, \bar{A_5}, ...$

The following table shows the first four polynomials of the nonlinear term y^2 generated by both the traditional Adomian polynomials formula (1) and El-Kalla polynomials formula (2). Clearly, the first four polynomials generated by El-Kalla formula (2) include the first four polynomials generated by the traditional formula (1) in addition to other terms that should appear in $\bar{A_4}, \bar{A_5}, ...$ using formula (2). Thus, the solution obtained using El-Kalla polynomials converges faster than the solution obtained using the traditional polynomials

n	Adomian polynomials formula (1) A_n	and El-Kalla polynomials formula (2)$\bar{A_n}$
0	y_0^2	y_0^2
1	$2y_0y_1$	$2y_0y_1 + y_1^2$
2	$y_1^2 + 2y_0y_2$	$2y_0y_2 + 2y_1y_2 + y_2^2$
3	$2y_1y_2 + 2y_0y_3$	$2y_0y_3 + 2y_1y_3 + 2y_2y_3 + y_3^2$

In this work we use the accelerated Adomian decomposition method proposed by El-kalla [14] coupled with Ramadan group transform for factional derivative [[15], [16], [17]], to solve nonlinear Riccati fractional differential equations. The following are some of the key benefits of El-kalla polynomials: First, El-Kalla polynomials converge faster than traditional Adomian polynomials. Second, El-Kalla polynomials are recursive and do not have derivative terms, making programming easier and saving time on the same processor as the traditional Adomian polynomials formula. So all of El-kalla's advantages inspired us throughout this study to achieve a more valuable, easily programable technique to solve RFDEs.

2 Materials and methods

In this section, we provide definitions and mathematical foundations for fractional calculus theory, and Ramadan group integral transform

Definition 1.(*See [18]) The exponential function is expanded by the Mittag-Leffler function. The implementation of one-parameter functions comes first.*

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{R}, z \in \mathbb{C}.$$

Using two parameters, The Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha > 0$, $\beta > 0$ *and* z *belongs to the complex plane* $\mathbb{C}$.

Definition 2. *In the perspective of Caputo, the fractional-order derivative is defined as([18]):*

$$D_*^{\rho} h(x) = \begin{cases} \frac{1}{\Gamma(m-\rho)} \int_0^x (x-t)^{m-\rho-1} h^{(m)}(t)dt, \; m-1 \leqslant \rho < m, \; m \in \mathbb{N}, \\ \frac{d^m}{dx^m} h(x), \;\; \rho = m \in \mathbb{N} \end{cases} \tag{3}$$

The last definition enables one to write

$$D_*^{\rho} x^k = \begin{cases} 0, & \text{if } k \in \mathbb{N}_0 \text{ and } k < \lceil \rho \rceil, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\rho)} x^{k-\rho}, & \text{if } k \in \mathbb{N}_0 \text{ and } k \geq \lceil \rho \rceil, \end{cases} \tag{4}$$

where $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}_0 = \{0,1,2,\ldots\}$.

Definition 3(See [15]). *Suppose a set A described by*

$$A = \{f(t) | \exists, t_1, t_2 > 0, |f(t)| \leq M \exp^{\frac{|t|}{t_i}}, if t \in (-1)^i \times [0,\infty)\},$$

then definition of the RGT is

$$K(s,u) = RG(f(t)) = \begin{cases} \int_0^\infty e^{-st} f(ut)dt, \;\; 0 \leq u < -t_2, \\ \\ \int_0^\infty e^{-st} f(ut)dt, \;\; -t_1 \leq u < 0. \end{cases}$$

Definition 4(See [15]). *If $F(s)$ and $G(u)$ are the Laplace and Sumudu integrals transforms respectively of $f(t)$, then, we have the following relationships*

$$F(s) = K(s,1), G(u) = K(1,u) \text{ and } K(s,u) = \frac{1}{u} F(\frac{s}{u}).$$

Definition 5(See [16, Theorem 3.1]). *If $K_1(s,u)$ and $_2(s,u)$ are the respective RGTs for the functions $f(t)$ and $g(t)$, then*

$$RG[(f * g)(t), (s,u)] = u K_1(s,u) K_2(s,u), \tag{5}$$

where $*$ represents the convolution of f(t) and g(t).

Definition 6. *[17] If $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n-1 \leq \alpha < n$ and $K(s,u)$ be the RGT of $f(t)$, then the RGT formula of fractional derivative in the Caputo sense of order α for $f(t)$, takes the form*

$$RG[D_*^{\alpha} f(t); (s,u)] = \frac{s^\alpha}{u^\alpha} K(s,u) - \sum_{k=0}^{n-1} \frac{u^{k-\alpha} f^{(k)}(0)}{s^{k-\alpha+1}}. \tag{6}$$

