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The Integrals of Fractional Operators with Non-Singular Kernels: A Conceptual Approach, Formulations, and Normalization Functions ⋆

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Abstract: Integrals pertinent to fractional operators with non-singular kernels (Caputo-Fabrizio and Atanagana-Baleanu) have been considered and analyzed. Special attention has been paid to the definitions of associate and constitutive integrals, clearly defining the principal differences between them. A conceptual application of the constitutive integrals to the construction of models, applying both the fading memory concept and Volterra equations, has been exemplified by one-dimensional transient heat conduction yielding fractional operators of Caputo-Fabrizio and Atangana-Baleanu in both Caputo and Riemann-Liouville sense. A special focus of the study is the definition and analysis of the normalization functions ($M(\alpha)$ and $B(\alpha)$) pertinent to these operators, systematically neglected in the literature after the conjecture of Losada and Nieto in 2015, that defining the Caputo-Fabrizio operator $M(\alpha)$ has to be accepted conventionally equal to unity.

Keywords: fractional associated integrals,fractional constitutive integrals, Caputo-Fabrizio, Atangana-Baleanu , fractional derivatives,normalization functions

1 Introduction

In this work, we address integrals (and normalization functions) related to two fractional operators with non-singular kernels forming a specific trend in modern fractional calculus, that is the Caputo-Fabrizio (3)-(4) [1,2] and Atangana-Baleanu derivatives (5)-(6) [3] defined as general constructions (1)-(2) of Caputo (1) and Riemann-Liouville (2) operators

$$^{C}D_t^{\alpha} f(t) = \frac{M(\alpha)}{N(\alpha)} \int_0^t R(t-\tau)\frac{df(\tau)}{d\tau}d\tau, \quad 0<\alpha<1, \quad t>0 \tag{1}$$

$$^{R}D_t^{\alpha} f(t) = \frac{M(\alpha)}{N(\alpha)} \frac{d}{dt}\int_0^t R(t-\tau)f(\tau)d\tau, \quad 0<\alpha<1, \quad t>0 \tag{2}$$

$$^{CFC}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\frac{df(\tau)}{d\tau}d\tau, \quad 0<\alpha<1, \quad t>0 \tag{3}$$

$$^{CFR}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \frac{d}{dt}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f(\tau)d\tau, \quad 0<\alpha<1, \quad t>0 \tag{4}$$

⋆ This article is dedicated to 8th anniversary of the seminal article of Caputo and Fabrizio published in *Progress in Fractional Differentiation and Applications*, No 1, 2015

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$$^{ABC}D_t^{\alpha}f(t)=\frac{B(\alpha)}{1-\alpha}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\frac{df(\tau)}{d\tau}d\tau,\quad 0<\alpha<1,\quad t>0 \tag{5}$$

$$^{ABR}D_t^{\alpha}f(t)=\frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)f(\tau)d\tau,\quad 0<\alpha<1,\quad t>0 \tag{6}$$

In general, in all these operators the normalization function $\Gamma(1-\alpha)$ known from the Riemann-Liouville and Caputo derivative with power-law memory kernels [4] is replaced by $(1-\alpha)$. The definition (1) relies on general operators of Caputo-type based on various versions of the Mittag-Leffler function [5,6], while (2) is of Riemann-Liouville type.

The principal problems discussed in this article are the definitions of *associated* (AI) and *constitutive* (CI) integrals related to these fractional operators. The problem was motivated initially by the fact that in the seminal articles [1,2] and [3] the definitions of $M(\alpha)$ and $B(\alpha)$, albeit the general requirements $M(0)=M(1)=1$ and $B(0)=B(1)=1$ were imposed, these functions remained undefined. The question of what $M(\alpha)$ in the construction (3) should be defined was raised by Losada and Nieto [7];the problems emerging are discussed in the sequel. Moreover, as a consequence, this resulted in definitions of two different types of integrals mentioned above, thus clarifying how constitutive equations of models related to transport phenomena should be formulated.

Note: In general, the article is organized and developed as a systematic analysis of the existing situation with the aforementioned fractional operators and as a conceptual approach defining two different types of integrals related to them: *Associate fractional integrals* and *Constitutive integrals*. The directions analyzed in the text that follows need a specific article organization, as mentioned in the following comments (see Aim 1 (1.2) and the remarks explaining this approach).

1.1 The next analysis's tasks

The targets of this article can be outlined as follows:

Task 1: An analysis of the associated integrals of the Caputo-Fabrizio and Atangana-Baleanu operators, elucidating existing problems in their evaluations.

Task 2: Definition of the concept of constitutive integrals and the a generalized approach to construct models with memories applying the Fading Memory formalism and the Volterra equations.

Task 3: To consider problems emerging in definitions of the normalization functions $M(\alpha)$ and $B(\alpha)$ and their versions .

These major tasks will allow the better formulation of the two principle directions of this study:

1) An analysis of existing problems in definition of associated integrals (as outlined in sections 1.2 and 1.3).

2) Formulations of new problem to be resolved regarding the constitutive integrals and analysis of the normalization functions in Section 6 (as outlined in 6.1 and 6.2).

1.2 Aim 1

The first aim of this study addresses attempts to formulate associated integrals of the Caputo-Fabrizio and Atangana-Baleanu fractional operators and the emerging problems in model formulations.

*Remark.*The formulation of two aims in this article is an unconventional approach but it is done to facilitate the reading naturally following the detailed analysis performed first and followed by formulation of new problems in Section 6.1.

1.3 Further text organization addressing Aim 1

Following the Aim 1, the first Section 2 in the sequel considers the Caputo-Fabrizio operator and the attempts to define its associated integrals (Sections 2.1, 2.2 and 2.3). The original and alternative approaches towards formulations of the Caputo-Fabrizio integrals are analyzed in Section 3. The problems and features of the associated integrals of the Atangana-Baleanu operator are analysed in Section 4. A brief outlining of the analyzes performed with drawing the consequently following tasks is presented in Section 5.

2 Caputo-Fabrizio operator: Origin of the problem and developed results

Considering the main problem, we are obliged to discuss the results already developed and see what has been done and have formulated new problems or not. This section addresses two approaches of Losada and Nieto [7,8] and the attempt of Bekkouche et al. [9], the results developed, and some thoughts invoked as a step towards formulating new problems solved in this article.

2.1 Losada-Nieto: Attempt 1

To define the associated fractional integral of the Caputo-Fabrizio derivative (3), Losada and Nieto [7] started from the equation

$$ {}^{CF}D_t^{\alpha} f(t) = u(t), \quad t > 0 \tag{7} $$

which to a greater extent is common in fractional calculus and we may trace it back to the applications of fractional operators with singular kernels [4]. Applying the Laplace transform to (7) as $\mathscr{L}\left[{}^{CF}D_t^{\alpha} f(t)\right](s) = \mathscr{L}[u(t)](s)$ the result is

$$ M(\alpha)\frac{(2-\alpha)}{2(s+\alpha(1-s))}\left(\mathscr{L}\left[{}^{CF}D_t^{\alpha} f(t)\right](s) - f(0)\right) = \mathscr{L}[u(t)](s), \quad s > 0 \tag{8} $$

The inverse Laplace transform yields (more detailed steps are available in [7])

$$ f(t) = \frac{1}{M(\alpha)}\left(\frac{2(1-\alpha)}{(2-\alpha)}[u(t)-u(0)] + \frac{2\alpha}{(2-\alpha)}\int_0^t u(s)\,ds\right) + f(0) \tag{9} $$

The result (9) induced the definition of an associated fractional integral related to ${}^{CF}D_t^{\alpha} f(t)$ as [7]

$$ {}^{CF}_{LN}I_t^{\alpha} f(t) = \frac{1}{M(\alpha)}\left[\frac{2(1-\alpha)}{(2-\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)}\int_0^t u(s)\,ds\right], \quad 0<\alpha<1, \quad t \geq 0 \tag{10} $$

We can see that for $\alpha = 0$ from (7) we get $f(t) = u(t)$. Then, equation (10) provides $\frac{1}{M(0)} = 1$ and therefore we should have $M(0) = 1$. Otherwise, for $\alpha = 1$, equation (7) yields $f(t) = \int_0^t u(s)\,ds$. Therefore, for $\alpha = 1$ we have $\frac{2}{M(1)} = 2$ which is leading to $M(1) = 2$. Hence, the requirement imposed on $M(\alpha)$ by the original definitions (3) and (4), as well as by (10) are obeyed only for $\alpha = 0$.

Nevertheless, Losada and Nieto [7] interpreted *the defined fractional integral as an average between the function $f(t)$ and its associated integral*, resulting in the imposing (see Definition 1 in [7])

$$ \frac{1}{M(\alpha)}\left[\frac{2(1-\alpha)}{(2-\alpha)} + \frac{2\alpha}{(2-\alpha)}\right] = 1 \Rightarrow \left[\frac{2(1-\alpha)}{(2-\alpha)} + \frac{2\alpha}{(2-\alpha)}\right] = 1 \tag{11} $$

Thus, the emerging difficulty was avoided, by a convention that $M(\alpha) = 1$, and the redefinition (10), but the more general problem concerning the function $M(\alpha)$ and its behavior within the interval $[0,1]$ was not resolved and remained open.

*Remark.*The problem, to our point of view, the idea to accept $M(\alpha) = 1$ is motivated to abandon the pursuit of defining $M(\alpha)$ and resides in the definition of the associated fractional integral ${}^{CF}_{LN}I_t^{\alpha} f(t)$ in a manner known from the integer-order calculus and the fractional calculus with singular kernels; this is, to a greater extent, the reason to use the term "associated fractional integral" instead "fractional integral" as it will be done in the sequel. In addition, we have to refer to [10] where the same convention was introduced after definition of *Caputo-Fabrizio fractional delta-integral* from the condition $\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)} = 1$, as an extension of the concept of Losada and Nieto [7].

*Remark.*To this point, as a continuation of the preceding remark, we have to remember that the first step in the definition of the Riemann-Liouville derivative is the definition of the fractional integral [4]. The Riemann-Liouville derivative is an integer-order (local) derivative of the fractional integral while the Caputo derivative can be obtained from it through integration by parts (this is from a mathematical point of view even though both derivatives have different physical backgrounds). Following this comment, it is logical to construct the Caputo-Fabrizio integral as

$$ {}^{CF}_{M(\alpha)}\Im^{\alpha}_{t} f(t) = \frac{M(\alpha)}{1-\alpha} \int_{0}^{t} e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(t)\, d\tau 0 \leq \alpha \leq 1 \tag{12} $$

where $M(\alpha)$ has to be defined; however, for the analysis in the following sections we will assume $M(\alpha) = 1$, thus coming to the definition used by Atanackovic et al. [11] (see Section 8.4).

$$ {}^{CF}\Im^{\alpha}_{t} f(t) = \frac{1}{1-\alpha} \int_{0}^{t} e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(t)\, d\tau \tag{13} $$

In general, we will call it a **constitutive (basic) construction of the Caputo-Fabrizio integral**, irrespective of the normalization function $M(\alpha)$ (see the definition in section 7.2)

2.2 Losada-Nieto: Attempt 2

The case of the associated fractional integral of Caputo-Fabricio was reconsidered in the second attempt of Losada and Nieto [8] starting from (in terms used in this text) the integral equation (with the new definition, that is with $M(\alpha) = 1$).

$$ {}^{CF}I^{\alpha}_{t} g(t) = 0 \tag{14} $$

resulting in

$$ (1-\alpha)[g(t) - g(a)] + \alpha \int_{a}^{t} g(s) ds \tag{15} $$

and $\frac{dg(t)}{dt} = -\frac{\alpha}{1-\alpha} g(t)$, so that

$$ g(t) = e^{-\frac{\alpha}{1-\alpha}(t-a)} \tag{16} $$

Recall, the time interval of the action of the correlation (memory) function is $(t-a)$, that is if $a = 0$ (setting at zero the lower terminal of the integral in (15), we will get $g(t) = e^{-\frac{\alpha}{1-\alpha}t}$ (no memory).

Then, using that

$$ {}^{CF}_{LN}I^{\alpha}_{t} f(t) = (1-\alpha)[f(t) - f(0)] + \alpha \int_{0}^{t} f(s)\, ds \tag{17} $$

and

$$ \frac{d}{dt}\left[{}^{CF}_{LN}I^{\alpha}_{t} f(t)\right] = (1-\alpha)\frac{d}{dt} f(t) + \alpha f(t) \tag{18} $$

it was derived that

$$ {}^{CF}_{LN}I^{\alpha}_{t} D^{\alpha}_{t} f(t) = f(t) - f(a)\left[e^{-\frac{\alpha}{1-\alpha}(t-a)}\right] \tag{19} $$

*Remark.*The derivation (19) raises too many questions, among them: What is the reason the fractional Caputo-Fabrizio integral to be derived starting from equation (7) and (14) which never can be related to a real physical phenomenon? Why do we have to look for a result of ${}^{CF}_{LN}I^{\alpha CF}_{t} D^{\alpha}_{t} f(t)$ mimicking the semi-group properties of the fractional operators with singular kernels? Albeit the correctness of the mathematical construction of (14)- (19) the main question is about the physical correctness of the postulation of (15), which can be considered as a mathematical experiment rather than related to any physical model. To be precise, the construction of any models with heredity, involving fractional operators, should start with a constitutive equation with an integral term with memory kernel [6]; the construction of the fractional derivative as an operator is a consequence, not the primary step. With this last comment, we stress the attention on the fact that the construction of the Caputo-Fabrizio fractional derivative has a strong physical basis [1,2] and the definition of an integral operator (fractional integral) with exponential memory exists in [2] (see Section 3).

2.3 Attempt of Bekkouche et al.[9]

Bekkouche et al. [9] introduced a new associated integral, left-sided, ${}^{CF}I_{a+}^{\alpha} f(t)$ of order $\alpha + n$, defined as

$$ {}^{CF}I_{a+}^{\alpha} f(t) = \frac{1}{M(\alpha)} \frac{1}{n!} \int_a^t \frac{\alpha (t-s) + n(1-\alpha)}{(t-s)^{1-n}} f(s)\, ds \tag{20} $$

where the normalization function $M(\alpha)$ obeys the same conditions as in the definition of the Caputo-Fabrizio derivative.

The main approach [9] is based on the development of a derivative $D^{(\alpha+n)} f(t)$ of order $n+\alpha$, $n \geq 1$, $\alpha \in [0,1]$, namely

$$ D^{(\alpha+n)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f^{(n+1)}(s) \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right] ds \tag{21} $$

where $D^{(\alpha+n)} f(t) = D^{(\alpha)}\left(D^{(n)} f(t)\right)$.

The approach of Bekkouche et al. [9] started from the equation (7) assuming that $u(t) = 0$, that is (in terms of the present text)

$$ D^{(\gamma)} f(t) = 0, \quad \forall t \in [a,b] \tag{22} $$

with a unique solution $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, $a_i \in \mathbb{R}$, $i = 0, 1, \cdots, n$ such that $f^{(n+1)} = 0$. The fractional order $\gamma \in [n, n+1]$ can be presented as $\gamma = n + \alpha$ where $\alpha \in [0,1]$ and $n = [\gamma]$.

Applying the Leibniz integral rule concerning $\frac{d}{dt} D_t^{(\gamma)} f(t)$, and skipping the intermediate calculations steps, the result is

$$ \frac{d}{dt} D_t^{(\gamma)} f(t) = \frac{M(\alpha)}{1-\alpha} f^{(n+1)}(t) - \frac{\alpha}{1-\alpha} D_t^{(\gamma)} f(t) \tag{23} $$

Then, from (22) we get $f^{(n+1)}(t) = 0$ and thus $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ is a solution of (22).

Now, starting from (22) and (23) we have

$$ \frac{d}{dt} D_t^{(\alpha+n)} f(t) = \frac{M(\alpha)}{1-\alpha} f^{(n+1)}(t) - \frac{\alpha}{1-\alpha} \frac{M(\alpha)}{1-\alpha} \int_a^t f^{(n+1)}(s) \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right] ds \tag{24} $$

That is

$$ \frac{d}{dt} D_t^{(\alpha+n)} f(t) = \frac{M(\alpha)}{1-\alpha} f^{(n+1)}(t) - \frac{\alpha}{1-\alpha} D_t^{(\alpha+n)} f(t) \tag{25} $$

Developing an expression concerning $f^{(n+1)}(t)$ whether we get

$$ f^{(n+1)}(t) = \frac{1}{M(\alpha)} \left[(1-\alpha) \frac{d}{dt} D_t^{(\alpha+n)} f(t) + \alpha D_t^{(\alpha+n)} f(t) \right] \tag{26} $$

The key step now is the application of the Cauchy formula for $(n+1)^{th}$ integration (the same resulting in the Riemann-Liouville integral), that is

$$ f(t) = \frac{1}{M(\alpha)} \frac{1}{n!} \int_a^t (t-s)^n \left[(1-\alpha) \frac{d}{ds} D_t^{(\alpha+n)} f(s) + \alpha D_t^{(\alpha+n)} f(s) \right] ds \tag{27} $$

Suggesting that $g(t) = D_t^{(\alpha+n)} f(t)$ and therefore $I_t^{n+\alpha} g(t) = f(t)$ we may present (27) as

$$ f(t) = \frac{1}{M(\alpha)} \frac{1}{n!} \int_a^t \frac{[\alpha (t-s) + n(1-\alpha)]}{(t-s)^{1-n}} g(s)\, ds \tag{28} $$

Further, if $f(t) \in [a,b]$ and $\gamma \in [n, n+1]$ can be presented as $\gamma = n + \alpha$ where $\alpha \in [0,1]$ and $n = [\gamma]$ (Lemma 3 in [9]) we have

$$ I_a^{\gamma}\left[D_t^{\gamma} f(t)\right] = f(t) + \sum_{i=0}^{n} a_i t^i, \quad a_i \in \mathbb{R}, \quad i = 0, 1 \cdots, n. \tag{29} $$

And if $f(a=0)=0$, then $D_t^{\gamma}\left[I_a^{\gamma}f(t)\right]=f(t)$, following from (the result of Lemma 3 in [9])

$$D_t^{\gamma}\left[I_a^{\gamma}f(t)\right]=f(t)-f(a)\exp\left[-\frac{\alpha}{1-\alpha}(t-a)\right] \tag{30}$$

This coincides with the result of Losada and Nieto [8] (see (19)).

*Remark.*We can see that two different attempts yielded two different associated integrals providing $D_t^{\gamma}\left[I_0^{\gamma}f(t)\right]=f(t)$. And, we may stress the attention again, that these integrals can facilitate the calculations, but are not related to fractional modelling where the Caputo-Fabrizio derivative is applied. However, we appreciate these efforts, since they provide answers to the classic problem $D_t^{\alpha}f(t)=u(t)$, where $D_t^{\alpha}f(t)$ could be of any kind. To this end, we have to mention that the definition of $M(\alpha)$ still remains unresolved. In the numerical experiments performed in [9] $M(\alpha)$ was defined *ad hoc* as $M(\alpha)=1-0.2sin(2\pi\alpha)$.

3 Fractional Caputo-Fabrizio Integral: The original and alternatives

Now we stress the attention on an approach how to construct constitutive equations involving convolution integral of Caputo-Fabrizio type that finally results in definitions of Caputo-Fabrizio fractional derivatives.

3.1 Fractional Caputo-Fabrizio Constitutive Integral: The original concept

Caputo and Fabrizio defined the following fractional integral with exponential kernel [2]

$${}^{CF1}I_t^{\alpha}f(t)=\frac{1}{\alpha}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}f(\tau)\,d\tau \tag{31}$$

where for $\beta=0$ we obtain $f(t)$, while for $\alpha=1$ we get $\int_0^t f(\tau)\,d\tau$.

Moreover, the time derivative of ${}^{CF1}I_t^{\alpha}f(t)$ is

$$\frac{d}{dt}{}^{CF1}I_t^{\alpha}f(t)=\frac{1}{\alpha}f(t)-\frac{1-\alpha}{\alpha^2}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}f(\tau)\,d\tau=\frac{1}{\alpha}f(t)-\frac{1-\alpha}{\alpha}{}^{CF1}I_t^{\alpha}f(t) \tag{32}$$

Now, suggesting the heat flux (a transient heat conduction problem) constitutive equation as [2]

$$q(x,t)=-k_0{}^{CF1}I_t^{\alpha}f(t)=-\frac{k_0}{\alpha}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}\frac{\partial\theta(x,\tau)}{\partial x}d\tau \tag{33}$$

The time derivative $\frac{dq(x,t)}{dt}$ can be presented as [2]

$$\frac{dq(x,t)}{dt}=-\frac{k_0}{\alpha}\frac{\partial\theta(x,t)}{\partial x}+\frac{k_0(1-\alpha)}{\alpha^2}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}\frac{\partial\theta(x,\tau)}{\partial x}d\tau \tag{34}$$

This result allows writing

$$\frac{dq(x,t)}{dt}=-\frac{k_0}{\alpha}\frac{\partial\theta(x,t)}{\partial x}+\frac{k_0(1-\alpha)}{\alpha}q(x,t) \tag{35}$$

Therefore, we got the construction of the Cattaneo-Maxwell equation [2]

$$\frac{\alpha}{1-\alpha}\frac{dq(x,t)}{dt}=-q(x,t)-\frac{k_0}{1-\alpha}\frac{\partial\theta(x,t)}{\partial x} \tag{36}$$

which recovers the Fourier law for $\alpha=1$

$$q(x,t)=-k_0\frac{\partial\theta(x,t)}{\partial x} \tag{37}$$

Then, applying the energy conservation equation (see next (39)) we get the Fourier equation (see next (41)).

3.2 Fractional Caputo-Fabrizio Constitutive Integral: An interpretation

Now, with $f(t) = \frac{\partial \theta(x,t)}{\partial x}$, we will apply a slightly different approach through differentiating (33) for x, namely

$$\frac{d}{dx}q(x,t) = -k_0 \frac{d}{dx} {}^{CF1}I_t^{\alpha} f(t) = -\frac{k_0}{\alpha}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}\frac{\partial^2\theta(x,\tau)}{\partial x^2}d\tau \tag{38}$$

Then, applying the energy conservation equation (the First Law of Thermodynamics)

$$\frac{\partial \rho C_p \theta(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x} \tag{39}$$

we obtain (assuming constant density ρ and heat capacity C_p)

$$\frac{\partial \theta(x,t)}{\partial t} = \frac{a_0}{\alpha}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}\frac{\partial^2\theta(x,\tau)}{\partial x^2}d\tau, \quad a_0 = \frac{k_0}{\rho C_p} \tag{40}$$

That is, we got a time-fractional heat conduction equation which for $\alpha = 1$ it recovers the Fourier model

$$\frac{\partial \theta(x,t)}{\partial t} = a_0 \frac{\partial^2 \theta(x,t)}{\partial x^2} \tag{41}$$

*Remark.*It is quite clear that the definition of the fractional integral (31) allows easy to construct adequate constitutive equations with memory. However, their alternative approaches to the definitions of both the Caputo-Fabrizio fractional integral and derivative [6] are starting from well-known and thermodynamically consistent constitutive equations about the heat flux, taken as a basic example. Some alternative approaches are demonstrated in Section 8.

4 Atangana-Baleanu operator: Origin of the problem and developed results

4.1 Definitions and properties related to the problem associated integral

Let us consider again the operators (5) and (6) with more details pertinent to the main concept developed in this article

Riemann–Liouville sense (ABR derivative)

$$^{ABR}D_{a+}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^z f(z)E_\alpha\left[\frac{-\alpha}{1-\alpha}(t-z)^\alpha\right]dz \tag{42}$$

with$0 < \alpha < 1$, $a < t < b$ and

$$f(t) \in L^1[a,b]$$

.

Caputo sense (ABC derivative)

$$^{ABC}D_{a+}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^z \frac{df(z)}{dz}E_\alpha\left[\frac{-\alpha}{1-\alpha}(t-z)^\alpha\right]dz \tag{43}$$

with $0 < \alpha < 1$, $a < t < b$ and $f(x,t)$ is differentiable function on $[a,b]$ such that $df/dt \in L^1[a,b]$.

The normalization function $B(\alpha)$ can be any function satisfying the conditions $B(0) = B(1) = 1$ [12] and the function $B(\alpha)$ is defined *ad hoc* as, $B(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$ [13]. In these definitions E_α is one-parameter Mittag-Leffler function [4]

$$E_\alpha = \sum_0^\infty \frac{z^k}{\Gamma(\alpha k+1)} \tag{44}$$

4.1.1 Laplace transforms and relationships

The Laplace transform of ABR derivative is [3]

$$\mathscr{L}\left\{{}_{0}^{ABR}D_{t}^{\alpha}\left[f\left(t\right)\right]\right\}\left(p\right)=\frac{B\left(\alpha\right)}{1-\alpha}\frac{p^{\alpha}}{p^{\alpha}+\frac{\alpha}{1-\alpha}}L\left\{f\left(t\right)\right\}\left(p\right) \tag{45}$$

Similarly for the ABC derivative

$$\mathscr{L}\left\{{}_{0}^{ABC}D_{t}^{\alpha}\left[f\left(t\right)\right]\right\}\left(p\right)=\frac{B\left(\alpha\right)}{1-\alpha}\frac{p^{\alpha}}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\left[\mathscr{L}p\left\{f\left(t\right)\right\}\left(p\right)-p^{\alpha-1}f\left(0\right)\right] \tag{46}$$

Consequently, the relation between ABR and ABC is [3])

$${}_{0}^{ABC}D_{t}^{\alpha}\left[f\left(t\right)\right]={}_{0}^{ABR}D_{t}^{\alpha}\left[f\left(t\right)\right]-\frac{B\left(\alpha\right)}{1-\alpha}f\left(0\right)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right) \tag{47}$$

Hence, with zero initial conditions, both derivatives are identical, a property already known from the classical Riemann-Liouville and Caputo-Liouville derivatives.

4.2 Associated integrals to ABC and ABR

Now, we address the associated ABR integral and emerging problems in its applications.

4.2.1 ABR derivative and the related associated fractional Integral

As was demonstrated by Atangana and Baleanu [3] the following fractional differential equation

$${}_{0}^{ABR}D_{t}^{\alpha}\left[f\left(t\right)\right]=u\left(t\right) \tag{48}$$

has a unique solution

$$f\left(t\right)=\frac{1-\alpha}{B\left(\alpha\right)}u\left(t\right)+\frac{\alpha}{B\left(\alpha\right)}\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}u\left(\tau\right)\left(t-\tau\right)^{\alpha-1}d\tau \tag{49}$$

where for $\alpha=0$ we recover the initial function, while for $\alpha=1$ we get the ordinary Riemann integral.

The ABR fractional derivative can be expressed as [12]

$$\begin{aligned}{}^{ABR}D_{a+}^{\alpha}f\left(t\right)&=\frac{B\left(\alpha\right)}{1-\alpha}\frac{d}{dt}\int_{a}^{t}f\left(z\right)\sum_{k=0}^{\infty}\frac{\left(-\alpha\right)^{k}}{\left(1-\alpha\right)^{k}}\frac{\left(t-z\right)^{\alpha k}}{\Gamma\left(\alpha k+1\right)}dz=\\&=\frac{B\left(\alpha\right)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{k}\frac{d}{dt}\left[{}^{RL}I_{a+}^{\alpha k+1}f\left(t\right)\right]\end{aligned} \tag{50}$$

where ${}^{RL}I_{a+}^{\alpha k+1}f\left(t\right)$ is the Riemann-Liouville fractional integral [4]

The ABR associated fractional integral operator ${}^{AB}I_{a+}^{\alpha}f\left(t\right)$ follows directly from the solution of (49) and can be precisely defined as [12]

$${}^{AB}I_{a+}^{\alpha}f\left(t\right)=\frac{1-\alpha}{B\left(\alpha\right)}f\left(t\right)+\frac{\alpha}{B\left(\alpha\right)}{}^{RL}I_{a+}^{\alpha}f\left(t\right) \tag{51}$$

The relation (51) can be easily developed by applying the Laplace transform to equation (48) as it was demonstrated by Baleanu and Fernandez [12]. Further, we have the following left and right inverse properties [12]

$${}^{AB}I_{a+}^{\alpha}\left[{}^{ABR}D_{a+}^{\alpha}f\left(t\right)\right]=f\left(t\right) \tag{52}$$

$$^{ABR}D^{\alpha}_{a+}f(t)\left[^{AB}I^{\alpha}_{a+}f(t)\right]=f(t) \tag{53}$$

and the commutative properties for $\beta \in (0,1)$

$$^{ABR}D^{\alpha}_{a+}f(t)\left[^{ABR}D^{\beta}_{a+}f(t)\right]={}^{ABR}D^{\beta}_{a+}\left[^{ABR}D^{\alpha}_{a+}f(t)\right] \tag{54}$$

$$^{AB}I^{\alpha}_{a+}\left[^{AB}I^{\beta}_{a+}f(t)\right]={}^{AB}I^{\beta}_{a+}\left[^{AB}I^{\alpha}_{a+}f(t)\right] \tag{55}$$

$$^{ABR}D^{\alpha}_{a+}\left[^{AB}I^{\alpha}_{a+}f(t)\right]={}^{AB}I^{\alpha}_{a+}\left[^{ABR}D^{\alpha}_{a+}f(t)\right] \tag{56}$$

4.2.2 Some generalization of the AB associate integrals

Now, we try to present the associated integrals of the AB derivatives in a form allowing more general formulations, especially to the concept of the fading memory (see Section 8.2.1) . For the sake of simplicity, assuming hereafter $B(\alpha)=1$, we get from (51) [14]

$$^{AB}I^{\alpha}_{a+}u(t)=(1-\alpha)u(x,t)+\alpha^{RL}I^{\alpha}_{a+}u(x,t)=f(x,t) \tag{57}$$

or equivalently

$$^{AB}I^{\alpha}_{a+}u(t)=m(\alpha)u(x,t)+\lambda(\alpha)\,^{RL}I^{\alpha}_{a+}u(x,t) \tag{58}$$

The constructions of (57) and (58) are the same as that of the Boltzmann linear superposition functional [15] expressing the fading memory concept (see Section 8.2.1) with a time-dependent memory (influence) function) $R(t,z)$, namely

$$\varphi(x,t)=m\left[v_x(x,t)\right]+\lambda\int_0^t R(t,z)\,v_x(z)\,dz \tag{59}$$

The memory integral,the 2nd terms in (57) or (58), is the standard Riemann-Liouville fractional integral $^{RL}I^{\alpha}_{a+}u(x,t)$. In (59) $v_x(z)=\nabla v(z)$. The coefficients m and λ are weighting functions (transport coefficients) which depend on the character of the modeled diffusion process (heat or mass). The basic idea of the fading memory concept is explained in Section 8.2.1.