2.1 Procedure

To improve the clarity and understanding of the suggested method, we will present a succinct step-by-step description of our computational methodology for solving the FDEs under consideration. First, we provide an outline of the proposed algorithm as follows:
we will consider a class of nonlinear fractional partial differential equation of the form

$$D_t^{\alpha} y(t) + Ry(t) + Ny(t) = f(t), \;\; n-1 < \alpha \leq n, \tag{7}$$

where R is a linear operator, N is a nonlinear term function, and f is the source function. With the following initial conditions
$y(0) = w_0, y^{(1)}(0) = w_1$, $y^{(2)}(0) = w_2, ..., y^{(n-1)}(0) = w_{n-1}$.
Step 1 Applying Ramadan group transform to both sides of equation (7) and using the linearity of RG-transform, we obtain

$$RG[D_t^{\alpha} y(t)] + RG[Ry(t)] + RG[Ny(t)] = RG[f(t)].$$

Step 2 Now using RG-transform definition for fractional derivative in the Caputo sense (6), we have

$$\frac{s^\alpha}{u^\alpha}Y(s,u)-\sum_{k=0}^{n-1}\frac{u^{k-\alpha}y^{(k)}(0)}{s^{k-\alpha+1}}=RG[f(t)-Ry(t)]-RG[Ny(t)], \tag{8}$$

hence, we have

$$Y(s,u)=\frac{u^\alpha}{s^\alpha}[\sum_{k=0}^{n-1}\frac{u^{k-\alpha}y^{(k)}(0)}{s^{k-\alpha+1}}]+\frac{u^\alpha}{s^\alpha}RG[f(t)]-\frac{u^\alpha}{s^\alpha}RG[Ry(t)+Ny(t)]. \tag{9}$$

Step 3 Now the solution $y(x)$, is defined by the series

$$y(t)=\sum_{n=0}^{\infty}y_n(t), \tag{10}$$

and the nonlinear term is decomposed as

$$Ny(t)=\sum_{n=0}^{\infty}\bar{A_n}, \tag{11}$$

where $\bar{A_0},\bar{A_1},\bar{A_2},.....$ are the new El-kalla polynomials, $\bar{A_n}$ can be computed from (2).

Step 4 Taking inverse RG-transform to (9), then substituting equations (10) and (11) into equation (9), we have

$$\sum_{n=0}^{\infty}y_n(t)=RG^{-1}[\frac{u^\alpha}{s^\alpha}[\sum_{k=0}^{n-1}\frac{u^{k-\alpha}y^{(k)}(0)}{s^{k-\alpha+1}}]]+RG^{-1}[\frac{u^\alpha}{s^\alpha}RG[f(t)]-RG^{-1}[\frac{u^\alpha}{s^\alpha}RG[R\sum_{n=0}^{\infty}y_n(t)+\sum_{n=0}^{\infty}\bar{A_n}]]. \tag{12}$$

Step 1 Comparing both sides of equation (12) yield the following iterative algorithm:

$$\begin{aligned}
y_0&=RG^{-1}[\frac{u^\alpha}{s^\alpha}[\sum_{k=0}^{n-1}\frac{u^{k-\alpha}y^{(k)}(0)}{s^{k-\alpha+1}}]]+RG^{-1}[\frac{u^\alpha}{s^\alpha}RG[f(t)]=x(t),\\
y_1&=-RG^{-1}[\frac{u^\alpha}{s^\alpha}RG[Ry_0+\bar{A_0}]],\\
y_2&=-RG^{-1}[\frac{u^\alpha}{s^\alpha}RG[Ry_1+\bar{A_1}]],\\
&\dots\\
&\dots\\
&\dots\\
y_n&=-RG^{-1}[\frac{u^\alpha}{s^\alpha}RG[Ry_{n-1}+\bar{A_{n-1}}]],\ \ n\geq 1.
\end{aligned} \tag{13}$$

3 Results

Three examples are provided in this section to demonstrate the practicality and precision of our technique. In each case, we created a new MATLAB programme to handle difficulties addressed by this study.

*Example 1.*Take into account the FDE [9]

$$D_t^\alpha y(t)+y^2(t)=\sqrt{2},\ \ t>0,\ 0<\alpha\leq 1, \tag{14}$$

with initial condition $y(0)=0$, and exact solution for $\alpha=1$ is $y(t)=\sqrt[4]{2}tanh(\sqrt[4]{2}t)$.

*Proof.*Applying Ramadan group transform to (14) and using the fractional derivatives properties of the RG-transform, we have

$$\frac{s^\alpha}{u^\alpha}Y(s,u)-\frac{s^{\alpha-1}}{u^\alpha}y(0)+RG[y^2(t)]=\frac{\sqrt{2}}{s},$$

hence,

$$Y(s,u)=\frac{\sqrt{2}u^{\alpha}}{s^{\alpha+1}}-\frac{u^{\alpha}}{s^{\alpha}}RG[y^{2}(t)], \tag{15}$$

applying inverse Ramadan group transform to the equation (15), we get

$$y(t)=RG^{-1}[\frac{\sqrt{2}u^{\alpha}}{s^{\alpha+1}}-\frac{u^{\alpha}}{s^{\alpha}}RG[y^{2}(t)]]. \tag{16}$$

Based on the recursive formula, the approximate solution of the fractional differential equation (14)

$$y_0(t)=RG^{-1}[\frac{\sqrt{2}u^{\alpha}}{s^{\alpha+1}}]=\frac{\sqrt{2}t^{\alpha}}{\Gamma(\alpha+1)}, \tag{17}$$

and

$$y_n(t)=-RG^{-1}[\frac{u^{\alpha}}{s^{\alpha}}[RG[\bar{A}_{n-1}]]], \text{ for } n\geq 1, \tag{18}$$

where, $\sum_{n=0}^{\infty}\bar{A}_n=y^2(t)$, given that for Ramadan group transform coupled with a variant of accelerated Adomian decomposition method (ARGADM), $\bar{A}_n$ are El-kalla polynomials (see [14]), then Evaluating (17) and (18) using Matlab, we obtain the approximate solution in the following form:

$$\begin{aligned} y(t)=&\frac{\sqrt{2}t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}\\ &-\frac{4t^{5\alpha}\Gamma(1+2\alpha)\left(\frac{-\sqrt{2}\Gamma(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}+\frac{t^{2\alpha}\Gamma(1+2\alpha)\Gamma(1+6\alpha)}{\Gamma(1+7\alpha)}\right)}{\Gamma^4(1+\alpha)\Gamma^2(1+3\alpha)}.... \end{aligned} \tag{19}$$