4.3 Some generalization of the AB associate integrals to the diffusion models

4.3.1 Conjectures pertinent to the diffusion flux

Now, let us turn to the basic formulation of the rate equation (48) expressed through either ABR or ABC derivative [3, 12] allowing the time-fractional derivative to be related to the flux expressed as a function of the gradient $\partial f/\partial x$. The following model development is based on two constitutive conjectures [14]

Conjecture 1: Assume that in (48) $u(x,t)=(-\partial f/\partial x)$, without loss of the generality of this equation. Consequently, the AB fractional integral operator $^{AB}I^{\alpha}_{a+}u(t)$(see (57) as a construction) can be expressed as

$$^{AB}I^{\alpha}_{a+}\left[u(x,t)\right]={}^{AB}I^{\alpha}_{a+}\left[-\frac{\partial f(x,t)}{\partial x}\right]=-\left\{m(\alpha)\frac{\partial f(x,t)}{\partial x}+\lambda(\alpha)\,^{RL}I^{\alpha}_{a+}\left[\frac{\partial f(x,t)}{\partial x}\right]\right\} \tag{60}$$

Conjecture 2: The flux relaxation follows the fading memory concept expressed by (59), that is

$$j_{\alpha}(x,t)={}^{AB}I^{\alpha}_{a+}\left[-D_0\frac{\partial f(x,t)}{\partial x}\right]=-D_0\left\{m(\alpha)\frac{\partial f(x,t)}{\partial x}+\lambda(\alpha)\,^{RL}I^{\alpha}_{a+}\left[\frac{\partial f(x,t)}{\partial x}\right]\right\} \tag{61}$$

In the formulation (61) D_0 is the transport coefficient (the diffusivity) for $\alpha=1$ (instantaneous flux with an infinite speed).

Now, the weighting functions $m(\alpha)=1-\alpha$ and $\lambda(\alpha)=\alpha$ depend on the degree of fractionality of the modeled diffusion process. For $\alpha=1$ we get $m(\alpha)=0$ and $\lambda(\alpha)=1$. The expressions (60) and (61) coincide with the Coleman-Noll definitions [16,17,18,19] (see Section 8.2.1) and the others related to it) about the flux of simple materials (where the flux is proportional to the gradient $j(x,t)\equiv-\partial f/\partial x$ [20,21,22], following the Boltzmann superposition principle (see section 8.2.1).

4.4 Diffusion model based on the ABR associated integral [14]

Hence, we may rewrite (48), without loss of generality, as

$$\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = -\frac{d}{dx} j_{\alpha}(x,t) \tag{62}$$

Differentiation concerning x in (48) yields

$$\begin{aligned} -\frac{d}{dx} j_{\alpha}(x,t) &= {}^{AB}I^{\alpha}_{a+}\left[D_0 \frac{\partial^2 f(x,t)}{\partial x^2}\right] = \\ &= D_0 \left\{ m(\alpha) \frac{\partial^2 f(x,t)}{\partial x^2} + \lambda(\alpha)\, {}^{RL}I^{\alpha}_{a+}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] \right\} \end{aligned} \tag{63}$$

Therefore, using the above conjecture the diffusion equation takes the form

$$\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = D_0 \left[{}^{AB}I^{\alpha}_{a+} \frac{\partial^2 f(x,t)}{\partial x^2} \right] \tag{64}$$

or

$${}^{ABR}D^{\alpha}_{a+} f(x,t) = D_0 \left[{}^{AB}I^{\alpha}_{a+} \frac{\partial^2 f(x,t)}{\partial x^2} \right] \tag{65}$$

Setting the lower terminal the ABR derivative and the Riemann-Liouville integral as $a = 0$, we get

$${}^{ABR}_{0}D^{\alpha}_{t} f(x,t) = D_0 \left\{ m(\alpha) \frac{\partial^2 f(x,t)}{\partial x^2} + \lambda(\alpha)\, {}^{RL}_{0}I^{\alpha}_{t}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] \right\} \tag{66}$$

As commented above, for $\alpha = 1$ we have $m(\alpha) = 0$ and $\lambda(\alpha) = 1$, and the memory integral in (ABM-3) becomes

$$\left[{}^{AB}_{0}I^{\alpha}_{t} \left(\frac{\partial^2 u(x,\tau)}{\partial x^2} \right) \right]_{\alpha=1} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{67}$$

Hence, for $\alpha = 1$ from (62) we recover the Fourier (Fick) law $j_{\alpha=1} = -D_0(\partial f/\partial x)$ and equation (66) reduces to the classical diffusion equation. Alternatively, for the sake of clarity of this statement, we may express (66) (taking into account that $m(\alpha) = 1 - \alpha$ and $\lambda(\alpha) = \alpha$) as

$${}^{ABR}_{0}D^{\alpha}_{t} f(x,t) = D_0 \left\{ \frac{\partial^2 f(x,t)}{\partial x^2} + \alpha \left[{}^{RL}_{0}I^{\alpha}_{t} \left(\frac{\partial^2 f(x,t)}{\partial x^2} \right) - \frac{\partial^2 f(x,t)}{\partial x^2} \right] \right\} \tag{68}$$

For $\alpha \to 1$ the second term in the right-hand side (in the squared brackets) tends to zero and we recover the classical diffusion equation.

4.5 Some pitfalls emerging in a formal application of the associate integral

Now, we turn to a problem emerging from the formal fractionalization of the diffusion model and the application of the associated integral [14].

4.5.1 Formal Diffusion equation in terms of ABR derivative

Let us consider (48) which to some extent could be considered as a kinetic equation, as well as, without loss of generality assume that $u(t) = \frac{\partial f(x,t)}{\partial x}$. Then, the diffusion equation could be constituted (postulated) as

$${}^{ABR}_{0}D^{\alpha}_{t} f(x,t) = D_0 \frac{\partial^2 f(x,t)}{\partial x^2} \tag{69}$$

For $\alpha = 1$ equation (69) formally recovers the classical diffusion equation, but when $0 < \alpha < 1$ the corresponding integral is (see eq.(49))

$$f(t) = \frac{1-\alpha}{B(\alpha)}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] + \frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t \left[\frac{\partial^2 f(x,t)}{\partial x^2}\right](t-\tau)^{\alpha-1}d\tau \tag{70}$$

Then the flux and its space derivative can be defined as two unphysical relationships with high-order space derivatives

$$j_{FF}(t) \equiv -\frac{\partial f}{\partial x} \equiv D_0\left\{\frac{1-\alpha}{B(\alpha)}\left[\frac{\partial^3 f(x,t)}{\partial x^3}\right] + \frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t \left[\frac{\partial^3 f(x,t)}{\partial x^3}\right](t-\tau)^{\alpha-1}d\tau\right\} \tag{71}$$

$$\frac{d}{dx}j_{FF}(t) \equiv -\frac{\partial^2 f}{\partial x^2} \equiv \left\{\frac{1-\alpha}{B(\alpha)}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right] + \frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t \left[\frac{\partial^4 f(x,t)}{\partial x^4}\right](t-\tau)^{\alpha-1}d\tau\right\} \tag{72}$$

It is hard to explain why from (72), that is for $\alpha \to 1$, we get

$${}_{0}^{ABR}D_t^{\alpha} f(x,t) = D_0 \frac{\partial^4 f(x,t)}{\partial x^4} \tag{73}$$

with $\frac{\partial^4 f(x,t)}{\partial x^4}$ in the right-hand side, since no physical meaning could be found to support this;in contrast, for $\alpha \to 1$, the equation (63) simply leads to $\frac{d}{dx}j_a(t) = \frac{\partial^2 f}{\partial x^2}$ following the classical diffusion concept.

4.5.2 Formal application of the associated ABR integrals

In addition, applying formally the right inverse property (53) to the diffusion equation (65) we get [14]

$${}^{ABR}D_{a+}^{\alpha}\left[{}^{ABR}D_{a+}^{\alpha} f(x,t)\right] = {}^{ABR}D_{a+}^{\alpha}\left[D_0 {}^{AB}I_{a+}^{\alpha}\frac{\partial^2 f(x,t)}{\partial x^2}\right] \tag{74}$$

This operation leads to

$${}^{ABR}D_{a+}^{\alpha}\left[{}^{ABR}D_{a+}^{\alpha} f(x,t)\right] = D_0 \frac{\partial^2 f(x,t)}{\partial x^2} \tag{75}$$

This result mimics the classical subdiffusion equation with a power-law memory kernel, but we have to remember that with the ABR derivative the index law does not work (see (54) and we have

$${}^{ABR}D_{a+}^{\alpha}\left[{}^{ABR}D_{a+}^{\alpha} f(x,t)\right] \neq {}^{ABR}D_{a+}^{\alpha+\alpha} f(x,t) \tag{76}$$

Hence, the formalistic construction of the diffusion equation in terms of AB derivatives (69) leads to unphysical results. For $\alpha = 1$ the left-hand side of (75) becomes [14]

$${}^{ABR}D_{a+}^{\alpha}\left[{}^{ABR}D_{a+}^{\alpha} f(x,t)\right]_{\alpha\to 1} \to D^{(2)} f(x,t) \tag{77}$$

that differs from (69) which naturally reduces to the classical diffusion equation.

5 Briefs outlining the problem analyzed

Despite the different approaches employed, all of these studies on the associate integrals of the Caputo-Fabrizio derivative show certain resolved issues with converging results.

The associate integrals are relevant to solutions of equations including the Caputo-Fabrizio derivative, but their origin and applications in the model-build-ups are not obvious, practically making them inapplicable, albeit in some cases they could facilitate the solutions.

The previous points of these briefs appeal to a correct definition of the Caputo-Fabrizio derivative construction as a step of the model formulations. This envisages definitions of constitutive integrals which differ from the associate integrals (this problem is resolved in Section 7).

6 New problem to be resolved

After this initial analysis, and the briefs in Section 5 we may formulate the main problems that resolved in this article and outlined in the following Aim-2

6.1 Aim-2

1. Analysis of the original constitutive integrals with non-singular kernels and their physical relevance.

2. Definitions of associate and constitutive integrals thus distinguishing them and showing their levels of applications in mathematical modeling.

3. Applications of constitutive integrals to modeling of diffusion problems by employments of the Fading Memory Concepts and the Volterra equations approach.

4 Analysis of emerging in the literature and conceived functional relationships of the normalization functions $M(\alpha)$ and $B(\alpha)$.

5 Application of thermodynamic analysis of the fractional operators towards proper determination of the normalization functions $M(\alpha)$ and $B(\alpha)$.

6.2 Further text organization following Aim 2

The first step addresses definitions of Associated and Constitutive integrals and the main differences between them (Section 7). Alternative formulations of the fractional Caputo-Fabrizio Constitutive Integral are presented in Section 8 starting from the original formulations of the Caputo and Fabrzio [1,2] (Section 8.1) and new approaches (Sections 8.1.1 and 8.1.2). Formulation of the Caputo-Fabrizio Constitutive Integral based on the Fading Memory Approach is presented in Section 8.2.3: Formulation of the Fading Memory formalism (Section 8.2.1), the main approach to model build-up (Section 8.2.2) with examples in Sections 8.2.3 and 8.2.4.The Caputo-Fabrizio Constitutive Integral formulated by the Volterra Equation Approach is demonstrated in Section 8.3. The Atangana-Baleanu constitutive integrals are discussed in Section 9: By the Fading Memory approach 9.1 and Volterra equations 9.2. The analysis of the normalization function $M(\alpha)$ and $B(\alpha)$, their existing and *ad hoc* formulations are addressed in Section 10.

7 Definitions of related integrals

7.1 Associated fractional integral (AI)

The associated integral of a certain fractional derivative can be obtained through the solution of eq. (7) . In the case of the Caputo-Fabrizio operator we have historically (see the preceding section) two definitions: The first Losada-Nieto's definition (10) and their second formulation (19). Thus, we make the following definition

Definition 1(Associate Integral). *The associate integral I_t^{α} is the solution of the basic fractional equation*

$$D_t^{\alpha} f(t) = u(t), \quad t > 0 \tag{78}$$

irrespective of the type of memory kernel used.

7.2 Constitutive fractional Integral (CI)

Definition 2(Constitutive Integral). *The constitutive integral has the same structure as the convolution integral of the related fractional operator (derivative). In the case of the Caputo-Fabrizio operator, we have two formulations of constitutive integrals: see (31) in section 3.1 and (88) in section 8.1.*

Thus, if the fractional derivative is defined generally as (of Caputo type for example)

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{N(\alpha)} \int_0^t R(t-\tau) \frac{df(\tau)}{d\tau} d\tau \tag{79}$$

Then the constitutive integral is defined as

$$^{C}\Im_t^{\alpha} f(t) = \frac{M_I(\alpha)}{N_I(\alpha)} \int_0^t R(t-\tau) \frac{df(\tau)}{d\tau} d\tau \tag{80}$$

or as a Riemann-Liouville integral as

$$^{RL}\Im_t^{\alpha} f(t) = \frac{M_I(\alpha)}{N_I(\alpha)} \int_0^t R(t-\tau) f(\tau) d\tau \tag{81}$$

*Remark.*The **constitutive integral**, sometimes mentioned here as the **Basic integral**, has to be formulated at the level of the constitutive equations which model physical law-see the results analyzed in section 3.1. This approach will be developed in the sequel by applying the fading memory formalism and the Volterra equations.

7.3 The case where the associated and constitutive fractional integrals coincide

The principal question arising naturally from the two different definitions is: Are there situations where the associated and the constitutive integrals coincide? To answer this principal question let us consider the classical case of the fractional diffusion equation with a singular memory kernel, starting from the definition of the Riemann-Liouville integral [4]

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy \tag{82}$$

Then, the non-local (in time) constitutive equation relating the flux to the gradient is

$$j = -D_{\alpha} D^{1-\alpha} \left[\frac{\partial}{\partial x} C(x,t) \right] = -D_{\alpha} I_t^{\alpha} \left[\frac{\partial}{\partial x} C(x,t) \right] \tag{83}$$

The form of (83) is very informative because it reveals that the flux (current) can be expressed dependent on a gradient of a certain potential function. Moreover, at each point the flux depends on the previous history, thus this formulation is exactly corresponding to the causality principle.

The potential form of (83) allows some useful properties to be outlined, among them:

i) The flux is proportional to the potential of $\frac{\partial}{\partial x} C(x,t)$ thus indicating that for heavy tail distributions with (long waiting times) the initial condition contributes the flux time evolution at all time even though it decays in time;

ii) This term is a convolution operator corresponding to the construction of the left Caputo derivative of order $1-\alpha$ or of the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$.

Taking into account the relationship

$$^{C}D_x^{\alpha} f(x) = I_x^{1-\alpha} \left[\frac{df(x)}{dx} \right] \tag{84}$$

because $^{C}D_t^{-\gamma} f = {}^{C}I^{\gamma} f$ and with $\gamma = 1-\alpha$ we get (84); moreover since $^{C}D_t^{\gamma}$ is left inverse of the fractional integral we have

$$^{C}D_t^{\gamma} I^{\gamma} = {}^{C}D_t^{\gamma C} D_t^{-\gamma} f = D_t^0 f = f \tag{85}$$

Further, if we assume that the continuity equation is valid, then with zero initial condition (i.e. $C(x,0) = 0$), for the sake of simplicity, we get directly that

$$\frac{\partial C}{\partial t} = D_{\alpha} I_t^{1-\alpha} \left[\frac{\partial^2 C(x,\tau)}{\partial x^2} \right] = D_{\alpha} \left\{ {}^{C}D_t^{\alpha-1} \left[\frac{\partial^2 C(x,\tau)}{\partial x^2} \right] \right\} \tag{86}$$

Applying to both sides of (86) the Caputo operator of order $1-\alpha$ we get the time-fractional diffusion equation

$$\frac{\partial^{\alpha} C(x,t)}{\partial t^{\alpha}} = D_{\alpha} \frac{\partial^2 C(x,t)}{\partial x^2} \tag{87}$$

Recall, that (83) recovers the Fourier (Fick's) law for $\alpha = 1$, and then (87) reduces to the Fourier (Fick's) equation.

*Remark.*Therefore, as a principal outcome of this example, we see that the constitutive integrals in (83) coincides with the associated integral (82) only in the case when there are semi-group properties of the fractional operators (85). If this is not the case, then the associated and the constitutive integral are different, following the definitions formulated above. At this point is worth noting to stress the attention that formulation of the constitutive integral is a fundamental step in fractional modeling of non-local physical laws, irrespective of the function used as a memory kernel and the existence of semi-group properties The associated integral, as defined above, is an outcome of the artificially defined "kinetic equation" (7) and the coincidence just explained is an exception related to the existence of semi-group properties. The constitutive integrals are fundamental (the starting points) in fractional modeling.

8 Fractional Caputo-Fabrizio Constitutive Integral:Alternative approaches

Now, we can see that using the basic construction of ${}^{CF}\mathfrak{I}_t^{\alpha} f(t)$ (13) it is possible to construct different versions of non-local flux-gradient constitutive equations and relevant heat conduction (diffusion) models.

8.1 Fractional Caputo-Fabrizio Constitutive Integral: Alternative integral formulations

Let us consider the following construction of an alternative constitutive integral

$$ {}^{CFA}\mathfrak{I}_t^{\alpha} f(t) = b_0 \frac{\alpha}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\, d\tau \tag{88} $$

and suggest that a certain physical quantity $A(x,t)$ can be related to the following constitutive equation

$$ A(x,t) = -b_0 {}^{CFA}\mathfrak{I}_t^{\alpha} f(t) = -b_0 \frac{\alpha}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\, d\tau \tag{89} $$

Formally, this formulation resembles (33), but now the rate constant in the exponential memory and the prefactors are defined as $\frac{\alpha}{1-\alpha}$. Now we will demonstrate two approaches resulting in physical sound equations.

8.1.1 Approach 1

Let us integrate by parts the integral in (89)

$$ \begin{aligned} & -b_0 \frac{\alpha}{1-\alpha} \int_{-\infty}^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\, d\tau = \\ & = -b_0 \frac{\alpha}{1-\alpha} \left[\frac{1-\alpha}{\alpha} \left(f(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \right) \Big|_{\tau=0}^{\tau=t} + \frac{1-\alpha}{\alpha} \int_{-\infty}^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \frac{df(\tau)}{d\tau} d\tau \right] \end{aligned} \tag{90} $$

and we get

$$ -b_0 \frac{\alpha}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\, d\tau = \left[f(t) - f(0) e^{-\frac{\alpha}{1-\alpha} t} \right] - b_0 \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \frac{df(\tau)}{d\tau} d\tau \tag{91} $$

Assuming for the sake of simplicity of the exposition that $f(0) = 0$ we may re-arrange (91) as

$$ \begin{aligned} A(x,t) &= -b_0 f(t) - b_0 (1-\alpha) \left[\frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \frac{df(\tau)}{d\tau} d\tau \right] = \\ &= -b_0 f(t) - b_0 (1-\alpha) D_t^{\alpha} f(\tau) \end{aligned} \tag{92} $$

For $\alpha = 1$ we have $A(x,t) = -b_0 f(t)$.

If now, we assume that $f(x,t)$ represents temperature (or concentration) gradient while $A(x,t)$ denoting the flux of heat (or mass), we get

$$\begin{aligned} q(x,t) &= -b_0 \frac{\partial \theta(x,\tau)}{\partial x} - b_0(1-\alpha)\left[\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \frac{d}{d\tau}\left(\frac{\partial \theta(x,\tau)}{\partial x}\right) d\tau\right] = \\ &= -b_0 \frac{\partial \theta(x,\tau)}{\partial x} - b_0(1-\alpha)^2 {}^{CF}D_t^{\alpha}\left[\frac{\partial \theta(x,\tau)}{\partial x}\right] \end{aligned} \tag{93}$$

As a consequence, after the application of the energy (mass) conservation equation, and with $b_0 = k_0$, we get

$$\frac{\partial \theta(x,\tau)}{\partial t} = a_0 \frac{\partial \theta(x,\tau)}{\partial x} + a_0(1-\alpha)^{CF} D_t^{\alpha}\left[\frac{\partial \theta(x,\tau)}{\partial x}\right], \quad a_0 = \frac{k_0}{\rho C_p} \tag{94}$$

For $\alpha = 1$ we get the Fourier (Fick) heat (mass) diffusion equation.

8.1.2 Approach 2

Alternatively, we may differentiate directly (89) concerning the time

$$\frac{d}{dt}A(x,t) = -b_0\alpha \frac{d}{dt}\left[{}^{CFA}I_t^{\alpha} f(t)\right] = -b_0\alpha \frac{d}{dt}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\,d\tau \tag{95}$$

and the result is

$$\frac{d}{dt}A(x,t) = -b_0\alpha \frac{1}{1-\alpha}\frac{d}{dt}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)\,d\tau = -b_0\alpha^{CFR} D_t^{\alpha} f(t) \tag{96}$$

Further, with the relationships between ${}^{CF}D_t^{\alpha} f(t)$ and ${}^{CFR}D_t^{\alpha} f(t)$ (89) we get

$$\frac{d}{dt}A(x,t) = b_0\alpha\left[{}^{CF}D_t^{\alpha} f(t) + f(0)\,\varphi_{\alpha}(t)\right] \tag{97}$$

or alternatively from (95) [11]

$$\begin{aligned} {}^{CFR}D_t^{\alpha} f(t) &= \frac{1}{1-\alpha} f(t) - \frac{\alpha}{1-\alpha}{}^{CF}\mathfrak{I}_t^{\alpha} f(t) \Rightarrow \\ &\Rightarrow -b_0 \frac{\alpha}{1-\alpha} f(t) + b_0 \frac{\alpha^2}{1-\alpha}{}^{CF}I_t^{\alpha} \end{aligned} \tag{98}$$

Then, in terms of $A(x,t)$ we have

$$\frac{d}{dt}A(x,t) = -b_0\alpha\left[\frac{1}{1-\alpha} f(t) - \frac{\alpha}{1-\alpha}{}^{CF}\mathfrak{I}_t^{\alpha} f(t)\right] = -b_0 \frac{\alpha}{1-\alpha} f(t) + \frac{\alpha}{1-\alpha} A(x,t) \tag{99}$$

Rearranging (99) and assuming that $A(x,t) = q(x,t)$, and $f(t) = -\frac{\partial \theta(x,t)}{\partial x}$ the result is

$$\frac{1-\alpha}{\alpha}\frac{dq(x,t)}{dt} = b_0 \frac{\partial \theta(x,t)}{\partial x} + q(x,t) \tag{100}$$

For $\alpha = 1$ we get the Fourier (Fick) law.

*Remark.*The structures of fractional integrals (31) and (88) allow easy to construct adequate constitutive equations with memory. To a greater extent, the preceding examples demonstrate an artistic approach when intuition and imagination are driving forces. However, there are alternative approaches to the definitions of both the Caputo-Fabrizio fractional integral and derivative [6] starting from well-known and thermodynamically consistent constitutive equations about the heat flux, taken as a basic example, and some of them are demonstrated next.

8.2 Fractional Caputo-Fabrizio Constitutive Integral: Fading Memory Approach

8.2.1 Fading memory concept

The fading memory concept relating the flux to its gradient, for simple materials [23,17,22] is modeled by the following integro-differential equation, with a time-dependent memory kernel (correlation function) $R(t,\tau)$,

$$j(x,t) = -A_1 \frac{\partial C(x,t)}{\partial x} - A_2 \int_{-\infty}^{t} R(t-\tau) \frac{\partial C(x,\tau)}{\partial x} d\tau \tag{101}$$

as a manifestation of the Boltzmann linear superposition functional [15] . In (101) the transport coefficients A_1 and A_2 are diffusivities.

In (101) the lower terminal of the convolution integral is set to zero since two principal conditions should be accounted for: 1) The memory kernel is a casual function, that is $R(t<0)=0$ and the *time of the process modeled* is not *the chronological time* (*instant time*),as mentioned by Hilfer [24], but the *time from the process onset till its end*, i.e. the *intrinsic time*.

From (101) it follows that

$$\frac{\partial}{\partial x} j(x,t) = -A_1 \frac{\partial^2 C(x,t)}{\partial x^2} - A_2 \int_{0}^{t} R(t-\tau) \frac{\partial^2 C(x,\tau)}{\partial x^2} d\tau + \frac{\partial}{\partial x} d(t) \tag{102}$$

The deep thermodynamic sense of the fading memory formulation is that the non-locality represented by the convolution term works for short times, while for long times we get the first term in (102), i.e. the instant reaction of the system. Moreover, models constructed with the fading memory principle obey: the *causality principle* (through the convolution term) [25,26,27], thermodynamic consistency [19,28], and *model observability* (objectivity) [29,30,18].

Further, if $A_1 = 0$, that is, the process modeled has no long-time asymptotically stable state, the result is a model close to the Continuum Time Random Walk (CTRW) without a stationary state [31,32], and applying the continuity equation to such a flux, we get

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} j(x,t) \Rightarrow \frac{\partial C(x,t)}{\partial t} = A_2 \int_{0}^{t} R(t-\tau) \frac{\partial C(x,\tau)}{\partial x} d\tau \tag{103}$$

To a greater extent the model (103) mimics the master equation [32,33] ; this is only a remark and we will not elaborate on it further in this study but refer readers to [34,35,36] where this issue was developed in detail regarding Prabhakar-type memory.

*Remark.*Fading memory concept in the case of heat conduction is well applicable to the so-called *simple materials* [22, 20] where *the flux is proportional to the temperature gradient* (the term was coined after the work of Storm [20]), as in all examples studied here .

8.2.2 Model build-up concept based on fading memory

Following the flux definition of (101) where the relaxation (memory) function can be presented as $M(t) = \delta(t) + R(t)$ such that (101) expressed as a convolution product, namely

$$j(x,t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * M(t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * [A_1 \delta(t) + A_2 R(t)] \tag{104}$$

The two main requirements for $R(t)$ are: it *should be completely monotone*, regardless of whether it is singular or non-singular at $t0+$, and that *its first derivative*$dR(t)/dt = R_t(t)$ should be also a completely monotone function.

Then, the main step in the construction of the flux constitutive model is the use $R_t(t)$ instead $R(t)$, that is

$$j(x,t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * M(t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * [A_1 \delta(t) + A_2 R_t(t)] \tag{105}$$

Next, the integration by parts of the convolution integral yields.

$$\int_0^t R_t(t-\tau)\frac{\partial C(x,\tau)}{\partial x}d\tau = -\int_0^t R(t-\tau)\frac{d}{d\tau}\left[\frac{\partial C(x,\tau)}{\partial x}\right]d\tau \tag{106}$$

Denoting $\frac{\partial}{\partial x}C(x,t) = f(x,t)$, we get a constriction of Caputo-type fractional derivative that is

$${}^{C}D_t^{\alpha} f(x,t) = \frac{M(\alpha)}{N(\alpha)}\int_0^t R(t-\tau)\frac{df(x,\tau)}{d\tau}d\tau \tag{107}$$

The prefactor $M(\alpha)/N(\alpha)$ takes only into account the fractional order, and, in general, it is determined by the type of memory kernel (see the analysis in [6] and [37], and the developments in the sequel of this article).

The flux can therefore be expressed as

$$j(x,t) = -A_1\frac{\partial C(x,t)}{\partial x} - A_2{}^{C}D_t^{\alpha}\left[\frac{\partial C(x,t)}{\partial x}\right] \tag{108}$$

In this case, the hereditary term is generically expressed as a fractional derivative of the Caputo type without any kernel-specific information. Then, applying the continuity equation $\frac{\partial C}{\partial t} = -\frac{\partial q}{\partial x}$ (The First Law of Thermodynamics) we get

$$\frac{\partial C(x,t)}{\partial t} = A_1\frac{\partial C(x,t)}{\partial x} + A_2{}^{C}D_t^{\alpha}\left[\frac{\partial C(x,t)}{\partial x}\right] \tag{109}$$

For $\alpha = 1$ this model reduces to the local diffusion equation.

*Remark.*To this point and recalling the results developed in section 7.3, we refer to the model considered by Fujita [38] (in the original notations)

$$u(x,t) = \phi(x) + \int_0^t h(t-s)\Delta u(x,s)\,ds, \quad \Delta = \partial^2/\partial x^2, \quad t>0, \quad x\in\Re \tag{110}$$

This is not his original construction but taken from [21] and [17]

Fujita accepted the Riesz distribution $R_n(t) = \frac{t^n}{\Gamma(n+1)}$ [39], with $n = \alpha - 1$ (see more details about this distribution related to fractional polynomial operators in [40] and [41])

$$h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 1\le\alpha\le 2 \tag{111}$$

thus transforming (110) into

$$u(x,t) = \phi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Delta u(x,s)\,ds \tag{112}$$

that in terms of fractional operators, precisely as a Riemann-Liouville integral, becomes

$$u(x,t) = \phi(x) + I^{\alpha}\Delta u(x,s) \tag{113}$$

In the work of Fujita, as a step of the solution, the term $\phi(x)$ was abandoned, i.e. $\phi(x) = 0$, and this led to

$$u(x,t) = I^{\alpha}\Delta u(x,s) \tag{114}$$

Applying to both sides of (114) the operator D^{α} (see for the similar operation in Section 7.3 concerning (86)) we get (in the original Fujita's notations)

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \Delta u(x,t) \tag{115}$$

We especially commented on this result of Fujita, since the formulation (110) is the fading memory concept. The acceptance $\phi(x) = 0$ and the kernel (111) refer to the Continuous Random Time Walk (CTRW) concept (not interpreted in this way in the source [38]). To complete, this is an example, where a hereditary model is constructed with a constitutive integral (the Riemann-Liouville integral) which only in the case when the kernel is defined by (111) coincides with the associated to D^{α} integral.

*Remark.*Specific comments should be made about the next exposure before proceeding with the modeling method. Assuming that the kernel of the simplest scenario is constructed first, model compilations are inductively demonstrated. Model constructs are shown deductively, suggesting that the nucleus of the simplest scenario is built first. When the same approach is used for handling increasingly complex kernels, the resulting fractional operator structures gradually become more complex. The approach demonstrated here is more instructive than the one frequently used in the literature, which starts with the most complex kernel and gradually creates simpler kernels by lowering the complexity parameters.