While for Ramadan group coupled with regular Adomian decomposition method (RGADM), $\bar{A}_n$ are the Adomian polynomials (see [19]), hence Evaluating (17) and (18) using Matlab, we obtain the approximate solution in the following form:

$$\begin{aligned} y(t)=&\frac{\sqrt{2}t^{\alpha}}{\Gamma(\alpha+1)}-\frac{2\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}\\ &-\frac{4\sqrt{2}t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma^3(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)}.... \end{aligned} \tag{20}$$

Noting that, Table 1 demonstrates the superiority of the proposed accelerated Ramadan group Adomian decomposition (ARGADM) through comparing the approximate solution of Example 1 using Ramadan group transform coupled with accelerated Adomian decomposition and the approximate solution of Example 1 using Ramadan group transform coupled with regular Adomian decomposition (RGADM), although we use less iterations through ARGADM, but the obtained results of ARGADM are more accurate than those of RGADM. Besides, in Table 2 we demonstrate the superiority of our method in terms of accuracy through comparing it with recent method used in [9], noting that, in [9] the symbol m' represents the set of Block-Pulse functions where the author expanded the third kind Chebyshev wavelet matrix in terms of this set of functions. Its clear that the solution obtained by our method is very much close to the exact solution. Also, see demonstrative figures 1, 2 and 3.

Table 1: Comparison of the approximate solution between Regular RGADM and ARGADM at different values of alpha for Example 1

Proposed ARGADM using four iterations ($n=4$) $y_{approximate}=y0+y1+y2+y3+y4$									
t	Approximate solution at $\alpha=0.8$	Absolute error	Approximate solution at $\alpha=0.9$	Absolute error	Approximate solution at $\alpha=0.95$	Absolute error	Approximate solution at $\alpha=1$	Absolute error	Exact solution
0.1	0.236355	0.0955961	0.183403	0.0426445	0.160866	0.0201074	0.140758	1.21569×10^{-13}	0.140758
0.2	0.397375	0.119748	0.33462	0.0569924	0.305295	0.0276675	0.277627	2.39638×10^{-10}	0.277627
0.3	0.525712	0.118577	0.466558	0.0594224	0.436682	0.0295461	0.407136	1.94723×10^{-8}	0.407136
0.4	0.629396	0.102839	0.580604	0.0540473	0.554058	0.0275012	0.526557	4.23381×10^{-7}	0.526557
0.5	0.713573	0.0794893	0.677896	0.0438129	0.65693	0.0228467	0.634088	4.43353×10^{-6}	0.634083
0.6	0.782619	0.0537748	0.760011	0.0311677	0.745586	0.0167421	0.728873	0.0000290925	0.728844
0.7	0.840806	0.0300209	0.829001	0.0182151	0.82103	0.0102448	0.810923	0.000137822	0.810786
0.8	0.892767	0.0122774	0.887405	0.00691558	0.884896	0.0044066	0.881003	0.000513402	0.88049
0.9	0.943841	0.00488006	0.938373	0.000588322	0.939435	0.000474307	0.940551	0.00158995	0.938961
Proposed ARGADM using five iterations ($n=5$) $y_{approximate}=y0+y1+y2+y3+y4+y5$									
t	Approximate solution at $\alpha=0.8$	Absolute error	Approximate solution at $\alpha=0.9$	Absolute error	Approximate solution at $\alpha=0.95$	Absolute error	Approximate solution at $\alpha=1$	Absolute error	Exact solution
0.1	0.236355	9.56×10^{-2}	0.236355	9.56×10^{-2}	0.160866	2.011×10^{-2}	0.140758	2.22×10^{-16}	0.140758
0.2	0.397375	1.197×10^{-1}	0.397375	1.197×10^{-1}	0.305295	2.767×10^{-2}	0.277627	2.066×10^{-12}	0.277627
0.3	0.525707	1.186×10^{-1}	0.525707	1.186×10^{-1}	0.436682	2.955×10^{-2}	0.407136	3.735×10^{-10}	0.407136
0.4	0.629338	1.028×10^{-1}	0.629338	1.028×10^{-1}	0.554056	2.75×10^{-2}	0.526557	1.421×10^{-8}	0.526557
0.5	0.713219	7.914×10^{-2}	0.713219	7.914×10^{-2}	0.656916	2.283×10^{-2}	0.634083	2.282×10^{-7}	0.634083
0.6	0.781113	5.227×10^{-2}	0.781113	5.227×10^{-2}	0.745499	1.666×10^{-2}	0.728842	2.109×10^{-6}	0.728844
0.7	0.835835	2.505×10^{-2}	0.835835	2.505×10^{-2}	0.82065	9.865×10^{-3}	0.810772	1.327×10^{-5}	0.810786
0.8	0.879144	1.345×10^{-3}	0.879144	1.345×10^{-3}	0.883563	3.074×10^{-3}	0.880427	6.289×10^{-5}	0.88049
0.9	0.911494	2.747×10^{-2}	0.911494	2.747×10^{-2}	0.935504	3.457×10^{-3}	0.938721	2.398×10^{-4}	0.938961
Regular RGADM using five iterations ($n=5$) $y_{approximate}=y0+y1+y2+y3+y4+y5$									
t	Approximate solution at $\alpha=0.8$	Absolute error	Approximate solution at $\alpha=0.9$	Absolute error	Approximate solution at $\alpha=0.95$	Absolute error	Approximate solution at $\alpha=1$	Absolute error	Exact solution
0.1	0.236355	0.0955961	0.183403	0.0426445	0.160866	0.0201074	0.140758	1.24418×10^{-10}	0.140758
0.2	0.397387	0.119759	0.334621	0.0569933	0.305295	0.0276677	0.277627	6.47604×10^{-8}	0.277627
0.3	0.525938	0.118802	0.466583	0.0594471	0.43669	0.0295541	0.407138	2.55448×10^{-6}	0.407136
0.4	0.631277	0.10472	0.580868	0.0543112	0.554154	0.0275977	0.526592	0.0000351547	0.526557
0.5	0.723403	0.0893195	0.679572	0.045489	0.657605	0.0235215	0.634355	0.000271909	0.634083
0.6	0.820815	0.0919712	0.767664	0.0388206	0.748922	0.020078	0.730304	0.00145998	0.728844
0.7	0.961557	0.150772	0.856787	0.0460011	0.833995	0.0232098	0.816872	0.00608687	0.810786
0.8	1.22061	0.34012	0.972606	0.0921163	0.927103	0.0466132	0.90155	0.0210606	0.88049
0.9	1.73564	0.796681	1.16776	0.228796	1.05932	0.120356	1.00209	0.0631249	0.938961