8.2.3 Example 1:Heat flux with exponential (time domain) memory kernel

Consider a virgin semi-infinite material (medium) with initial and boundary conditions (signaling problem) that will be valid for all examples developed in the sequel.

$$T(x,0) = T(\infty,t) = T(-\infty,t) = 0, \quad T_x(x,0) = T_{xx}(x,0) = 0, \quad T(0,t) = T_s \tag{116}$$

Hence, we consider the kernel (following the results of [42,6]) as

$$e^1_{1,1}(-\lambda t) = \exp(-\lambda t), \quad \frac{d}{dt}\left[e^1_{1,1}(-\lambda t)\right] = -\lambda \exp(-\lambda t) \tag{117}$$

which is the exponential memory (Jeffrey's kernel) $R(t)$ [43] having a finite relaxation time τ (and $\lambda = 1/\tau$)

$$R(t-s) = \exp\left(-\frac{t-s}{\tau}\right), \quad R_t(t) = -\frac{1}{\tau}\exp\left(-\frac{(t-s)}{\tau}\right) \tag{118}$$

Then, we may can construct the relaxation kernel following (105) as

$$M(t) = k_1\delta(t) + k_2 R_t(t) \Rightarrow M(t-s) = k_1\delta(s) + k_2\frac{1}{\tau}\exp\left(-\frac{(t-s)}{\tau}\right) \tag{119}$$

Therefore, the memory term in the relaxation kernel is constructed as the constitutive (basic) integral, namely

$$M_{relaxing}(t) = k_2\frac{1}{\tau}\int_0^t exp\left(-\frac{t-s}{\tau}\right) f(s)\,ds, \quad f(s) = \frac{\partial T(x,s)}{\partial x} \tag{120}$$

Further, applying the technology of integration of the convolution integral, and the definition of the Caputo-Fabrizio derivative [1], in terms of $T(x,t)$, we get [42,6]

$${}^{CF}D_t^{\alpha} T(x,t) = \frac{N(\sigma)}{\sigma}\int_0^t e^{\frac{-\alpha}{1-\alpha}(t-s)}\frac{dT(x,s)}{dt}ds = \frac{M(\alpha)}{1-\alpha}\int_0^t e^{\frac{-\alpha}{1-\alpha}(t-s)}\frac{dT(x,s)}{dt} \tag{121}$$

In the terms used here $\sigma = 1/\lambda = (1-\alpha)/\alpha \in [0,\infty]$, while $N(\alpha)$ and $M(\alpha)$ are normalization functions [1] (see (107)).

Now, applying the energy balance equation and taking into account the last expression of (121), we get [42,6]

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\,{}^{CF}D_t^{\alpha}\frac{\partial^2 T(x,t)}{\partial x^2}, \quad t > 0 \tag{122}$$

Equation (122) models transient heat conduction with a damping term expressed through the Caputo-Fabrizio fractional derivative. For $\alpha = 1$ we get the Fourier equation.

As a result, we could see that the Caputo-Fabrizio derivative, as a damping term, was generated directly by application of the fading memory approach.

8.2.4 Example 2 : Heat flux with exponential spatial memory

In the context, of the general exposition of this chapter and the possibility to formulate spatial derivatives we refer to [44] where the constitutive flux equation is defined with spatial relaxation function

$$R^*(x) = k_1\delta_x(u) + \frac{k_{2x}}{\lambda_x}\exp\left(-\frac{u}{\lambda_x}\right), \quad \int_0^{\infty}\delta_x(u)du = 1 \tag{123}$$

constructed by analogy with the flux in Example 1.

In this case, the flux with a spatial memory is defined as

$$q(x) = -k_1 \frac{dT(x)}{dx} - \frac{k_{2x}}{\lambda_x} \int_0^x e - \frac{(x-u)}{\lambda_x} \frac{dT[u]}{du} du \tag{124}$$

where λ_x is a "spatial relaxation distance" defined in (125).

Skipping detailed derivation available in [44] and applying the energy conservation equation to (124), the following heat conduction model with exponential spatial memory was developed

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_{2x}(1-\mu)\,{}^{CF}D_x^{\mu} T(x,t), \quad \lambda_x = \frac{1-\mu}{\mu}, \quad 0<\mu<1 \tag{125}$$

The spatial derivative ${}^{CF}D_x^{\mu} T(x,t)$ is of Caputo-type and in steady state (125) reduces to

$$0 = a_1 \frac{d^2 T(x)}{dx^2} + a_{2x}(1-\mu)\,{}^{CF}D_x^{\mu} T(x) \tag{126}$$

In the case of short-range memory effects, i.e. when only the second term of the constitutive flux equation (124) is considered, the flux is formulated as

$$q_{2x} = -k_{2x}(1-\mu)\,{}^{CF}D_x^{\mu} T(x) \tag{127}$$

This formulation coincides with (151) and the only difference is in the formulation (definition) of the spatial derivatives.

8.3 Fractional Caputo-Fabrizio Constitutive Integral: Volterra Equation Approach

Now, the approach applied is based on the Volterra integral equations and we need some preliminary basic information. We have two main constructions of Volterra equations [45], namely

First kind integral equation

$$f(t) = \int_a^t K(t,s) y(s)\, ds \tag{128}$$

where the unknown function is $y(s)$

Second kind integral equation

$$f(t) = y(t) + \int_a^t K(t,s) f(s)\, ds \tag{129}$$

where the unknown function is $f(s)$.

In the context of the following exposition we consider the first kind of Volterra equation where the unknown function is the concentration or temperature gradient, that is

$$f(t) = \int_a^t K(t,s) \frac{\partial T(x,s)}{\partial x} ds \tag{130}$$

and the kernel $K(t,s)$ is preliminarily defined regarding the type of fractional operator considered.

It is easy to convert (130) as a second kind equation, namely [45]

$$f(x) = y(t) + \int_a^x \frac{K^{10}(t,s)}{K(x,x)} y(s)\, ds, \quad K^{i,j}(x,t) = \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial t^j} K(x,t) \tag{131}$$

or in terms of $\frac{\partial T(x,s)}{\partial x}$, as a more convenient for us construction

$$f(x) = A_1 \frac{\partial T(x,t)}{\partial x} + A_2 \int_a^x R(t,s) \frac{\partial T(x,s)}{\partial x} ds \tag{132}$$

and assuming the lower terminal at $a = 0$; A_1 and A_2 are transport coefficients.

8.3.1 The concept and Riemann-Liouville operators through Volterra Equation Approach

The main step in the construction of the flux constitutive model is the use $R(t)$ as a memory kernel, namely

$$j(x,t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * M(t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * [A_1\delta(t) + A_2R(t)] \tag{133}$$

This is a Volterra equation of first kind, as originally formulated (see (128)) , but converted into the form (132), for the purpose of the following analysis. For the sake of clarity of this formulation see the Boltzmann superposition principle presented above and the remarks at the end of this section. In contrast to the fading memory concept, here there is no restriction on the time duration of the memory function, but we may accept conditionally that the recent past is more effective on the current situation than the distant past.

In what follows we consider the same initial and boundary conditions as in the cases where the Caputo-time fractional operators were derived, that is (signaling problem):

$$C(x,0) = C(0,0) = C(\infty,t) = C(-\infty,t), \quad C_x(x,0) = C_{xx}(x,0) = 0. \tag{134}$$

Now, considering only the convolution integral and setting the lower terminal at zero, bearing in mind that $R(t)$ is a causal function, that is

$$\int_0^t R(t-\tau)\frac{\partial C(x,\tau)}{\partial x}d\tau \Rightarrow \int_0^t R(t-\tau)f(x,\tau)d\tau, \quad f(t) = \frac{\partial C(x,t)}{\partial x} \tag{135}$$

In this way, we got a construction of a convolution integral of the Riemann-Liouville type

$$I^\alpha f(x,t) = \int_0^t R(t-\tau)f(x,\tau)d\tau \tag{136}$$

with a kernel $R(t)$ that should be a completely monotonic function. Then, following the tradition, the time-fractional derivative can be formulated as

$$^{RL}D_t^\alpha f(x,t) = \frac{d^m}{dt^m}[I^\alpha f(x,t)] = \frac{d^m}{dt^m}\int_0^t R(t-\tau)f(x,\tau)d\tau, \quad m \in \mathbb{N} \tag{137}$$

or simply for $m = 1$

$$^{RL}D_t^\alpha f(x,t) = \frac{d}{dt}\int_0^t R(t-\tau)f(x,\tau)d\tau \tag{138}$$

Therefore, turning on the constitutive model (133) we may present it as

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - k_2 I^\alpha\left[\frac{\partial T(x,t)}{\partial x}\right] \tag{139}$$

Then, the energy balance equation yields a general form of the non-local heat conduction equation

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2 I^\alpha\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] \tag{140}$$

where the non-locality appears as a damping term, not affecting the time derivative of the energy balance equation (the First Law of Thermodynamics). Following this formulation, for $\alpha = 1$ we get

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2\int_0^t \frac{\partial^2 T(x,t)}{\partial x^2}dt \tag{141}$$

This model is also non-local, due to the integral in the second the term of the right-hand side, but this term is not fractional. Moreover, the second term in (141) has no physical meaning, thus we have to resolve this problem. In the above derivations, we did not insert a normalization function of the fractional integral, thus we have to reformulate it as

$$I^{\alpha} f(x,t) = \frac{1}{N(\alpha)} \int_0^t R(t-\tau) f(x,\tau) d\tau = \frac{1}{1-\alpha} \int_0^t R(t-\tau) f(x,\tau) d\tau, \quad N(\alpha) = 1-\alpha \tag{142}$$

In this context ${}^{RL}D_t^{\alpha} f(x,t)$ can be reformulated as

$${}^{RL}D_t^{\alpha} f(x,t) = \frac{d}{dt}\left[\frac{1}{1-\alpha}\int_0^t R(t-\tau) f(x,\tau) d\tau\right] \tag{143}$$

After these justifications, we may re-write the heat diffusion models as

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2 (1-\alpha) I^{\alpha}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] \tag{144}$$

Hence, we got a similar construction of the heat conduction model as in the case when the fading memory approach was systematically applied. For $\alpha = 1$ we recover the Fourier model.

If we like the non-locality to be represented by the fractional derivative ${}^{RL}D_t^{\alpha} f(x,t)$, then a differentiation concerning the time t of both sides of (144) yields

$$\frac{\partial^2 T(x,t)}{\partial t^2} = a_1 \frac{\partial}{\partial t}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] + a_2 (1-\alpha) {}^{RL}D_t^{\alpha}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] \tag{145}$$

This is a hyperbolic equation that simply can be presented as

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \left[a_1 \frac{\partial}{\partial t} + a_2 (1-\alpha) {}^{RL}D_t^{\alpha}\right] \frac{\partial^2 T(x,t)}{\partial x^2} \tag{146}$$

and for $\alpha = 1$ we recover the Fourier model.

Now, after these general formulations, we can see what would be the effects of different versions of the relaxation function $R(t)$. For the systematic exposition of the ideas developed in this article, we will apply the same memory kernels as in the examples where the fading memory formalism was applied.

8.3.2 Example 3: Heat flux with exponential memory (time domain) and Volterra equation approach

With $R(t) = \exp(-\lambda t)$, applying (144) we get

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2 (1-\alpha) \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \frac{\partial^2 T(x,\tau)}{\partial x^2} d\tau, \quad 0 < \alpha < 1, \quad t > 0 \tag{147}$$

where the rate coefficient in exponential memory $\lambda \in (0,\infty)$ is mapped by $\lambda = \frac{\alpha}{1-\alpha}$.

Thus, we formulated a Riemann Liouville integral with an exponential memory, with a normalization function $(1-\alpha)$ as

$${}^{RL}_{\exp}I_t^{\alpha} = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau \tag{148}$$

and consequently the corresponding time-fractional derivative (with $m = 1$) is

$${}^{RL}_{\exp}D_t^{\alpha} f(x,t) = \frac{1}{1-\alpha} \frac{d}{dt} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau \tag{149}$$

This is the so-called Yang-Srivastava-Machado derivative [46], where the singular kernel $\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}$ in the classical Riemann-Liouville derivative was replaced by $\frac{\exp\left[\frac{-\alpha}{1-\alpha}(t-\tau)\right]}{1-\alpha}$, without any physical background. Then, it was applied as a spatial fractional derivative [46]

$$ {}^{RL}_{\exp}D^{\alpha}_{x} f(x) = \frac{1}{1-\alpha}\frac{d}{dx}\int_{0}^{t} e^{-\frac{\alpha}{1-\alpha}(x-z)} f(z)\,dz \tag{150} $$

to a steady-state heat conduction model [46]

$$ {}^{RL}_{\exp}D^{\alpha}_{x} f(x,t) = g \tag{151} $$

8.4 Caputo-Fabrizio integral-Distributional theory approach

Now, for the sake of completeness of the approaches defining Caputo-Fabrizio constitutive integral, we refer to the work of Atanackovic et al. [11] where the Caputo-Fabrizio fractional derivative, in the Riemann-Liouville sense, is defined through the kernel function $\varphi_{\alpha}(t)$ (i.e., the relaxation kernel is primarily defined (constituted))

$$ \varphi_{\alpha}(t) = \frac{1}{1-\alpha}\exp\left(-\frac{\alpha}{1-\alpha}t\right), \quad t \in \mathbb{R} = (-\infty,\infty), \quad \alpha \in (0,1) \tag{152} $$

Applying the distributional theory (see for example, [47,48] for the main ideas of this theory) where for any $\theta \in C_{0}^{\infty}(\mathbb{R})$[11]

$$ \langle H\varphi_{\alpha}, \theta \rangle = \frac{1}{1-\alpha}\int_{0}^{\infty} e^{-\frac{\alpha}{1-\alpha}\tau}\theta(\tau)\,d\tau = \frac{1}{1-\alpha}\int_{0}^{\infty} e^{-\alpha s}\theta((1-\alpha),s)\,ds \tag{153} $$

This means that $H\varphi_{\alpha} \to \delta$ in $S'(\mathbb{R})$ for $\alpha \to 1^{-}$. In (153) , H is the Heaviside function, right continuous at 0, that is $H(0) = 1$. Moreover, the function f is considered as [11]: $f \in AC_{loc}([0,\infty))$, s well as $f \in AC_{loc}((-\infty,\infty))$, so that their restriction on $[0,\infty)$ denoted by $(fH) \in AC_{loc}([0,\infty))$

Then, the Caputo-Fabrizio derivative is a Riemann-Liouville sense is defined as

$$ {}^{CFR}D^{\alpha}_{t} f(t) = \left((H\varphi_{\alpha})' * Hf(t)\right), \quad t \in (-\infty,\infty) \tag{154} $$

The relationship between ${}^{CFR}D^{\alpha}_{t} f(t)$ and original ${}^{CF}D^{\alpha}_{t} f(t)$ is [11]

$$ {}^{CF}D^{\alpha}_{t} f(t) = {}^{CFR}D^{\alpha}_{t} f(t) - f(0)\varphi_{\alpha}(t) \tag{155} $$

and,the Caputo-Fabrizio fractional integral is defined as

$$ {}^{CF}\mathfrak{I}^{\alpha}_{t} f(t) = (H\phi_{\alpha}(t)) * (Hf(t)), \quad t \in (-0,\infty) \tag{156} $$

The relations of ${}^{CFR}D^{\alpha}_{t} f(t)$ to the Caputo-Fabrizio fractional integral ${}^{CF}\mathfrak{I}^{\alpha}_{t} f(t)$ is:

$$ {}^{CFR}D^{\alpha}_{t} f(t) = \frac{1}{1-\alpha}f(t) - \frac{\alpha}{1-\alpha}{}^{CF}\mathfrak{I}^{\alpha}_{t} f(t), \tag{157} $$

This construction, to some extent, resembles both the associated integrals and the fading memory constitutive equation. The following asymptotics of the Caputo-Fabrizio operators ${}^{CFR}D^{\alpha}_{t} f(t)$ and ${}^{CF}D^{\alpha}_{t} f(t)$ hold [11]:

$$ \lim_{\alpha\to 0^{+}}\left[{}^{CF}\mathfrak{I}^{1-\alpha}_{t} f(t)\right] = f(t), \quad t \geq 0 \tag{158} $$

$$ \lim_{\alpha\to 0^{-}}\left[{}^{CF}\mathfrak{I}^{1-\alpha}_{t} f(t)\right] = \begin{cases} \int_{0}^{t} f(u)\,du, & t > 0 \\ 0, & t < 0 \end{cases} \tag{159} $$

$$ \lim_{\alpha\to 1^{-}}\left[{}^{CFR}D^{\alpha}_{t} f(t)\right] = (Hf)'(t) = (Hf')(t) + f(0)\delta(t), \quad t \in (-\infty,\infty) \tag{160} $$

$$\lim_{\alpha \to 1^-} \left[{}^{CF}D_t^{\alpha} f(t) \right] = \left(Hf' \right)(t) = f'(t), \quad t > 0 \tag{161}$$

$$\lim_{\alpha \to 0^+} \left[{}^{CFR}D_t^{\alpha} f(t) \right] = f(t), \quad t > 0 \tag{162}$$

$$\lim_{\alpha \to 0^+} \left[{}^{CF}D_t^{\alpha} f(t) \right] = f(t) - f(0), \quad t > 0 \tag{163}$$

where $f'(t)$ is locally integrable on $[0,\infty)$. For more deep analysis of this approach, we recommend systematic studies of [11,49] as well [50].