Table 2: The absolute error values for ARGADM Method for $\alpha=1.0$ and recent method at [9] for Example 1

t	Absolute Error for Proposed ARGADM method (4 iterations)	Absolute Error for Proposed ARGADM method (5 iterations)	Absolute errors for method in [9]		
			$m'=6$	$m'=12$	$m'=24$
0	0	0	4.494019E-03	6.773097E-04	8.893090E-05
0.1	1.21569E-13	2.4980E-16	5.035322E-04	1.286374E-04	6.421086E-05
0.2	2.39638E-10	2.0661E-12	1.059434E-03	4.707650E-04	1.139629E-04
0.3	1.94723E-8	3.7346E-10	2.690553E-03	5.094155E-04	1.274317E-04
0.4	4.23381E-7	1.4214E-08	2.670298E-03	6.297543E-04	1.452844E-04
0.5	4.43353E-6	2.2820E-07	1.333743E-03	5.284051E-04	1.442915E-04
0.6	2.9093E-5	2.1092E-06	2.117661E-03	5.116593E-04	1.248584E-04
0.7	1.3782E-04	1.3271E-05	1.789413E-03	3.883598E-04	9.652208E-05
0.8	5.1340E-04	6.2885E-05	9.298447E-04	3.016540E-04	7.396663E-05
0.9	0.00158995	2.3978E-04	5.437335E-04	1.342873E-04	4.458551E-05

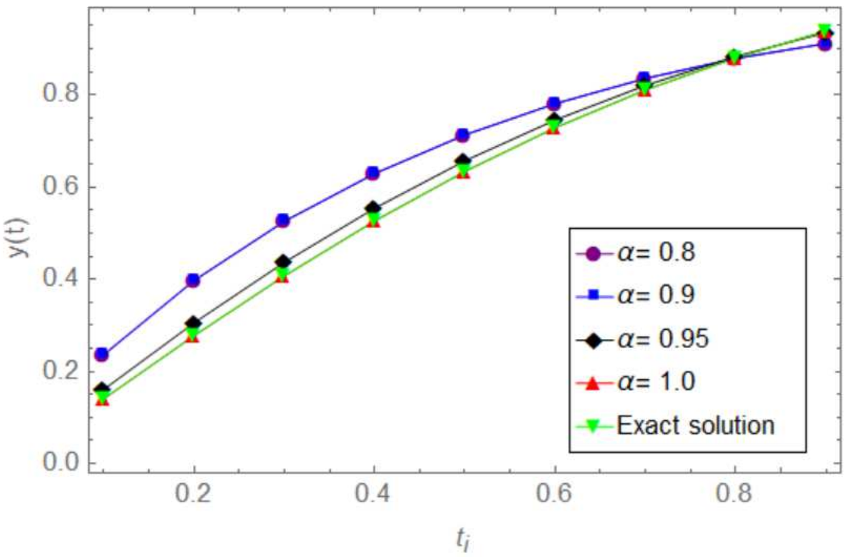

Fig. 1: The graph of the solution to the fractional differential equation in Example 1 using the combined method of Ramadan group transform and accelerated Adomian decomposition method (RGADM), using five iterations ($n = 5$) for $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 0.95$ and $\alpha = 1$ versus the exact solution