*Remark.*As a closing comment to this part of the article where the two conceptual approaches in fractional modeling build-up were demonstrated, we refer to the analysis of Fabrizio et al. [51]. Concisely, the analysis focuses attention on the fact that the transport processes in media (materials) with memory, also termed hereditary systems, have attracted the attention of mathematical modeling in the past and traced to the works of Boltzmann [15] and Volterra [52].

The Boltzmann concept of heredity [15] was later developed by Volterra [52] applying a laborious functional setting. However, as pointed out by Fabrizio et al. [51] there are some differences in these concepts, among them:

– The Boltzmann superposition principle involves a hereditary elasticity in constitutive equations through a fading memory related to stress-strain relationships in the Riemann-Stieltjes form (as frequently used in this article). In addition, Boltzmann [15] stressed the attention on a hereditary model of viscoelastic materials through fading memory kernels, named relaxation properties. For additional readings, we refer to [5,42,6,37] and [16].

– Volterra describes the same stress-strain relations through Lebesgue's representation of linear functionals in the history space with two basic postulates (in the framework of the hereditary elasticity): i) the principle of invariant heredity, and ii) the principle of the closed cycle. In the present analysis, we avoided this approach and used directly the final formulation of Volterra's equations to model hereditary effects.

*Remark.*As was commented by Fabrizio et al. [51], the Boltzmann and the Volterra constitutive relations are equivalent. In this context, we can see that the Riemann-Liouville derivatives generated through the Volterra equations are equivalent to the Caputo derivatives, upon zero-initial conditions, and only differentiation by parts in the convolution integrals is needed to reach this result.

In addition, we have to stress the attention on the history concept when an infinite time interval is defined, as in the Riemann-Stieltjes integral formulation is used. This formulation is only a good mathematical formulation rather than a correct model of the hereditary transport process because the time starts to count at the point $t = 0$ when the process begins. Therefore, the essence of this is well expressed by Hilfer in [24], as mentioned earlier, that in fractional operators the time *is not the chronological time* (*instant time*) but *the intrinsic time* of the process (the *time of duration*), *starting at the point accepted as* $t = 0$. Hence, in this context, the setting of the lower terminals of the hereditary integrals at zero is physically motivated, since *there is no process before* $t = 0$.

Now, we are going further, following the schedule defined in the Aim-2 and the main task of this study. Hence, we start with the Atangana-Baleanu constitutive integrals,discussed in Section 9, from two basic starting points: By the Fading Memory approach (Section 9.1) and Volterra equations (Section 9.2). The analysis of the normalization function $M(\alpha)$ and $B(\alpha)$, their existing, *ad hoc* and optimized formulations are addressed in Section 10.

9 Atangana-Baleanu operators

9.1 Example 4: Atangana-Baleanu operators: Definitions by the Fading Memory Formalism

Now, we constitute a new relaxation (memory) function for the heat flux [5,6]

$$R_{JP} = k_1 \delta(t) + k_2 \lambda \frac{E_{\alpha,0}\left(-\lambda t^{\alpha}\right)}{t} = k_1 \delta(t) + (k_2/\tau) \frac{d}{dt}\left[E_{\alpha,1}\left(-\lambda t^{\alpha}\right)\right], \quad \lambda = 1/\tau \tag{164}$$

For $\alpha = 1$ it is $R_{JP} = k_1 \delta(t) + k_2 \lambda \frac{d}{dt}\left[\exp k_2 \left(-\lambda t^{\alpha}\right)\right]$.

The second term in (164) is a monotonically decaying function and singular at $t \to 0$ and thus obeys all necessary properties to be used as a memory function. Then, following the fading memory construction for a model of the heat flux with memory (104) we can write

$$q(x,t) = -k_1 \frac{\partial T(x,t)}{\partial x} - k_2 \int_{-\infty}^{t} \left[\frac{E_{\alpha,0}\left(\lambda (t-s)^{\alpha}\right)}{(t-s)} \right] \frac{\partial T(x,s)}{\partial x} d\lambda s, \quad \lambda = 1/\tau \tag{165}$$

or following the main task of this study we get

$$q(x,t) = -k_1 \frac{\partial T(x,t)}{\partial x} - \frac{k_2}{\tau} \int_{-\infty}^{t} \frac{\partial T(x,s)}{\partial x} \frac{d}{ds} \left[E_{\alpha,1} \left(\lambda (t-s)^{\alpha} \right) \right] d\lambda s \tag{166}$$

Now, the integration by parts of the integral (denoting $F(x,t) = \partial T(x,t)/\partial x$) in (166) yields

$$\begin{aligned} &\int_{-\infty}^{t} E_{\alpha,1} \left(\lambda (t-s)^{\alpha} \right) F(x,s)\, d\lambda s = E_{\alpha,1} \left((t-s)^{\alpha} \right) \left[F(x,s) - F(x,t) \right] \Big|_{-\infty}^{t} + \\ &+ \int_{-\infty}^{t} E_{\alpha,1} \left(\lambda (t-s)^{\alpha} \right) \frac{\partial F(x,s)}{\partial s} d\lambda s \end{aligned} \tag{167}$$

The first term in the RHS of (167) is zero, while the second one matches the definition of the Atangana-Baleanu derivative fractional derivative (5). The coefficient $\lambda = 1/\tau \in (\infty, 0)$ $\tau \in (0, \infty)$ mapped by (see the same relations in [42]) [5,6]

$$\lambda = \frac{\alpha}{1-\alpha} \in [0, \infty], \quad \alpha = \frac{1}{1 + 1/\lambda} = \frac{\lambda}{1+\lambda} \in [0, 1] \tag{168}$$

Then, assuming the lower limit in the Stieltjes integral at zero (i.e, obeying the causality principle), we get a new expression for the second term of the heat flux with a memory

$$q_e = -a_2 \int_{0}^{t} E_{\alpha,1} \left[-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right] \frac{d}{ds} \left[\frac{\partial T(x,s)}{\partial x} \right] ds \tag{169}$$

The integral in (169) resembles the construction of the ABC derivative (5) and we can recast it following [5,6]. The subscript in q_e means *elastic* following the older definition of the time decaying memory terms in the Cattaneo approach [42].

$$\begin{aligned} q_e &= -a_2 (1-\alpha) \left\{ \frac{1}{1-\alpha} \int_{0}^{t} E_{\alpha,1} \left[-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right] \frac{d}{ds} \left[\frac{\partial T(x,s)}{\partial x} \right] ds \right\} = \\ &= -a_2 (1-\alpha)\, {}^{ABC}D_t^{\alpha} \left[\frac{\partial T(x,t)}{\partial x} \right] \end{aligned} \tag{170}$$

Therefore, for the RHS of the energy conservation equation, we obtain

$$\begin{aligned} &-\frac{\partial q_e}{\partial x} = a_2 (1-\alpha) \int_{0}^{t} E_{\alpha,1} \left[-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right] \frac{d}{ds} \left[\frac{\partial^2 T(x,s)}{\partial x^2} \right] ds = \\ &= -a_2 (1-\alpha)\, {}^{ABC}D_t^{\alpha} \left[\frac{\partial^2 T(x,t)}{\partial x^2} \right] \end{aligned} \tag{171}$$

Finally, the energy conservation equation with a hereditary flux through a Mittag-Leffler memory kernel yields

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2 (1-\alpha)\, {}^{ABC}D_t^{\alpha} \left[\frac{\partial^2 T(x,t)}{\partial x^2} \right] \tag{172}$$

For $\alpha = 1$ it reduces to the Fourier equation.

Hence, we saw that the ABC derivative naturally appears when a suitable memory kernel in the constitutive equation about the heat flux is constituted.

Remark. To complete this section we refer to [6] where fractional operators, with kernels based on various versions of the Mittag-Leffler function and their derivatives, have been developed from a common basis employing the Fading Memory formalism. Examples related to diffusion and viscoelasticity are available elsewhere [37].

9.2 Example 4: Atangana-Baleanu operators:Definitions by the Volterra equations

With a memory kernel $R(t)=E_{\alpha}\left(-\lambda t^{\alpha}\right)$, assuming $\lambda\in(0,\infty)$, mapped by $\lambda=\frac{\alpha}{1-\alpha}$, and applying (133) we get (skipping intermediate calculations) a heat conduction equation

$$\frac{\partial T(x,t)}{\partial t}=a_1\frac{\partial^2 T(x,t)}{\partial x^2}+a_2(1-\alpha)\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\frac{\partial^2 T(x,\tau)}{\partial x^2}d\tau, \quad 0<\alpha<1, \quad t>0 \tag{173}$$

Thus, we can define a fractional integral with a Mittag-Leffler function of one parameter as a kernel, namely

$$ {}^{RL}_{ML}I_t^{\alpha}=\frac{1}{1-\alpha}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)f(\tau)\,d\tau \tag{174}$$

and with a normalization function $(1-\alpha)$.

Differentiation to the time t in (174) yields

$$\frac{d}{dt}{}^{RL}_{ML}I_t^{\alpha}f(\tau)={}^{RL}_{ML}D_t^{\alpha}f(\tau)=\frac{1}{1-\alpha}\frac{d}{dt}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)f(\tau)\,d\tau \tag{175}$$

Then, we may present (173) (through differentiation concerning the time of both sides) as

$$\frac{\partial^2 T(x,t)}{\partial t^2}=a_1\frac{\partial}{\partial t}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right]+a_2(1-\alpha){}^{RL}_{ML}D_t^{\alpha}\left[\frac{\partial^2 T(x,\tau)}{\partial x^2}\right], \quad 0<\alpha<1, \quad t>0 \tag{176}$$

For $\alpha=1$, both equations, (173) and (176), reduce to the Fourier model.

9.3 Example 5: Prabhakar kernel as memory and Volterra equation approach

This is an additional example demonstrating how the main principle can be applied successfully and the feasibility of the generalized approach to the model (precisely the memory term) construction.

Applying the technology used in the previous point and assuming the memory kernel as the Prabhakar product $R(t)=e^{\gamma}_{\alpha,\beta}=t^{\beta-1}E^{\gamma}_{\alpha,\beta}\left(-\lambda t^{\alpha}\right)$, with$\lambda\in(0,\infty)$, we can map $\lambda=\frac{\alpha}{1-\alpha}$, and then applying (133) (skipping the intermediate steps demonstrated in the preceding sections) the result is

$$\frac{\partial T(x,t)}{\partial t}=a_1\frac{\partial^2 T(x,t)}{\partial x^2}+a_2(1-\alpha)\int_0^t e^{\gamma}_{\alpha,\beta}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\frac{\partial^2 T(x,\tau)}{\partial x^2}d\tau, \quad 0<\alpha<1, \quad t>0 \tag{177}$$

Thus, we can define a Prabhakar integral in the Riemann-Liouville sense as

$$ {}^{P}I^{\alpha}_{\alpha,\beta,\gamma,t}=\frac{1}{1-\alpha}\int_0^t e^{\gamma}_{\alpha,\beta}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)f(t)\,d\tau, \quad f(t)\in L^1(0,1) \tag{178}$$

It differs from the definition in [53] by the introduction of the normalization function $(1-\alpha)$.

Then, the fractional derivative in the Riemann-Liouville sense with a kernel $e^{\gamma}_{\alpha,\beta}=t^{\beta-1}E^{\gamma}_{\alpha,\beta}\left(-\lambda t^{\alpha}\right)$ can be defined as

$$\frac{d}{dt}\left({}^{P}I^{\alpha}_{\alpha,\beta,\gamma,t}\right)={}^{RL}_{P}D^{\alpha}_{\alpha,\beta,\gamma,t}=\frac{1}{1-\alpha}\frac{d}{dt}\int_0^t e^{\gamma}_{\alpha,\beta}\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)f(t)\,d\tau \tag{179}$$

Turning on (177) and differential both sides concerning the time, the result is

$$\frac{\partial^2 T(x,t)}{\partial t^2} = a_1 \frac{\partial}{\partial t}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] + a_2(1-\alpha)\,{}_p^{RL}D^{\alpha}_{\alpha,\beta,\gamma,t}\left[\frac{\partial^2 T(x,\tau)}{\partial x^2}\right], \quad 0<\alpha<1, \quad t>0 \tag{180}$$

For $\alpha = 1$, both integro-differential equations, the parabolic model (177) and the hyperbolic (180), reduce to the Fourier model.

To complete Example 6, we mention that a derivative with Prabhakar kernel as a memory, in the Caputo sense, applying the fading memory formalism, has been developed in [6].

*Remark.*Now we recall the remarks in section 8.4 with the comments of Fabrizio et al. [51] about the equivalence of the two approaches used here but we refer again to this problem concerning only the terms with memories of the developed models.

Hence, the convolution terms of the heat fluxes, with generalized (i.e.not specifically defined) memory kernel $R(t)$(a differentiable function with easy Laplace transform) are

$$j_{2(FM-AB)} = a_2 \frac{1}{1-\alpha}\int_0^t R(t-\tau)\frac{dF(x,\tau)}{d\tau}d\tau \tag{181}$$

$$j_{2(Volt-RL)} = a_2 \frac{1}{1-\alpha}\frac{d}{dt}\int_0^t R(t-\tau)F(x,\tau)\,d\tau \tag{182}$$

Now, let us apply the Laplace transforms and rearrange them, (omitting the pre-factor $a_2/(1-\alpha)$ for the sake of clarity) and assume that $F(0,x)=0$

For the flux defined through the Fading memory formalism and leading to Caputo-type operators we have

$$\mathscr{L}\left[j_{2(FM-AB)}\right] \equiv \mathscr{L}\left[\int_0^t R(t-\tau)\frac{dF(x,\tau)}{d\tau}d\tau\right] = = R(s)\,sF(x,s) = sR(s)F(x,s) = s\left[R(s)F(x,s)\right] \tag{183}$$

In the time domain, this can be presented as

$$j_{2(FM-AB)} \equiv \int_0^t R(t-\tau)\frac{dF(x,\tau)}{d\tau}d\tau \equiv \int_0^t R_t(t-\tau)F(x,\tau)\,d\tau \equiv \frac{d}{dt}\int_0^t R_t(t-\tau)F(x,\tau)\,d\tau \tag{184}$$

For the flux defined through the Volterra equation formalism and leading to Riemann-Liouville sense operators we have

$$\mathscr{L}\left[j_{2(Volt-RL)}\right] \equiv \mathscr{L}\left[\frac{d}{dt}\int_0^t R(t-\tau)F(x,\tau)\,d\tau\right] = = s\left[R(s)F(x,s)\right] = sR(s)F(x,s) = R(s)\,sF(x,s) \tag{185}$$

and in the time domain, this can be presented as

$$j_{2(Volt-RL)} \equiv \frac{d}{dt}\int_0^t R(t-\tau)F(x,\tau)\,d\tau \equiv \int_0^t R_t(t-\tau)F(x,\tau)\,d\tau \equiv \int_0^t R(t-\tau)\frac{F(x,\tau)}{d\tau}d\tau \tag{186}$$

Comparing the Laplace transforms and their originals in the time domain we can see the equivalence of the defined fractional operators as well as that of the Caputo-type and the Riemann-Liouville operators (derivatives) are equivalent upon zero condition, a fact known since the era of the power-law kernels [4].