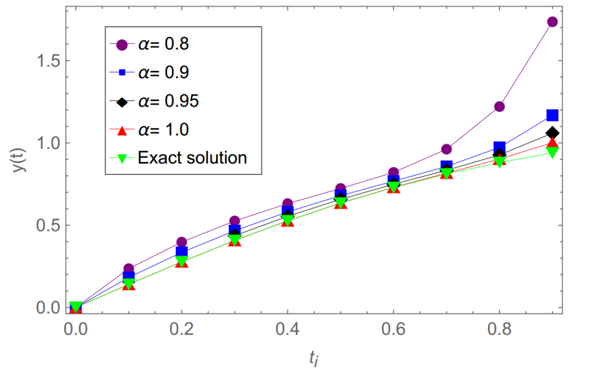

Fig. 2: The graph of the solution to the fractional differential equation in Example 1 using the combined method of Ramadan group transform and regular Adomian decomposition method (RGADM), using five iterations ($n = 5$) for $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 0.95$ and $\alpha = 1$ versus the exact solution

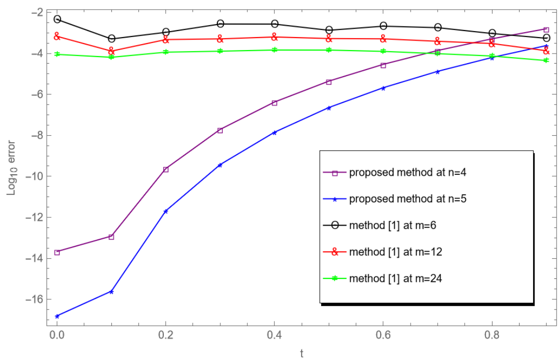

Fig. 3: Comparison of the absolute errors ,as shown in Table 2 for Example 1 to the proposed method (ARGADM) with 4 and 5 iterations against the method in [9] with $m' = 6$, 12 and 24

Example 2. Take into account the FDE [20,21]

$$D_t^{\alpha} y(t) = 1 - y^2(t), \ \ t > 0, \ 0 < \alpha \leq 1, \tag{21}$$

with initial condition $y(0) = 0$, and exact solution $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$ at $\alpha = 1$.

Proof. Applying Ramadan group transform to (21) and using the fractional derivatives properties of the RG-transform, we have

$$\frac{s^{\alpha}}{u^{\alpha}} Y(s,u) - \frac{s^{\alpha-1}}{u^{\alpha}} y(0) = \frac{1}{s} - RG[y^2(t)],$$

hence,

$$Y(s,u) = \frac{u^{\alpha}}{s^{\alpha+1}} - \frac{u^{\alpha}}{s^{\alpha}} RG[y^2(t)], \tag{22}$$

applying inverse Ramadan group transform to the equation (22), we get

$$y(t) = RG^{-1}[\frac{u^{\alpha}}{s^{\alpha+1}} - \frac{u^{\alpha}}{s^{\alpha}} RG[y^2(t)]]. \tag{23}$$

Based on the recursive formula, the approximate solution of the fractional differential equation (21) uses the combined method of the accelerated Adomian decomposition method and RG-transform can be presented as follows:

$$y_0(t) = RG^{-1}[\frac{u^{\alpha}}{s^{\alpha+1}}] = \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{24}$$

and

$$y_n(t) = -RG^{-1}[\frac{u^{\alpha}}{s^{\alpha}}[RG[\bar{A}_{n-1}]]], \text{ for } n \geq 1, \tag{25}$$

where, $\sum_{n=0}^{\infty} \bar{A}_n = y^2(t)$, given that $\bar{A}_n$ are El-kalla polynomials, then by Evaluation of (24) and (25) using Matlab, we obtain the approximate solution in the following form:

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} - \frac{\Gamma^2(2\alpha+1)t^{7\alpha}}{\Gamma^3(3\alpha+1)\Gamma^4(\alpha+1)\Gamma(7\alpha+1)} \ldots.. \tag{26}$$

Noting that, taking three iterations the approximate solution takes the form

$$\begin{aligned} y_{approx.} = & \frac{t^a}{\Gamma(a+1)} - \frac{t^{3a}\Gamma(2a+1)}{\Gamma(a+1)^2\Gamma(3a+1)} \\ & - \frac{t^{5a}\Gamma(2a+1)\left(t^{2a}\Gamma(2a+1)\Gamma(5a+1)\Gamma(6a+1) - 2\Gamma(a+1)\Gamma(3a+1)\Gamma(4a+1)\Gamma(7a+1)\right)}{\Gamma(a+1)^4\Gamma(3a+1)^2\Gamma(5a+1)\Gamma(7a+1)} \\ & - \frac{t^{7a}\Gamma(2a+1)}{\Gamma(a+1)^8\Gamma(3a+1)^4\Gamma(7a+1)^2}\left(\frac{4\Gamma(a+1)^4\Gamma(3a+1)^3\Gamma(4a+1)\Gamma(6a+1)\Gamma(7a+1)}{\Gamma(5a+1)}\right. \\ & - \frac{2t^{2a}\Gamma(a+1)^3\Gamma(2a+1)\Gamma(3a+1)^2\Gamma(7a+1)(\Gamma(5a+1)\Gamma(6a+1)+2\Gamma(4a+1)\Gamma(7a+1))\Gamma(8a+1)}{\Gamma(5a+1)\Gamma(9a+1)} \\ & + \frac{2t^{4a}\Gamma(a+1)^2\Gamma(2a+1)\Gamma(3a+1)\Gamma(7a+1)\left(2\Gamma(3a+1)\Gamma(7a+1)\Gamma(4a+1)^2 + \Gamma(2a+1)\Gamma(5a+1)^2\Gamma(6a+1)\right)\Gamma(10a+1)}{\Gamma(5a+1)^2\Gamma(11a+1)} \\ & - \frac{4t^{6a}\Gamma(a+1)\Gamma(2a+1)^2\Gamma(3a+1)\Gamma(4a+1)\Gamma(6a+1)\Gamma(7a+1)\Gamma(12a+1)}{\Gamma(5a+1)\Gamma(13a+1)} \\ & \left. + \frac{t^{8a}\Gamma(2a+1)^3\Gamma(6a+1)^2\Gamma(14a+1)}{\Gamma(15a+1)}\right). \end{aligned} \tag{27}$$