Table 1: Gamma function based $M(\alpha)$.Note: $\star$-present work suggestion

Code	$M(\alpha)$	*Comments*
$M1$	$1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$	Atangana [13]
$M2$	$1-\alpha+\frac{1}{\Gamma(\alpha)}$	$\star$
$M3$	$1-\alpha+\frac{\alpha}{\Gamma(2-\alpha)}$	$\star$
$M4$	$1-\alpha+\frac{1}{\Gamma(1-\alpha)}$	$\star$
$M5$	$\Gamma(2-\alpha)$	$\star$
$M6$	$\frac{1}{\Gamma(2-\alpha)}$	$\star$

9.4 Briefs on the AB derivatives formulations through constitutive equations

To complete this section we have to outline the main results developed. First of all, we demonstrated that both the ABR and ABC derivatives can be developed in straightforward ways by applications of thermodynamically consistent and obeying the causality conditions constitutive equations about the heat flux (this is valid also if a diffusion problem is considered): when the constitutive integrals are based on the Mittag-Leffler function (one-parameter) as memory kernel.

The examples are restricted to only these two fractional operators, as the most popular ones, and in the context of the present article, but the method applies to a wider range of functions from the Mittag-Leffler family [6] (see Example 5). Furthermore, both approaches are equivalent from either thermodynamic or formal (see the closing remark in 9.3) for points of view.

However, one element of the definitions of the fractional operators remains unresolved, i.e. the normalization functions $M(\alpha)$ and $B(\alpha)$, and this problem is developed in the remainder of this article.

10 Normalization functions $M(\alpha)$ and $B(\alpha)$

Frankly, the study began with an idea to resolve the problem concerning definitions of the normalization functions $M(\alpha)$ and $B(\alpha)$, but the systematic approach applied needed all issues considered in the preceding section to be thoroughly analyzed. Now, we go to definitions of versions of $M(\alpha)$, the normalization function belonging to the Caputo-Fabrizio derivative. With this, we envisage also $B(\alpha)$ of the Atangana-Baleanu derivatives since both of them should obey equal conditions at the border of the interval $\alpha \in [0.1]$.

10.1 Existing and new formulated normalization functions: a few remarks at a glance

It is hard to find a motivated definition of $M(\alpha)$ in the contemporary literature on modern fractional calculus, but there are only two short remarks (just claimed functions, without any motivations): in [13] about $B(\alpha)$-see Table 1 and of [9] concerning $M(\alpha)$-see Table 3. As an attempt to create more versions of $M(\alpha)$ the present work suggests three groups of functions, created *ad hoc*, summarized in Tables, 1, 2 and 3; the grouping is based on the basic functions used. The graphical presentations reveal that all of them are close to unity, located up or down concerning the line $M(\alpha)=1$ (see Figure 1); only one $M11(\alpha)$ oscillates about this line (see Figure 1-d). Although the behaviors of all these functions seem different, they are almost similar when normalized by the denominator $(1-\alpha)$ as shown in Figure 2: all of them begin at $M(\alpha)/(1-\alpha)=1$, at the left border of the interval $\alpha \in [0.1]$, and sharply go to almost infinity when $\alpha \to 1$.

Thus, we have families of functions obeying all conditions imposed in the general definitions of the non-singular fractional operators discussed here. However, we cannot say anything about their behavior within the interval $\alpha \in [0.1]$, and therefore efforts towards more conditions yielding strict definitions are highly required. The idea at issue is the focus of the points that follow.

10.2 Restrictions on the memory kernels and issues related to the normalization function $M(\alpha)$

Even though imagination in definitions of fractional operators with various kernels could have no limits, as is observed in the last decade of this century, we have to stress the attention that the important point in all these definitions is the missing thermodynamic analysis of the relaxation (memory) functions used. A systematic analysis in this direction is beyond the

Table 2: Algebraic function based $M(\alpha)$. Note: $\star$-present work suggestion

Code	$M(\alpha)$	Comments
$M7$	$(2-\alpha)^{\alpha}$	$\star$
$M8$	$\frac{1}{(2-\alpha)^{\alpha}}$	$\star$
$M9$	$\frac{1}{2-\alpha}+\frac{1-\alpha}{2}$	$\star$

Table 3: Trigonometric function based $M(\alpha)$.Note: $\star$-present work suggestion

Code	$M(\alpha)$	Comments
$M10$	$\sin\left(\alpha\frac{\pi}{2}\right)+\cos\left(\alpha\frac{\pi}{2}\right)$	$\star$
$M11$	$1-0.2\sin(2\pi\alpha)$	Bekkouche et al.[9]

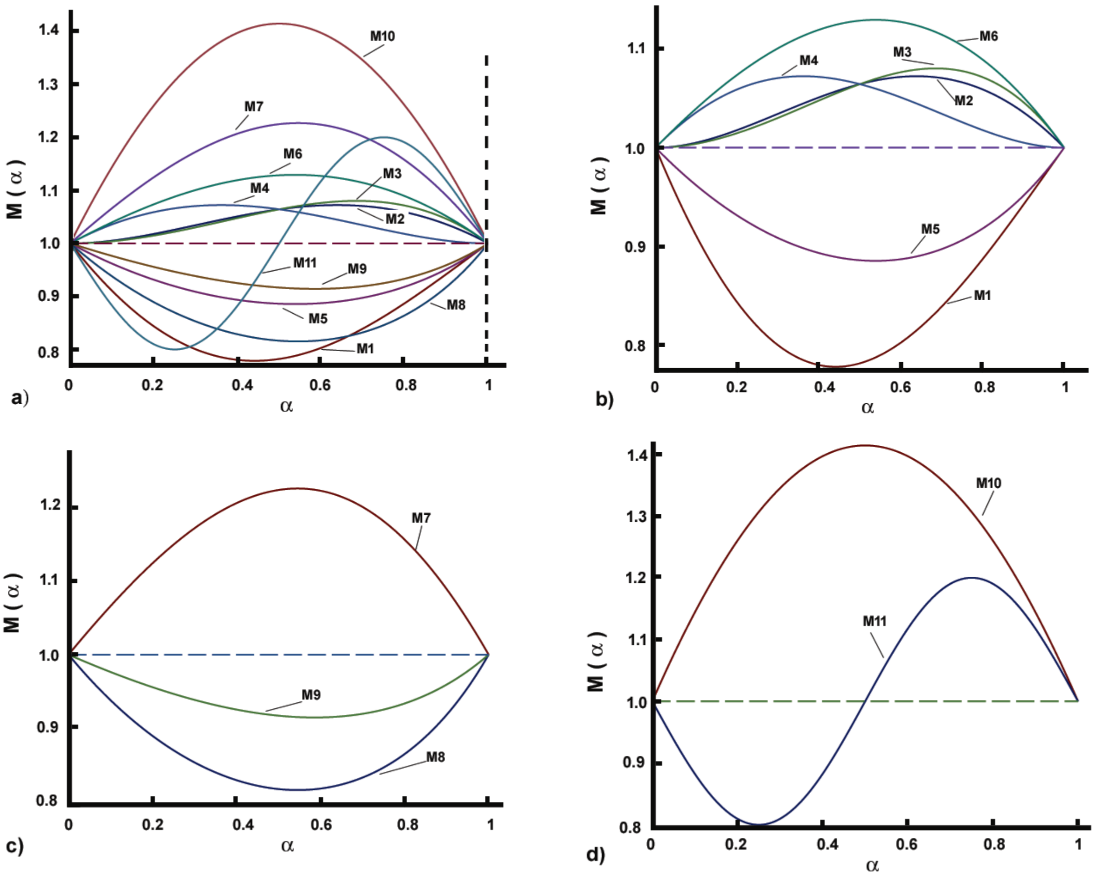

Fig. 1: Functions $M(\alpha)$: a) All functions from tables, 1, 2 and 3; b) Gamma function based; c) Algebraic functions; d) Trigonometric function; Note: Dotted line $M(\alpha)=1.0$

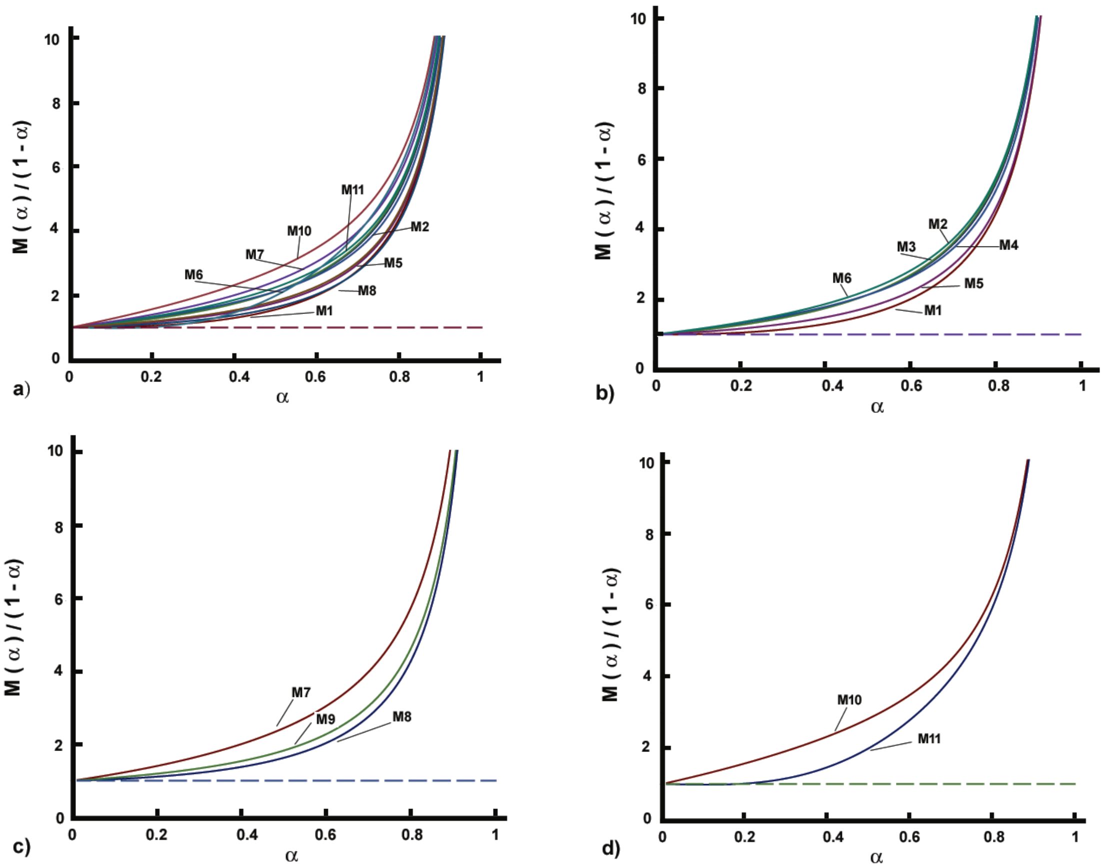

Fig. 2: Functions $M(\alpha)/(1-\alpha)$: a) All functions, as in Figure 1-a ; b) Gamma function based; c) Algebraic functions; d)Trigonometric function; Note: Dotted line $M(\alpha) = 1.0$

scope of this study, thus we limit ourselves to the exponential and Mitagg-Leffler functions, from a different angle looking for some definitions or restrictions that could be imposed on the normalization function $M(\alpha)$. This would be useful for the reduction of the number of possible versions that will facilitate the calculus with these operators.

10.2.1 Restrictions on the exponential memory kernel and results thereof

Exponential memory is a long-time applied kernel in non-local dynamics, but particularly to the models discussed here, we refer to the work of Fabrizio and Morro [28] where its thermodynamic compatibility is rigorously proved. Now, we will use the results of Fabrizio et al. [54]. Precisely, considering an exponential memory (exponential decaying history) of the heat flux (in the original notations)

$$\mathbf{q}\,(t-s) = \mathbf{q_0}\exp\left[-a\,(t-s)\right], \quad a>0, \quad s\in[0,\infty) \tag{187}$$

The sufficient condition for the thermodynamic restriction is [54]

$$\left[\mathbf{q}(t) - q(t-s)\right]^2 + \bar{q}\,(t,s)\cdot\partial_s q\,(t-s) \geq 0 \tag{188}$$

$$\mathbf{q}(t) - \mathbf{q}(t-s) = \frac{d}{dt}\int\limits_{t-s}^{t} q\,(\xi)d\xi, \quad \bar{q}\,(\xi) = \int\limits_{t-s}^{t}\mathbf{q}\,(\xi)d\xi \tag{189}$$

For example, for constant histories $q(t-s)=q_0$ with $s\in[0,T]$ satisfies (188) since in such a case $\mathbf{q}(t)-q(t-s)=0$ and $\partial_s q(t-s)=0$. Then, we get [54]

$$[\mathbf{q}(t)-q(t-s)]^2+\bar{q}(t,s)\cdot\partial_s q(t-s)=q_0^2\exp(2at)[1-3\exp(-as)+2\exp(-2as)] \tag{190}$$

and following (188) , it holds when [54]:$\exp(-as)\leq 1/2$ or $\exp(-as)>1$.

From the first condition, it follows that [54]: $s\geq\ln(2)/a$ and we get one-parameter family of histories, namely

$$q(t-s)=\begin{cases} q_0\exp(at-\ln 2), & s\in[0,\ln(2)/a\,) \\ q_0\exp(at(t-s)), & s\in[\ln(2)/a,\infty\,) \end{cases} \tag{191}$$

Further, skipping cumbersome expressions, available in Section 5 of [54], we have the condition

$$\tau_\alpha a^\alpha\frac{(\ln 2)^{1-\alpha}}{1-\alpha}\leq\frac{\Gamma(1-\alpha)}{\alpha} \tag{192}$$

Now, to make the analysis clearer, we translate (192) to terms and definitions used in the present study, namely

$$a=\frac{-\alpha}{1-\alpha},\quad \tau_\alpha=\frac{1}{a}=-\frac{1-\alpha}{\alpha} \tag{193}$$

That is, a is the rate factor of the exponential memory, while τ_α is the relaxation time (see Example 1 in Section 8.2.3).

Hence, taking into account, the definitions of the history flux we have a functional prefactor $\frac{M(\alpha)}{1-\alpha}$, although, for the sake of easier presentation of the results developed, we used up to this point $M(\alpha)=1$, following, by convention, the idea (definition) of Losada and Nieto [7].