In a particular case, when $\alpha = 1$, the approximate solution of our proposed method taking three iterations takes the form

$$y_{approxi.} = t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{t^7}{63} + \frac{-715t^{15} + 13860t^{13} - 109746t^{11} + 570570t^9 - 1621620t^7}{42567525} \tag{28}$$

which matches with the identical terms of the series expansions of the exact solution

$$y_{exact} = \frac{e^{2t}-1}{e^{2t}+1} = t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835} - \frac{1382t^{11}}{155925} + \frac{21844t^{13}}{6081075} - \frac{929569t^{15}}{638512875} + \dots \quad (29)$$

Table 3 shows that the more terms we include, the greater the accuracy we get. Furthermore, when the fractional order approaches to one, the approximate solution approaches the exact solution. Table 4, Table 5 and Table 6 demonstrate the superiority of our method through comparing it with methods used in [20] and [21], also a demonstrative Figure 4 is presented. Its clear that the solution obtained by our method is very much close to the exact solution.

Table 3: Values of approximate and exact solution of Example 2 at different values of the fractional order α using five iterations $n = 5$.

t	at $\alpha = 0.8$		at $\alpha = 0.9$		at $\alpha = 0.95$		at $\alpha = 1$		Exact
t	$y_{approx.}$	Absolute error	$y_{approx.}$	Absolute error	$y_{approx.}$	Absolute error	$y_{approx.}$	Absolute error	
0.1	0.168002	6.833×10^{-2}	0.130037	3.037×10^{-2}	0.11397	1.43×10^{-2}	0.099668	4.763×10^{-17}	0.099668
0.2	0.285241	8.787×10^{-2}	0.238789	4.141×10^{-2}	0.217405	2.003×10^{-2}	0.197375	1.858×10^{-13}	0.197375
0.3	0.381868	9.056×10^{-2}	0.335962	4.465×10^{-2}	0.313362	2.205×10^{-2}	0.291313	3.431×10^{-11}	0.291313
0.4	0.462944	8.299×10^{-2}	0.422583	4.263×10^{-2}	0.401418	2.147×10^{-2}	0.379949	1.343×10^{-9}	0.379949
0.5	0.531255	6.914×10^{-2}	0.499135	3.702×10^{-2}	0.481137	1.902×10^{-2}	0.462117	2.232×10^{-8}	0.462117
0.6	0.588874	5.182×10^{-2}	0.566169	2.912×10^{-2}	0.552364	1.531×10^{-2}	0.537049	2.145×10^{-7}	0.53705
0.7	0.637482	3.311×10^{-2}	0.624383	2.002×10^{-2}	0.615246	1.088×10^{-2}	0.604366	1.408×10^{-6}	0.604368
0.8	0.678428	1.439×10^{-2}	0.674573	1.054×10^{-2}	0.670179	6.142×10^{-3}	0.66403	6.973×10^{-6}	0.664037
0.9	0.712711	3.586×10^{-3}	0.717553	1.255×10^{-3}	0.717718	1.42×10^{-3}	0.71627	2.783×10^{-5}	0.716298
1.0	0.740928	2.067×10^{-2}	0.754065	7.529×10^{-3}	0.758488	3.106×10^{-3}	0.761501	9.357×10^{-5}	0.761594

Table 4: Comparison of the absolute error of Example 2 using the proposed method(ARGADM) for five iterations ($n = 5$) versus the absolute error obtained by [20]

t	Absolute error of RGT coupled with MADM method at			Absolute error for results of [20] at		
t	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.1	6.833×10^{-2}	3.037×10^{-2}	4.7627×10^{-17}	0.0701	0.0308	0.0000
0.2	8.787×10^{-2}	4.141×10^{-2}	1.85849×10^{-13}	0.0954	0.0438	0.0000
0.3	9.056×10^{-2}	4.465×10^{-2}	3.43134×10^{-11}	0.1065	0.0500	0.0003
0.4	8.299×10^{-2}	4.263×10^{-2}	1.34349×10^{-9}	0.1076	0.0508	0.0013
0.5	6.914×10^{-2}	3.702×10^{-2}	2.2319×10^{-8}	0.0993	0.0462	0.0038
0.6	5.182×10^{-2}	2.912×10^{-2}	2.14469×10^{-7}	0.0809	0.0350	0.0090
0.7	3.311×10^{-2}	2.002×10^{-2}	1.40759×10^{-6}	0.0511	0.0157	0.0187
0.8	1.439×10^{-2}	1.054×10^{-2}	6.97293×10^{-6}	0.0077	0.0137	0.0347
0.9	3.586×10^{-3}	1.255×10^{-3}	2.783×10^{-5}	0.0518	0.0557	0.0593
1.0	2.067×10^{-2}	7.529×10^{-3}	9.357×10^{-5}	0.1302	0.1130	0.0949

Table 5: Comparison of the approximate solution of Example 2 using Ramadan group transform coupled with accelerated Adomian decomposition method using five iterations ($n = 5$) versus the approximate solution obtained by [21]

t	Exact	RGT coupled with MADM		RESULTS OF [21]	
t	y_e	$y_{approx.}$	$y_{approx.}$	$y_{approx.}$	$y_{approx.}$
t		at $\alpha = 0.9$	at $\alpha = 1$	at $\alpha = 0.9$	at $\alpha = 1$
0.2	0.197375	0.238789	0.197375	0.238794	0.197375
0.4	0.379949	0.422583	0.379949	0.422593	0.379949
0.6	0.53705	0.566169	0.537049	0.566181	0.537049
0.8	0.664037	0.674573	0.66403	0.674636	0.664037
1.0	0.761594	0.754065	0.761501	0.754607	0.761614

Table 6: Comparison of the absolute error for Example 2 using Ramadan group transform coupled with accelerated Adomian decomposition method using five iterations ($n = 5$) versus the absolute error of the $11th$ approximate Laplace residual power series method (LRPS) obtained by [22] at $\alpha = 1$

t	RGT coupled with MADM		LRPS [22]	
t	$y_{approx.}$	Absolute error	$y_{approx.}$	Absolute error
0.1	0.099668	4.763×10^{-17}	0.099667	1.68079×10^{-11}
0.2	0.197375	1.858×10^{-13}	0.197375	2.11082×10^{-10}
0.3	0.291313	3.431×10^{-11}	0.291312	1.51948×10^{-8}
0.4	0.379949	1.343×10^{-9}	0.379949	3.4917×10^{-7}
0.5	0.462117	2.232×10^{-8}	0.462121	3.92967×10^{-6}

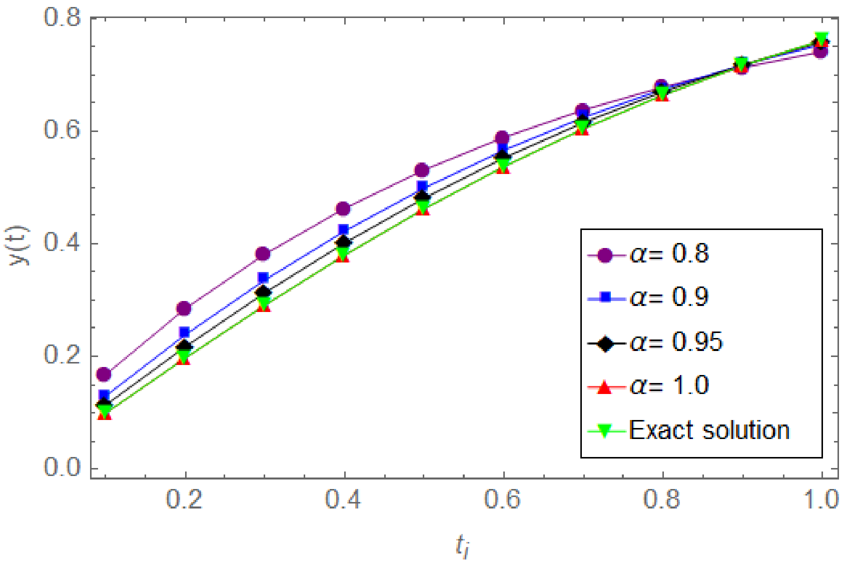

Fig. 4: Numerical results comparison for Example 2 using the proposed ARGADM at different values of α and the exact solution

*Example 3.*Take into account the FDE [23]

$$D_t^{\alpha} y(t) = 1 + y^2(t), \;\; t > 0, \; 0 < \alpha \leq 1, \tag{30}$$

with initial condition $y(0) = 0$, and exact solution for $\alpha = 1$ is $y(t) = \tan t$.

*Proof.*Applying Ramadan group transform to (30) and using the fractional derivatives properties of the RG-transform, we have

$$\frac{s^{\alpha}}{u^{\alpha}} Y(s,u) - \frac{s^{\alpha-1}}{u^{\alpha}} y(0) = \frac{1}{s} + RG[y^2(t)],$$

hence,

$$Y(s,u) = \frac{u^{\alpha}}{s^{\alpha+1}} + \frac{u^{\alpha}}{s^{\alpha}} RG[y^2(t)], \tag{31}$$

applying inverse Ramadan group transform to the equation (31), we get

$$y(t) = RG^{-1}[\frac{u^{\alpha}}{s^{\alpha+1}} + \frac{u^{\alpha}}{s^{\alpha}} RG[y^2(t)]]. \tag{32}$$

Based on the recursive formula, the approximate solution of the fractional differential equation (30) uses the combined method of the accelerated Adomian decomposition method and RG-transform can be presented as follows:

$$y_0(t) = RG^{-1}[\frac{u^{\alpha}}{s^{\alpha+1}}] = \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{33}$$

and

$$y_n(t) = RG^{-1}[\frac{u^\alpha}{s^\alpha}[RG[\bar{A}_{n-1}]]], \text{ for } n \geq 1, \tag{34}$$

where, $\sum_{n=0}^{\infty} \bar{A}_n = y^2(t)$, given that $\bar{A}_n$ are El-kalla polynomials (see [14]), then by Evaluation of (33) and (34) using Matlab, we obtain the approximate solution in the following form:

$$y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{\Gamma^2(2\alpha+1)t^{7\alpha}}{\Gamma^3(3\alpha+1)\Gamma^4(\alpha+1)\Gamma(7\alpha+1)} \ldots.. \tag{35}$$

Table 7 and Table 8 show that the more terms we include, the greater the accuracy we get. Also, when the fractional order approaches to one, the approximate solution approaches to the exact solution. Furthermore, Table 8 demonstrates the superiority of our method through comparing it with methods used in [23] and [24], its clear that the solution obtained by our method is very much close to the exact solution. Figure 5 shows that the error of our proposed method is less than that of [23], also, our proposed method is more accurate than the method in [24] till $t = 0.6$, but we can increase the accuracy of our proposed method by including more terms.