Now, with the normalization function $M(\alpha)$ and the denominator $(1-\alpha)$, and (193 we can be present (192) as

$$\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{1-\alpha}\right)^\alpha\left[\frac{M(\alpha)}{1-\alpha}\right]\frac{(\ln(2))^{1-\alpha}}{1-\alpha}\leq\frac{\Gamma(1-\alpha)}{\alpha} \tag{194}$$

Then, after simple re-arrangements, we get

$$\frac{M(\alpha)}{1-\alpha}\leq\frac{\Gamma(1-\alpha)}{(\ln(2))^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^\alpha} \tag{195}$$

The right-hand side of (195) has no simple approximation. Thus, we have to see what happens at the boundaries of the interval $\alpha\in[0-1]$. From (195), we have two extreme inequalities, namely:

$$\alpha=1\Rightarrow\frac{M(\alpha)}{1-\alpha}\leq\frac{\Gamma(0)}{1}\to\infty \tag{196}$$

$$\alpha=0\Rightarrow\frac{M(\alpha)}{1-\alpha}\leq\frac{\Gamma(1)}{\ln(2)}\approx 1.442 \tag{197}$$

The second condition (197) is obeyed, by all functions suggested in the preceding section, since by definition for $\alpha=0$ $\frac{M(\alpha)}{1-\alpha}=1\leq 1.442$, and this can be seen from the left corners of all panels in Figure 2. The condition (196) is also obeyed by all functions and this is well presented by their behaviors in the right corners of the panels of Figure 2.

10.2.2 Restrictions on the Mittag-Leffler memory kernel and results thereof

Now, we have to consider a flux with Mitag-Leffler memory, following the fading memory concept, namely

$$q_{ML}(t-s)=q_\infty+q_{0(ML)}E_\alpha\left(t^{-\alpha}\right) \tag{198}$$

It would be quite hard to carry out estimations as in the case with the simple exponential memory, i.e. a good approximation of $\left[q_{ML}(t-s)-q_{0(ML)}\right]^2$ that would be useful for estimation of the normalization function $B(\alpha)$ and we have to look for alternative approaches.

The Mittag-Leffler function (one-parameter) has two asymptotic approximations [55,56]

$$E_{\alpha}(t) \sim \begin{cases} E_{\alpha}^{\infty}(t) \doteq \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} = \dfrac{\sin(\alpha\pi)}{\pi}\dfrac{\Gamma(\alpha)}{t^{\alpha}}, & t \to \infty \\ E_{\alpha}^{0}(t) \doteq \exp\left[-\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right], & t \to 0 \end{cases} \tag{199}$$

From these asymptotics, it is hard to perform estimations based on the thermodynamic restrictions as was done in the preceding section. However, we recall the possibility of expressing the Mittag-Leffler as an approximation through Prony's series [57], as it is explained briefly next.

With the substitution $x = u^{\alpha}$, $E(-t^{\alpha})$ can be expressed as [58]

$$E(-t^{\alpha}) = \int_{0}^{\infty} \frac{\sin(\alpha\pi)}{x^2 + 2\cos(\alpha\pi)x + 1} \exp\left(-x^{1/\alpha}\right) dx \tag{200}$$

Expressing the integral as [58]

$$E_{\alpha}(-t^{\alpha}) = \int_{0}^{b^{-N}} + \sum_{j=1}^{N} \int_{b^{-j}}^{b^{-j+1}} + \sum_{j=1}^{M} \int_{b^{j-1}}^{b^{j}} + \int_{b^{M}}^{\infty} \tag{201}$$

and a consequent application of the Gauss-Legendre quadrature approximation to each subs-interval yields [58]

$$E_{\alpha}(-t^{\alpha}) \approx S(t) = \sum_{j=1}^{N+M} \sum_{i=1}^{n_j} w_{ij} \exp(-s_{ij}t) \tag{202}$$

where

$$w_{ij} = \omega_{ij}^{(n_j)} \frac{\sin c(\alpha\pi)}{x_{ij}^2 + 2\cos(\alpha\pi)x_{ij} + 1}, \quad s_{ij} = \left[x_{ij}^{(n_j)}\right]^{1/\alpha}, \quad \sin c(x) = \frac{\sin(x)}{x} \tag{203}$$

In (203), $\omega_{ij}^{(n_j)}$, $x_{ij}^{(n_j)}$ are the Gauss-Legendre quadrature nodes and weights of order n_j of the jth interval $\left[b^{j-N}, b^{j-N+1}\right]$.

The problem emerging immediately is about the number of exponentials involved in the approximation. Skipping the thorough analysis performed in [58] the numerical tests with a tolerance $\varepsilon = 2^{-23}$ (see Table 5 in [58]) that the number of exponentials increases as the fractional order α decreases by the estimate

$$N_{ex} = O\left(\log\left(\frac{\sin c(\alpha\pi)}{\varepsilon}\right)^2\right) \tag{204}$$

To close this point we may refer to [59,60] where asymptotic exponential expansions of E_{α} for $0 < \alpha < 1$ have been thoroughly investigated.

Now, referring to (202) we may write a single convolution constitutive equation (for the sake of simplicity) as an approximation

$$q(t) = E_{\alpha 1}\left[\left(-\frac{t}{\tau_k}\right)^{\alpha_k}\right] \approx E_{\alpha}(-t^{\alpha}) \approx \sum_{j=1}^{N+M} \sum_{i=1}^{n_j} w_{ij} \exp(-s_{ij}t) \tag{205}$$

The relaxation times $s_{ij} = 1/\tau_{ij}$ are related to fractional orders α_{ij} of the Caputo-Fabrizio operators by the simple relations $\alpha_{ij} = 1/(1 + \tau_{ij}/t_0)$.

In the convolution integral, the substitution of the Mittag-Leffler kernel by a Prony's series yields

$$\begin{gathered} q_{ML}(t) = q_{\infty} + \int_{0}^{t} E_{\alpha 1}\left(\frac{t-s}{\tau_k}\right)^{\alpha_k} \frac{d}{ds} F(s)\, ds \Rightarrow \\ \Rightarrow q(t) \approx q_{\infty} + \int_{0}^{t} \left[\sum_{j=1}^{N+M} \sum_{i=1}^{n_j} w_{ij} \exp(-s_{ij}t)\right] \frac{d}{ds} F(s)\, ds \end{gathered} \tag{206}$$

where $F(t)=\frac{\partial T(x,t)}{\partial x}$ and $F(s)=\mathscr{L}[F(t)]$ is the Laplace transform. Changing the order of integration and summations in (206) we get

$$q_{ML}(t)\approx q_{\infty}+\sum_{j=1}^{N+M}\sum_{i=1}^{n_j}\left[\int_0^t w_{ij}\exp(-s_{ij}t)\frac{d}{ds}F(s)\,ds\right] \tag{207}$$

Now, in terms of Caputo -Fabrizio operators with fractional orders α_{ij} we may express (207)

$$q_{ML}(t)\approx q_{\infty}+\sum_{j=1}^{N+M}\sum_{i=1}^{n_j}\left[(1-\alpha_{ij})D_t^{\alpha_{ij}}\frac{\partial T(x,t)}{\partial x}\right] \tag{208}$$

As an outcome of these thoughts, if the Mittag-Leffler function can be approximated by Prony's series, then we may write that

$${}^{ABC}D_t^{\alpha}f(t)\approx\sum_0^N(1-\alpha_i)D_t^{\alpha_i}\left[\frac{\partial T(x,t)}{\partial x}\right] \tag{209}$$

Therefore, we may suggest that for each term of the series approximations (207)-(209) the estimations developed for case of a simple exponential memory are also valid, thus we may further use the term $M(\alpha)$ instead $B(\alpha)$.

However, as was seen all suggested functions $M(\alpha)$ obeyed the restrictions imposed, and no specific one was selected. This needs more conditions to $M(\alpha)$ to be imposed and the problem, at this moment, is open for a solution and further developments.

10.2.3 Some simple versions of $M(\alpha)$ considered

As we have seen, there is a variety of normalization functions that cannot be unified toward creating fractional operators adequately applicable to the modeling of real data. Precisely, the function $M(\alpha)$ should contribute to the process of the inverse problems when the fractional order α should be defined. Because of that, the question is: Is it possible to find a simpler version of $M(\alpha)$ so that the following calculations would be carried out easier? From the functions summarized in Tables 1-3 we focus the attention on two simple versions: $M5=\Gamma(2-\alpha)$ and $M7=(2-\alpha)^{\alpha}$. The first candidate can be presented as,$M5=(1-\alpha)\Gamma(1-\alpha)$ and because the first term in this product coincides with the denominators in both fractional operators considered here we will abandon it; moreover, it is based on the Gamma function that would complicate the calculations. We suggest a modified version of $M7(\alpha)$ formulated as

$$M7n(\alpha)=(2-\alpha)^{n\alpha} \tag{210}$$

In this formulation, the dimensionless coefficient n controls the variations of$M7n(\alpha)$, as it is shown in Figure 3. Furthermore, in this way, we get an additional parameter that would facilitate the data fitting, as mentioned above. In this context, the Caputo-Fabrizio operator (in the Caputo sense) would look like

$${}^{CFC}D_t^{\alpha(n)}f(t)=\frac{(2-\alpha)^{n\alpha}}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\frac{df(\tau)}{d\tau}d\tau \tag{211}$$

Its Laplace transform is (assuming for the sake of simplicity $f(0)=0$)

$$\mathscr{L}\left[{}^{CFC}D_t^{\alpha(n)}f(t)\right](p)=(2-\alpha)^{n\alpha}\frac{p\mathscr{L}[f(t)]}{p+\alpha(1-p)} \tag{212}$$

Similarly, in the case of the ABC operator we get

$${}^{ABC}D_t^{\alpha(n)}f(t)=\frac{(2-\alpha)^{n\alpha}}{1-\alpha}\int_0^t E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\frac{df(\tau)}{d\tau}d\tau \tag{213}$$

and the Laplace transform is (assuming again for the sake of simplicity that $f(0)=0$)

$$\begin{aligned}\mathscr{L}\left[{}^{ABC}D_t^{\alpha(n)}f(t)\right](p)&=\frac{(2-\alpha)^{n\alpha}}{1-\alpha}\frac{p}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\mathscr{L}[f(t)]=\\&=(2-\alpha)^{n\alpha}\frac{p}{\alpha+(1-\alpha)p^{\alpha}}\mathscr{L}[f(t)]\end{aligned} \tag{214}$$

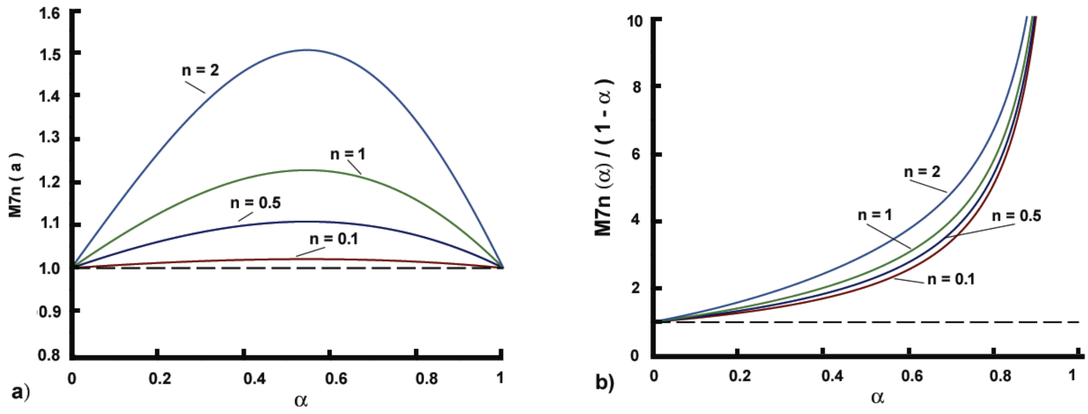

Fig. 3: Versions of the simplified function $M7n(\alpha)$ for various values of the parameter n: a) Non-normalized; b) Normalized as $M7n(\alpha)/(1-\alpha)$: Note: Dotted line $M(\alpha) = 1.0$

At this point, we have to close the discussions on the simplified versions of $M(\alpha)$, and $B(\alpha)$, since going deeply into the problem, many unresolved issues drawing problems beyond the format and the scope of this work appear. However, we formulated the problem with some details, and revealed that at the moment there is no unique solution to the functional relationships of $M(\alpha)$ and $B(\alpha)$. The conjecture that they are equal to unity is only a palliative approach but not a solution to the complete definitions of these operators. Thus, the problem is open and is waiting for further research.

*Remark.*The exercises with various versions of normalization functions revealed that at the moment there is no unique definition and more strict conditions on their properties should be defined. The attempts to apply the thermodynamic conditions to the case with exponential memory only demonstrated that additional restrictions are highly required. Therefore, there are two ways: To go deeply into the problem as defined above or to close the eyes assuming that it does not exist, and accept that $M(\alpha) = 1$ and $B(\alpha) = 1$. Otherwise, applying various versions, such as $M1(\alpha)$ [13] and $M11(\alpha)$ [9], will be a more artistic rather than systematic approach.

11 Final comments and outlines

The study on the integrals of the most popular fractional operators non-singular kernels went systematically through many issues related to their properties, especially on how they can be applied. The main outcomes of the performed analysis can be outlined as follows:

The definitions provided clearly distinguish the associated and constitutive integrals and mark their roles in fractional modeling:

a) The associated integrals try to find something that is already known from the era of fractional calculus with singular memories, mimicking the approach and the starting modeling equation.

b) The constitutive integrals are directly related, with their memory kernels to the relaxation function of the modeled transport phenomenon.

The application of two alternative modeling approaches, the fading memory formalism, and the Volterra equations, with correctly defined memory kernels, yield integrodifferential equations where the memory integral can be presented in terms of fractional operators with non-singular kernels. The transient heat conduction model served as a good template to demonstrate the applications of the associated integrals with their pitfalls and the successful modeling by applications of constitutive integrals.

The attempt to find clear definitions of the normalization function $M(\alpha)$ and $B(\alpha)$ was discussed for the first time since the definitions of the related fractional operators and the Losada-Nieto conjecture in 2015 [7]. The outcome of this attempt is definitions of various functions obeying the macroscopic conditions imposed on these normalization functions. However, the endeavor was not completely successful since, for example, applying thermodynamic restrictions the problems that emerged drew lines that should be resolved beyond the scope of this study. Even though, demonstrating that there are normalization functions different from unity, the efforts applied open new problems that are waiting for their solutions.

We suggest that the analysis performed as well as the results and problems highlighted through it would serve as both a focus on specific points in this new trend in fractional modeling and a generator of new problems to be resolved.

12 Conclusion

We draw short conclusions after too many comments and remarks in the text, and the main issues in a compact form can be formulated as:

–Associated and constitutive integrals related to fractional operators with non-singular kernels (Caputo-Fabrizio and Atangana-Baleanu) have been analyzed through a systematic analysis revealing their positions and applications in fractional modeling.

–The conceptual application of the constitutive integrals to the construction of models, applying both the fading memory concept and Volterra equations, using transient heat conduction as a modeling template demonstrated that fractional operators appear in modeling integrodifferential equations as separate memory terms. The approach has been demonstrated by 6 examples using well-known functions as memory kernels.

–Definitions and analysis of normalization functions related to these operators, systematically neglected in the literature after the conjecture of Losada and Nieto in 2015, have been carried out

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