Table 7: Values of approximate and exact solution of Example 3 at different values of the fractional order α using five iterations ($n = 5$).

t	$y_{approx.}$ at $\alpha = 0.7$	$y_{approx.}$ at $\alpha = 0.8$	$y_{approx.}$ at $\alpha = 0.9$	$y_{approx.}$ at $\alpha = 1$	y_e Exact
0.1	0.22522608417	0.17240527248	0.13177173872	0.10033467209	0.10033467209
0.2	0.38245488611	0.30854098232	0.25006774775	0.20271003551	0.20271003551
0.3	0.53936435917	0.44401261941	0.36976714628	0.30933624957	0.30933624961
0.4	0.71297676901	0.58897431884	0.4966044541	0.42279321684	0.42279321874
0.5	0.92135983941	0.75301208032	0.63618880833	0.54630245154	0.54630248984
0.6	1.1928052494	0.94907497557	0.79564391688	0.6841363405	0.68413680834
0.7	1.5788295024	1.1978345262	0.98530465616	0.84228429205	0.84228838046
0.8	2.1807778525	1.5354764299	1.2215008891	1.0296102176	1.0296385571
0.9	3.2114444507	2.0294563855	1.5317539171	1.2599910975	1.2601582176
1.0	5.1560549859	2.8126769295	1.9650276242	1.5565238405	1.5574077247

Table 8: Comparison of the approximate solutions for Example3 using the proposed method (ARGADM) using six iterations ($n = 6$) versus the methods in [23] and [24].

t	Exact solution	Proposed Method	Absolute Error	Method in[23]	Absolute Error in [23] at m=12	Method in[24]	Absolut error in [24]
0.1	0.100334672085	0.1003346721	1.3878×10^{-17}	0.1003346714	2.8897×10^{-8}	0.1003346713	8.162E-10
0.2	0.202710035509	0.2027100355	1.0825×10^{-15}	0.2027100349	5.1478×10^{-8}	0.2027099297	1.0580E-7
0.3	0.30933624961	0.3093362496	5.077×10^{-13}	0.3093362509	4.6086×10^{-8}	0.3093343442	1.9050E-6
0.4	0.422793218738	0.4227932187	4.1615×10^{-11}	0.4227932186	3.4828×10^{-8}	0.4227777155	1.5500E-5
0.5	0.546302489844	0.5463024885	1.3345×10^{-9}	0.5463024891	2.3389×10^{-8}	0.5462212762	8.1210E-5
0.6	0.684136808342	0.6841367844	2.3967×10^{-8}	0.6841368110	5.0755×10^{-8}	0.6838056920	3.3110E-4
0.7	0.842288380463	0.8422880878	2.9268×10^{-7}	0.8422883779	7.3355×10^{-8}	0.8411449022	1.1430E-3
0.8	1.02963855705	1.029635819	2.7377×10^{-6}	1.0296385599	4.4578×10^{-8}	1.0261001110	3.5380E-3
0.9	1.26015821755	1.260136941	2.1277×10^{-5}	1.2601582184	5.8748×10^{-8}	1.2499664940	1.1090E-3
1.0	1.55740772465	1.557261577	1.4615×10^{-4}	1.5574077258	2.2418×10^{-8}	1.5293009690	2.8110E-3

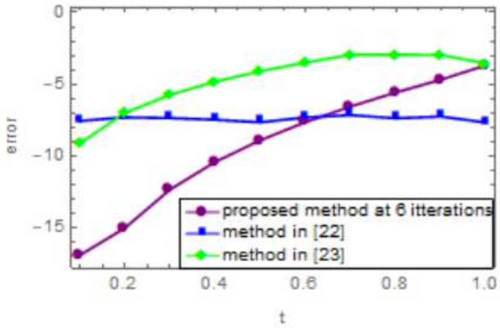

Fig. 5: Comparison of the absolute errors ,as shown in Table 8 for Example 3 to the proposed method (ARGADM) with 6 iterations against methods in [24] and [23].

4 Conclusion

This paper investigates a numerical solution for quadratic Riccati differential equations of fractional order using a hybrid method combining the Ramadan group integral transform and the accelerated Adomian decomposition method (ARGADM). The method has been shown to be more accurate and faster than the standard Adomian polynomial with Ramadan group transform, as well as more contemporary techniques reported in [[9], [20], [21], [14]]. The numerical results obtained, which are presented in tables and figures, show that the approximate solutions obtained using the proposed method are highly consistent with the exact solutions and are an effective and simple method when compared to other methods mentioned in the study or elsewhere that are not mentioned in this work.. We conclude that the proposed method can be utilized to solve a wide range of nonlinear fractional differential equations in a variety of applications.

5 Conflict of interest

The authors declare no conflict of interest.

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